SCHOOL MATHEMATICS AS A SPECIAL KIND OF MATHEMATICS

ANNE WATSON [1]

It has long been an aim of mine that school students should be introduced to authentic mathematical activity such as is practised by professional mathematicians, and those forms of exploration that contribute to the development of the subject. Through this kind of activity, students get a sense of mathematics as human invention, as certain habits of mind, that is more engaging and meaningful than learning a procession of given facts, methods and question-types. For example, in one school well known to me, all mathematics classrooms display a text derived from Burton's (2004) descriptions of professional mathematical behaviour:

> Mathematicians …
>
> - have imaginative ideas
> - ask questions
> - make mistakes and use them to learn new things
> - are organised and systematic
> - describe, explain and discuss their work
> - look for patterns and connections
> - keep going when it is difficult
>
> Together we can learn to be mathematicians.

How far does it make sense to expect that school students can act like mathematicians? Before I start this exploration, let me make it clear that my own experience as a teacher convinces me that nearly all students can indeed act in the ways described above, and can also generate mathematical ideas, identify and describe relationships between properties, construct effective algorithms. It is also my experience that those who have the opportunity to do this become better mathematical learners than those drilled in more traditional methods. This is now so well supported by research that it hardly needs stating. But to say that mathematical learning is better if learners 'act like mathematicians' is not to say that it is possible to do this in institutional settings. In this article I take 'school mathematics' to be the institutionalization of mathematical knowledge for novices, and it is the relationship between this and the discipline of mathematics that I am setting out to describe.

I shall claim that school mathematics is not, and perhaps never can be, a subset of the recognised discipline of mathematics, because it has different warrants, authorities, forms of reasoning, core activities, purposes and unifying concepts, and necessarily truncates mathematical activity in ways that are different from those of the discipline. By 'discipline of mathematics' I mean the activities that advance mathematical knowledge: the forms of engagement, kinds of questions, and standards of argument that are accepted as contributing to the conventional canon of pure or applied mathematics. 'School mathematics' means the forms of engagement in mathematics in formal teaching contexts for the novitiate, including some undergraduates, or for those who do not even see themselves as novices but have mathematics thrust upon them.

In its worst form, school mathematics can be a form of cognitive bullying that neither develops students' natural ways of thinking in advantageous directions, nor leads obviously towards competence in pure or applied mathematics as practised by adult experts. The relationship of school mathematics to adult competence is similar to the relationship between doing military drill and military leadership; between being made to eat all your spinach and becoming a chef; between being forced to practise scales and becoming a pianist. There are some connections, but they are about having a focus on a narrow subset of semi-fluent expertise in negative social and emotional contexts, without full purpose, context and meaning. That some people become effective military leaders, beautiful pianists or inspiring cooks is interesting, but what is more interesting is the fact that most people who go through these early experiences do not achieve these levels; instead, they merely follow orders, or hate green vegetables, or give up practising their instruments.

The image around which I hang this paper is the cafeteria at the heart of the mathematics faculty at Cambridge. It is a large room with coffee at one end, small tables covered with papers and laptops, each surrounded by four chairs, students (mainly male), some undergraduate, some graduate, pure and applied, some alone, some in casual groups, some in self-organised study groups with their own internal disciplines and plans, taking time, talking, arguing. There are many furrowed brows, people leaning forward with arms and hands working away to express some thought; bodies, words and minds hauling together to communicate how they see relationships and properties, and offering each other handles to work with abstract objects and multi-dimensional extensions. No one is doing exercises or practising techniques; no one is interrupted by instructions or bells telling them to change to the next task; no one is taking their work to teachers to be marked. Every now and then there are whoops of excitement or groans of frustrated realisation.

I recognise that behind such an apparently idyllic scene there are a variety of pressures and frustrations that arise from institutional demands and the lived experience of mathematicians and mathematics students. One might also say that Cambridge is not typical, or question the dominance of men and ask who is excluded, or simply laugh at how anyone could want to replicate such behaviour. Nevertheless, this picture is of people doing mathematics together, in ways that fit the description 'acting like mathematicians'. It is also possible to replicate much of this scene in school classrooms. But even in these environments, where teachers may believe that their students are 'working like mathematicians', there are major significant differences between school and 'the discipline'.

