AFTERWORD

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All the responses confirm that there are issues here about which we need to talk – using 'we' rather vaguely – but I hope we do not merely define *mathematics* and *discipline* in ways that serve to maintain our favourite dichotomies. Zazkis sees that I have not suggested that school mathematics and the discipline of mathematics are totally disjoint. So, to dichotomise further, what are my commitments and on what am I prepared to hedge a little?

I commit myself to a view of mathematics as recognised by research mathematicians, and by historical and published artefacts, and of mathematical habits of mind as described by Poincaré, Polya and others. These do not exist *in vacuo*, devoid of emotional and social relationship, yet it is possible to speak about them separately and hence to think about how they might be found among the complexities of school mathematics. Without these habits one is not doing mathematics even though what one is doing might be worthwhile from other perspectives, such as developing fluency with tools, or learning to learn together. This is why it is so hard for those teachers who have not experienced sustained engagement with such habits to teach mathematics in ways which are authentically related to the discipline. There are many who can and do, but also many who cannot and/or do not. True apprenticeship models of learning mathematics are rare, although some respondents show how closely designed tasks and pedagogy can lay foundations for shared mathematical argument to be the authority.

I talk of the shifts that have to be made in order to do harder mathematics (these are not one-way shifts, not permanent changes of perspective, but the growth of fluency in new perspectives so that one can choose which to use). These shifts are unlikely to be made without intervention. Ironically, many of them restore ways of thinking that may have been edited out of learners' repertoire by school, namely: dealing with uncertainty, working with ill-defined problems, seeing quantities as scaled and compared rather than added, and so on. Others have a special place in mathematics, such as dealing with abstract and idealised objects, reliance on deductive reasoning, distinguishing between discrete and continuous variables. Central concepts needed for quantitative understanding of global problems include some counter-intuitive ideas, such as risk, dependency of variables, acceleration/deceleration of growth rates, and opportunity cost. Counterintuitive ideas need expert intervention. Some so-called traditional teachers help students make these shifts; many so-called 'reform' teachers do not.

Rather than saying that school *should* produce people who act like mathematicians, I tried to call into question whether it makes sense that school students should be expected to act like mathematicians, while maintaining that they should be introduced to authentic mathematical activity so they know what it is. In some schools I see some aspects of authenticity (exploration, discussion, conjecturing, problem-solving) but not others (focusing on structure and relationships, looking at special cases, expressing generality, using symbolic tools fluently). My argument is that the institutional characteristics of schooling render the full canon of authentic mathematical behaviour inappropriate – they take it, mangle it, impose inadequate measures on it, and devalue it. It can also be devalued when the essential shifts of mathematics such as those between empirical and deductive reasoning, *ad hoc* and general methods, and enactive to abstract representations, are ignored in order to include and value all students' ideas or to focus on workable answers. It is also devalued by rule-based teaching in which the only acceptable curiosity is that suggested by the teacher and displayed by already-successful students. The full canon includes not only various social modes of working but habitual shifts of understanding between concrete and abstract, examples and generalities, special cases and new concepts, objects and representations, between graphs, relations, functions and cases. Most of these shifts take time to make, so schools tend to avoid them in favour of rule-based approaches. I commit strongly to the view that this devaluing is not about ways of teaching, but about the content of teaching.

School is almost always about coercion. Coercion to practise techniques is very different to that described by Peatfield who shows that mathematicians may practise techniques, but this practising is purposeful, self-chosen, and may use special rather than random examples to learn more about how a method works. The difference is not only in the element of choice but also in the nature of reflection on effects. Coercion and empowerment are not mutually exclusive and teachers might use the coercive powers of the institution to empower learners. Mathematics as a discipline includes many empowerment opportunities, not only about future employment but also in ways of thinking, for example: to construct powerful abstract models, *e.g.* exponential structures, which make it possible to think about hitherto inaccessible ideas; to take a mathematical object and find out for yourself the effects of changing one aspect of it; to be able to state rigorous 'truths' based on accepted axioms.

On the face if it, de-schooling seems an attractive option. My argument also suggests subversion through teachers understanding the power of schools to distort disciplines and working within systems to establish mathematical integrity.