Mathematical enquiry

The students in the cafeteria are in various stages of transition between school maths and academic maths. With a few exceptions they have rarely worked like this at school and only a few may have been told explicitly that this is how they could be working now. Mathematics as a discipline includes the social and cultural characteristics that have contributed to its genesis. Most mathematical advancement does not arise in an isolated, independent way but is a product of its time, within the current paradigms, co-emergent with current technological and economic needs and tools, co-emergent with the *zeitgeist*. Locally and globally, it is a product of social interaction, even if that interaction does not take place until there is a result (Burton, 2004; Davis & Hersh, 1981).

In school, things are generally different. I am not assuming any homogeneity about school practices – it is always possible to read such assumptions and argue to exclude your particular patch of the world, or particular teachers, or a particular curriculum. Instead, I focus on the mathematical practices in which, I presume, the students described above are engaged but which cannot be observed in the short-term, nor do they fit neatly into socio-cultural analyses of the learning environment. For me, the starting point is what it means to do mathematics and to be mathematically engaged.

In the discipline of mathematics, mathematics is the mode of intellectual enquiry, and effective methods of enquiry become part of the discipline – so much so that mathematics theses do not have chapters explaining methodology and methods. This is not just about the availability of coffee, social groupings, the published discourse, or choices between Maple and Mathematica, although these are crucial parts of the picture. 'Doing mathematics' is predominantly about empirical exploration, logical deduction, seeking variance and invariance, selecting or devising representations, exemplification, observing extreme cases, conjecturing, seeking relationships, verification, reification, formalisation, locating isomorphisms, reflecting on answers as raw material for further conjecture, comparing argumentations for accuracy, validity, insight, efficiency and power. It is also about reworking to find errors in technical accuracy, and errors in argument, and looking actively for counterexamples and refutations. Mathematics is about creating methods of problem-presentation and solution for particular purposes, tinkering between physical situations and their models, and it also involves proving theorems. Of course all these can be done in school, but it is the way in which these are coordinated and aligned that makes the overall activity different.

Different kinds of mathematics

The practices of mathematics just described are very far indeed from the concerns of psychologists who want to construct efficient methods of instruction, seeking for the fastest and most productive ways to teach students how to find answers to broadly isomorphic problems. Research and development in psychology and neuroscience do not even begin to offer ways to induct students into the practices listed above. My perspective on the discipline of mathematics is also different from the concerns of the socio-cultural take on classrooms and learners, which tells convincing and robust stories about the existence of differences in practice, and the process of development of identity in different cultural settings, but which cannot put detailed flesh on these stories in terms of the development of specific mathematical ideas.

Of course we would expect to see different kinds of learning and different classroom practices where there are different views of mathematics and different curriculum goals, and in particular we would expect to see a political dimension to this as countries connect the outcomes of education with their future workforce. As Radford (2003) has said,

> While the humanist view of mathematics emphasizes the role this discipline plays in the development of logical thinking, abstraction, rigor and other highly prized faculties that have been the clear marks of men of sophisticated spirit since the Enlightenment, the socialist trend stressed the applicability of mathematics. Its importance was seen in terms of the utilitarian ability of mathematics to master nature in the interests of mankind. (p. 552)

If the curriculum takes a 'humanist' approach, rigorous argument and proof are taken to be important aspects of the subject, yet many teachers treat 'proof' as a topic within mathematics rather than as the way truth is examined and warranted throughout the subject. They also take argument as a specific focus for particular tasks or lessons. If this is combined with fallibilist-inspired teaching styles such as investigative work, 'What if…?' questions, and algebra introduced as an expression of generality, then in many classrooms two standards of argument are being offered. One is empirical, arising from specific cases, tables of values, inductive construction of formulae and testing special cases, while the other requires deductive reasoning and a willingness to engage in formal logical argument. The shift learners have to make between inductive to deductive argument is hard and demands some 'giving up' of forms of reasoning which serve students well elsewhere. For example, it is well documented that students who can produce examples easily, in an inductive mood, find the production and understanding of counter-examples, with their deductive role, much harder (Zaslavsky & Ron, 1998). To use mathematical argument as the natural and normal method of enquiry in mathematics it needs to be fully embedded in day-to-day lessons as part of classroom discourse. In reality, a curriculum written in accordance with what Radford calls 'humanist' principles is likely to be presented as a taught curriculum in which topics are presented in a developmental order and few students will see the role for logical thinking embedded in the development of mathematical understanding. If their own logical thinking *does* play a part in mathematics lessons, as it does with some teachers, these humanist curriculum aims might be realized in part.

Learners exposed to utilitarian mathematics have a different experience compared to those taught with a more abstract emphasis, but solving realistic and everyday problems need not lead them to understand the role of mathematics beyond providing *ad hoc* methods for real problem-solving, or as a service subject that holds tools for moving forward in other domains.

Mathematical shifts of understanding in school mathematics

But the choice between different possible 'mathematics' is, to my mind and in my experience, not the core problem about mismatch between school mathematics and the discipline of mathematics, as I intend to show. I am going to describe several shifts of understanding that have to be made for learners to be successful within and beyond school mathematics, whatever the dominant view of the discipline:

Additive to multiplicative reasoning

A shift from seeing additively to seeing multiplicatively is expected to take place during late primary or early secondary school. Not everyone makes this shift successfully, and multiplication seen as 'repeated addition' lingers as a dominant image for many students. This is unhelpful for learners who need to work with ratio, to express algebraic relationships, to understand polynomials, to recognise and use transformations and similarity, and in many other mathematical and other contexts. Eventually another shift has to be made to exponential reasoning.

Probabilistic reasoning

The concept of probability, understood mathematically, offers a different warrant for truth than is associated with either deductive logic or induction from empirical evidence. To understand probability as a tool, or to see the world probabilistically, requires abstraction and imagination well beyond observable phenomena. Moreover, one cannot merely follow algorithms to get answers except in very simple contexts; often learners have to decide for themselves whether events are independent, exclusive or not. The shift here is from being told everything about a situation to having to identify characteristics and properties for oneself, before conceptually based action.

Integration

In the UK, integration used to be the first context in the school curriculum where learners could not merely apply methods and be sure they would get some kind of answer. However students now are told not only what method to use, but also what substitutions to make if substitution is the method examiners wish to see. The shift from being told what methods and tools to use, to developing sensible selection criteria for what is possible, has to be made to become a mathematician.

Geometrical reasoning

Questions involving application of theorems can be avoided in UK national tests at 16+ and students can still be awarded the highest grades. Theorems and proofs of any kind, let alone geometrical contexts, do not play a part in higher school examinations. In countries where reasoning from axioms still has a place in the school curriculum, these may be taught mechanically, as a kind of memory or question-spotting activity rather than as a demonstration of deductive reasoning to explore phenomena and to establish a particular kind of truth. A shift from knowing what to look for, to selecting what to look at, then to deciding what to use and constructing multi-stage arguments for oneself, has to be made to become a mathematician.

Getting answers to gaining insight

A problem (in Polya's sense) can be solved and nothing new be learnt, even about problem-solving. A mathematician will usually have a purpose in mind when solving a problem, so that the outcome is used to reflect on the context in which the problem arose, to decide if something unexpected has arisen, to raise further questions, or in some other way to enrich or extend knowledge. The answer, if there is one, is not the end of the process. A shift from getting answers to gaining insight or constructing arguments has to be made. In non-mathematical contexts, once the problem is solved there is no motive for extending the work hypothetically.

Modelling

Mathematical modelling of realistic or artificial situations is a feature of many mathematics curricula. While generalisation might take place in order to create the model, and this might be explored further to look for extreme cases or further variation, the modelling process itself does not require more abstraction or structural understanding than the situation being modelled.

The shifts of perception, attention and engagement just described do not take place without the learner being in an environment of tasks, language, representation, deliberately designed to support those shifts – that is, a schooling environment whose purpose is to ensure learners make these shifts.

The discipline of school mathematics

It is not mere coincidence that these are all inherently 'hard-to-teach' aspects of mathematics. They all offer epistemological obstacles that require shifts of reasoning, or new ways to act with imagery, or encapsulation of previous experience to be overcome.

Importantly, it is becoming more and more the case that, in an effort to ensure that more students can gain school mathematics qualifications, the difficult shifts that would have to be made for school mathematics to be a subset of the discipline of mathematics are being edited *out* of mathematics as a school subject, rather than edited *in* as the discipline itself becomes more complex, more post-modern and less certain. I am talking here of all kinds of classrooms: reform and traditional; classrooms in which students are expected to behave like little mathematicians and those where they are expected to behave like acquiescent cognitive machines; classrooms in the developing world and those in high-achieving countries. I am not arguing for or against particular kinds of school curriculum; I am saying that the task of schooling the mathematical mind to make the shifts that have to be made to become a mathematician is not, by and large, undertaken in schools.

In all the curriculum contexts I described above, and in many others, there are features that are peculiar to school mathematics, and the way it is generally taught, which are not part of the discipline as practised by adult experts. For example:

- there is a strong focus on answers and generalisations rather than structural insight and abstraction;
- there is avoidance by teachers, tests, and curricula, of the need for uncertain choices;
- curricula seem to cling to topics and approaches that can be represented experientially, diagrammatically or in concrete ways, rather than in abstract and imaginary ways;
- in school, inductive, empirical, and ad hoc reasoning are privileged over deductive or probabilistic reasoning.

These characteristics, peculiar to school mathematics, can hinder the progress of students at university on pure and applied courses, alongside inappropriate work habits, challenges to identity, and unrealistic expectations of the subject (sometimes promulgated by popularisers). Mathematics as a discipline, by contrast to school mathematics, is concerned with thought, structure, alternatives, abstract ideas, deductive reasoning and an internal sense of validity and authority. It is also concerned with uncertainties about ways forward in its own realms of enquiry. To do maths includes holding nagging questions in the mind while carrying on with life, and not expecting answers to be found, problems to be solved, within the confines of a particular room or timescale. The concerns of school mathematics pull learners in directions that differ from these. The core activity in school mathematics is to learn to use mathematical tools and ways of working so that these can be used to learn more tools and ways of working later on.

Whichever approach is taken by the curriculum, most school students are taught and examined on mathematics of a kind that is done, both in the academy and in other workplaces, by machines. They are taught not as an apprenticeship to adult mathematics users, but as bottom-up preparation for future mathematical activity that might one day be meaningful, either as an intellectual or an economic activity. Furthermore, they are taught this in regimented settings, with short time-scales, by teachers who themselves have limited experience of the mathematical practices described above.

Those students who continue beyond school might be motivated by a so-far satisfied need to have right answers, or by getting a kick from the resolution of puzzles, or by discovering special features which intrigue them, or by anticipating the delayed gratification of getting a further qualification, or of finally being able to study at the cutting edge. They might also have what Krutetskii (1976) identified as common features of mathematically gifted students: propensities to see the world and organize their mental activity in certain mathematical ways. The list he identified has little in common with the contents of formal taught lessons and assessment regimes, and much to do with the list of practices above, and what Cuoco and his colleagues (1996) have called mathematical 'habits of mind'. Neither a curriculum based on rigour, historical genesis and conceptual development, nor a curriculum based on modelling and authentic contextual questions, can achieve the education of the mind required to engage consistently, habitually, with the normal intellectual practices of mathematicians.

The roles of unifying concepts

A further striking difference is in the role of unifying concepts in mathematics as a discipline and in school mathematics. In mathematics as a discipline these orientate much of what is studied and researched. Such theories provide opportunities for links and connections within the discipline, new languages for discussing ideas, and new questions to be explored. In school mathematics, however, the shaping of a curriculum with unifying concepts could be unhelpful. A unifying concept makes sense to mathematicians precisely because it offers unification of previously disparate ideas of which they have a range of experience. For school students, to be told about a unifying concept before experiencing several widely differing examples of it reduces the concept to 'something else to be learnt' or 'something else to be taught' rather than an enabling encapsulation.

Linearity, for example, crops up throughout school and could have been included in the 'hard to teach' list above, because it requires a shift from looking at functions as representations of relationships to comparing properties of functions. However, this only makes sense when a learner has experienced several different linear and non-linear situations so that there is a need for a word for those that behave in certain ways. Older mathematicians are supposed to have had a range of experiences to draw on when they meet, for example, vector spaces so that they have examples which might be activated by hearing definitions, theorems and questions (Watson & Mason, 2002). To use the idea of a unifying concept successfully in school mathematics would require vertical coordination of teaching across years by teachers who have constructed a shared understanding of how that concept will be approached and learnt. In the discipline of mathematics, unifying concepts arise through the published and broadcast insights of mathematicians drawing on the processes of historical cultural genesis – they move the field on.

Constraints on teaching that alter the discipline

It certainly is the role of school mathematics to provide a range of experiences of various kinds so that students understand the usefulness of mathematics, and can do various necessary calculations and estimations. It would also be helpful for students to understand the purpose and value of practice, and of basic standards of accuracy. School could introduce students to ways of working on mathematics, to the kinds of questions mathematicians ask and the subject matter worked on, but would have to include in that some of the practices of number systems, algebraic manipulation, and ways of using diagrams, which are assumed within the discipline. At the most extreme, schools use entirely different kinds of questions, enquiry, warrants and work habits than those of the discipline. At best, they introduce what is to come, while moulding it to fit the institutional constraints, rather than to fit the development of mathematical ideas.

Many teachers and projects try to make their classrooms more and more like mathematical workshops, but the school context overlays these with purposes that are not about the development of mathematics, but are more about learning what to do and how to be. In the best of these classrooms, teachers are explicit about appropriate forms of statement and discussion in ways that are not made explicit in mathematics faculties, but even the best teachers cannot be present at every student's side to be explicit about the many practices of professional mathematics listed above. Mathematics as a discipline includes discussion and critique of its own modes of enquiry, but does not include explicit teaching of these modes of enquiry, or practising core techniques, as school mathematics has to do. Mathematicians decide for themselves what they need to practise, and how to practise it, and how to validate their techniques.

In school the truncation of mathematical activity takes place when enough has been learnt for now, or at the end of a lesson, or when the curriculum requires moving on, or a syllabus has been 'covered', or students leave school. In maths the truncation of enquiry happens when a problem has been solved, a proof has been accepted, a model has been produced, or everyone is tired of the enquiry – but all of these can be the starting points for new enquiry. Of course temporary truncation takes place because of teaching, administrative and family commitments, but the mathematician is not expected simultaneously to worry about academic problems in geography, language, science, citizenship as well as mathematics.

Another aspect of school mathematics that differs from the discipline of mathematics is the nature of authority. Freudenthal (1973), Vergnaud (1997) and others have emphasised that mathematics includes its own methods of validation, and a good teacher can enable students to use the structures of mathematics to verify their own work and ideas. In the education system as a whole, however, validation and authority are highly structured through textbooks, examinations, and curriculum systems rather than by mathematicians working as a community. Formal axiomatic proof is inaccessible for most school-age children, although many children of all ages can develop robust arguments with appropriate scaffolding. Even so, for most of what is taught in school the correct application of procedures and checking answers using inverse procedures, or checking reasonableness, are available as warrants for truth, and the prevailing warrants in classrooms are usually teacher's approval, or checking answers.

Conclusion

In conclusion, therefore, I find myself claiming that school mathematics is necessarily not a subset of the discipline of mathematics, whatever the nature of mathematics being taught, whatever the way it is taught. Essentially, this is for the following reasons:

Schools have to prepare students for future study and/or employment and this requires them to be explicit about ways to develop recall, fluency, accuracy, and ways of working; these constitute the major part of the goals of school mathematics.

Mathematics requires shifts away from the everyday thinking of mathematics lessons to special forms of mathematical thinking. Someone has to do this work, which requires knowledgeable teachers in school and university. We have to recognise that many schoolteachers of mathematics do not have personal experience of what it means to do mathematics over time, exploring questions that have intellectual purpose, not pedagogic purpose.

Limited time slots, curricular pressures, and assessment regimes constrain or prevent the development of the kinds of questions and ways of working which characterise the discipline.

Authority in school mathematics lies with teachers, textbooks and assessment regimes, not with mathematical argument.

It is not only ways of working and goals that are different between school and disciplinary maths; it is the way that these shape the available forms of mathematical enquiry that makes school mathematics a different discipline, with its own rules, purposes, authorities and warrants.

Note

[1] The substance of this paper was presented originally at a working group on "Disciplinary Mathematics and School Mathematics" at the Symposium on the Occasion of the 100th Anniversary of ICMI (International Commission on Mathematical Instruction), Rome, March 5-8, 2008.

References

Burton, L. (2004) *Mathematicians as enquirers: learning about learning mathematics,* Dordrecht, NL, Kluwer.

Cuoco, A., Goldenberg, P. and Mark, J. (1996) 'Habits of mind: an organizing principle for mathematics curricula', *Journal of Mathematical Behavior* **15**, 375-402.

Davis, P. and Hersh, R. (1981) *The mathematical experience,* Brighton, UK, Harvester.

Freudenthal, H. (1973) *Mathematics as an educational task,* Dordrecht, NL, Kluwer.

Krutetskii, V. A. (1976, translated by J. Teller; edited by J. Kilpatrick and I. Wirszup), *The psychology of mathematical abilities in school children,* Chicago, IL, University of Chicago Press.

Radford, L. (2003) 'From truth to efficiency: comments on some aspects of the development of mathematics education', *Canadian Journal of Science, Mathematics and Technology Education / Revue canadienne de l'enseignement des sciences, des mathématiques et des technologies,* **4**(4), 551-556.

Vergnaud, G. (1997) 'The nature of mathematical concepts', in Nunes, T. and Bryant, P. (eds.) *Learning and teaching mathematics: an international perspective,* London, UK, Psychology Press.

Watson, A. and Mason, J. (2002) 'Extending example spaces as a learning/teaching strategy' in Cockburn, A. and Nardi, E. (eds.), *Proceedings of the 26th Annual Conference of the International Group for the Psychology of Mathematics Education,* Norwich, UK, University of Norwich, **4**, 377-384.

Zaslavsky, O. and Ron, G. (1998) 'Students' understandings of the role of counter-examples', in Olivier, A. and Newstead, K. (eds.), *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education,* Stellenbosch, South Africa, University of Stellenbosch, **4**, 225-233.

Responses to Watson

Looking for a possible intersection

RINA ZAZKIS

I concur with Watson in referring to mathematics as an activity and a way of engagement, rather than as a body of knowledge. Her main claim is that "school mathematics is necessarily *not a subset* of the discipline of mathematics", as there are "major significant differences".

Focusing on these differences, a superficial reading might prompt one to interpret her claim to mean that school mathematics and disciplinary mathematics are disjoint sets. However, a more attentive reading will acknowledge claims such as "it is also possible to replicate much of this scene in school classrooms" and "Of course all these can be done in school" – where the phrase "all these" refers to activities involving mathematicians advancing disciplinary mathematics. Thus, while I accept the "not a subset" claim, I prefer to consider intersecting rather than disjoint sets, and this intersection is of my interest here.

I recognize, considering historical trends and cultural settings, that school mathematics and disciplinary mathematics, as well as the intersection thereof, are constantly changing. However, my central claim is that teacher education plays a pivotal role in determining the nature and the size of this intersection, in any context. That is, guiding prospective teachers through the experience of "working and thinking like a mathematician" may eventually result in instilling these ways of working in students and thus create a larger intersection between approaches practiced in teaching and learning school mathematics and approaches employed in developing disciplinary mathematics. While Watson recognizes teachers' experience and motivation as one of the obstacles, I suggest that this also is a key to the solution.

In describing school mathematics, Watson mentions "two standards of argument" and acknowledges the demanding shift from inductive to deductive argument. However, rather than presenting this as a dichotomy and a need for a shift, school mathematics may focus on considering inductive argument as an introduction to a deductive one. That is, empirical exploration of special cases is conducted not to draw conclusions, but to conjecture and find out what is there to prove. I exemplify this approach in what follows.

Example: Exploring tridians and conjecturing

Similarly to a median, we define a *tridian* as a segment that connects a vertex of a triangle to a point on the opposite side that is marking $^{1}/_{3}$ of the side's length. As shown in Figure 1, BX, BY, AW, AZ, CS and CT are tridians in a triangle ABC.

Inspired by Brown and Walter's (1983) "what if not" strategy, one of the tasks I present to prospective secondary teachers is as follows:

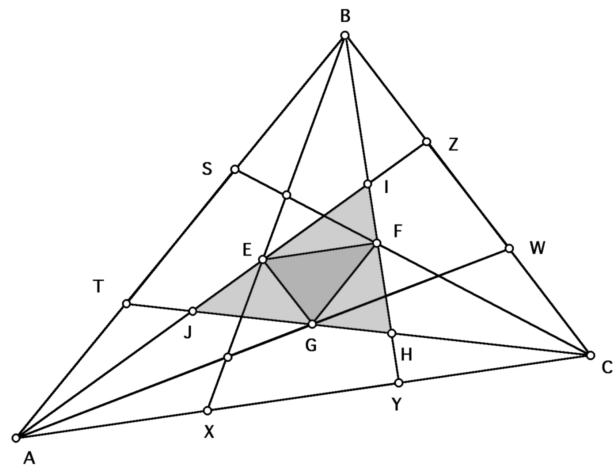

Figure 1: Tridians in a triangle

> Tridians generate many interesting relationships at various points of intersection and of various areas they cut. Compare for example the areas of triangles ABC, AZW, IJH and BIZ. With the help of the Geometer's Sketchpad, make several conjectures about tridians in a triangle. Formulate and prove at least 2 of your conjectures.
>
> What if not? What if not a triangle? What if not tridians? What if … Explore. Present at least one variation on a theme of tridians in a triangle. Prove at least some of your observations.

There are many possible relationships. While some are quite obvious and easy to prove from the definition of a tridian, such as the ratio of areas ABX:ABC = 1:3, others are rather challenging, such as the ratio of areas IJH:ABC = 1:7. [1]

What's in a task?

The tridians task exemplifies several important features of "thinking like a mathematician". First, there is an invitation to observe relationships, to notice, to wonder and then to formulate and test conjectures. The software helps in exploring and verifying conjectures: preservation of relationship when "dragging", while not a proof, is a very good indicator of a relationship to be proven. This activity is similar to the work of mathematicians in constructive mathematics: the computer is used not to prove theorems but to find out what is there that may be proven (Borwein, Beiley & Girgensohn, 2004). Of note, while computer feedback presents wonderful encouragement for testing conjectures, it is not a necessary component in this kind of activity.

Second, there are choices of engagement and of levels of difficulty. Though the task directs learners to one particular triangle, different individuals will observe different properties and will wonder about and justify different patterns and relationships. Working with students at different levels of mathematical sophistication, these choices, while limited, are extremely important – after all, mathematicians chose for themselves what would be their next investigation or their next proof.

And third, there is an incentive in the task – that would hopefully develop into a habit – not to leave the problem when it is solved, but to continue with explorations. Once the initial question is answered, the task is not left to rest. Answers generate further questions that involve either variation or generalization of the previous ones, allowing new observations and conjectures to emerge.

Conclusion

The task discussed above presents an opportunity for many activities – such as empirical exploration, logical deduction, seeking variance and invariance, conjecturing, seeking relationships, verification or proof – that Watson attributes to disciplinary mathematics. Similar engagements do not require one to be working at the frontiers of the discipline; they can be presented at any level of difficulty in helping students develop mathematical habits-of-mind. "Working like a mathematician" experiences in teacher education open the gate for similar experiences for students and increase the intersection between school mathematics and disciplinary mathematics.

Note

[1] When one proves this conjecture using linear algebra, vectors or affine coordinates, the challenge is to prove it using only methods of Euclidian geometry. I thank Guershon Harel for presenting me with this challenge.

References

Borwein, J., Beiley, D. and Girgensohn, R. (2004) *Experimentation in mathematics: computational paths to discovery,* Natick, MA, A. K. Peters.

Brown, S. and Walter, M. (1983) *The art of problem posing,* Philadelphia, PA, Franklin Institute Press.

Mason, J., Burton, J. and Stacey, K. (1982) *Thinking mathematically,* Reading, MA, Addison-Wesley.

Is all course-based mathematics special?

DAVID W. HENDERSON

Wow! Watson set off contradictory reactions in me: I find myself at the same time both strongly agreeing and strongly disagreeing with her arguments. I do not mean agreeing with some of her arguments and disagreeing with others; rather I am in the position of strongly agreeing and strongly disagreeing with her overall argument. I recognize this as a potentially very creative place that I find myself in; so let me try to examine where I am standing.

Here is what I find myself strongly agreeing and strongly disagreeing with: Watson describes what she means by the "discipline of mathematics" and what she means by "school mathematics". She then argues that school mathematics is, for reasons she delineates, "necessarily not a subset of the discipline of mathematics". After internalizing my conflict, I find that I have trouble with the large (I think I would even say 'huge') gap between 'school mathematics' and the 'discipline of mathematics'. In particular, there is 'college/university mathematics' that engages mathematics majors but also many other student who take mathematics courses. I think that Watson's description of 'school mathematics' as "the forms of engagement in mathematics in formal teaching contexts for the novitiate … or for those who do not even see themselves as novices but have mathematics thrust upon them" is a description that also applies (at least in USA universities) to almost all undergraduate courses and graduate courses up to the level of the graduate comprehensive exams (in the USA most of the first-year and some second-year graduate courses are geared to these exams) – I shall call these pre-comprehensives courses.

For example, the students in these graduate first-year courses may not have "simultaneously to worry about academic problems in geography, language, science, citizenship as well as mathematics"; but they *do* have simultaneously to worry about academic problems in algebra, analysis, topology, and, maybe, logic, probability, or geometry. In undergraduate years students normally take at least four courses at a time, at least one (usually two) of which is not mathematics.

Watson states in her first page: "let me make it clear that my own experience as a teacher convinces me that nearly all students can indeed act in the ways described [in Burton's 'professional mathematical behavior']" and then describes in what ways this does not in practice generally happen in school courses. I would make the same statement about pre-comprehensives courses. Note that in her Cambridge mathematics cafeteria there is no mention of courses – and, if they are taking courses, are these courses different in kind from their school courses? Do not many (most?) pre-comprehensives courses have "authority" that "lies with teachers, textbooks and assessment regimes, not with mathematical argument". If not, why have comprehensive exams? Maybe Cambridge is very different; but I suspect that, even in Cambridge, most students who study mathematics are not mathematics majors and are not among those that Watson observed in the cafeteria.

I agree with Watson's descriptions of the difficult shifts in students that are necessary for them to participate in the discipline of mathematics. Of course, some mathematics students make these shifts and become mathematicians; but, is there a "typical" college or university where this happens in *courses* because of "an environment of tasks, language, representation, deliberately designed to support those shifts, that is – a schooling environment whose purpose is to ensure learners make these shifts" or is the most common occurrence that the student (at all pre-comprehensives levels) somehow makes this shift outside of courses? – one reason that I have often heard cited by mathematicians is the inspiration from a special teacher (school or university) – I do not recall any mathematician saying that their inspiration came from a specific course.

So what are my conclusions? I look at Watson's statement: "But to say that mathematical learning is better if learners 'act like mathematicians' is not to say that it is possible to do this in institutional settings". However, Watson has described in other publications examples of school classrooms that go in this direction; and I suspect that we all know, at the school and undergraduate levels, examples of special courses that accomplish this to a greater or lesser degree. So, I am left with the question that Watson's article does not resolve: *Would it be possible for typical pre-comprehensives mathematics courses to contain "an environment of tasks, language, representation, deliberately designed to support those shifts – that is, a schooling environment whose purpose is to ensure learners make these shifts"?* I think the answer is 'yes' – it is possible; but I agree that it does not, at present, happen in the typical pre-comprehensives course. Some of the reasons that it does not happen are the limitations delineated by Watson; but, I believe, that these limitations apply at all pre-comprehensives levels, though less at the university than in schools.

Maintaining the mathematical integrity of school curricula: the challenge

GUERSHON HAREL

In essence, Anne Watson's paper can be interpreted in terms of and organized around four fundamental questions. The first three are: (1) What is mathematics? (2) Do current school curricula maintain the mathematical integrity of their content? and (3) Can, in principle, school curricula maintain the mathematical integrity of their content? Watson's justifications as to why her answer to the third question is negative raises a fourth question: (4) What are the current belief systems and social and institutional constraints that make school mathematics not mathematical, or, using Watson's words, "not a subset of the discipline of mathematics"?

Regarding the first question, let me begin with the premise that there is one and only one mathematics. In this respect – and despite my inference that Watson shares this premise – the sentence "school mathematics is not a subset of the discipline of mathematics" is logically inconsistent. This premise is critical because it entails that curricular objectives of school mathematics must be formulated in terms of the elements that define mathematics. Elsewhere, I define *mathematics* as the discipline consisting of all the *ways of understanding* and *ways of thinking* that have been institutionalized by communities of mathematicians throughout history. The terms "ways of understanding" and "ways of thinking" have technical meanings, but for our purpose here, they can be thought of as two categories of knowledge, the first refers to "subject matter," such as particular definitions, theorems, and proofs, and the second to "conceptual tools," such as heuristics and beliefs about what constitutes proof in mathematics (Harel, 2008). Watson offers many examples of *mathematical* ways of thinking, and she convincingly argues that these have nothing or little to do with the ways of thinking targeted and promoted by schools. There are two significant aspects to her approach. First, collectively, her list of examples typifies, rather than just exemplifies, mathematical practice. Second, for Watson these ways of thinking are not *a priori* constructs but products of human activities – they are what mathematicians do to learn, engage in, and produce mathematics.

An implication of the first aspect is that mathematical ways of thinking should be identified and studied by researchers and, accordingly, targeted as instructional objectives by curriculum developers and teachers. The second aspect is even more critical: it implies that learning – as a human activity – necessarily involves the construction of imperfect and even erroneous ways of understanding and deficient, or even faulty, ways of thinking. The history of mathematics is rich of cases that attest to this fact.

The shortcomings of current curricula and instruction that Watson points to cannot be justified on the basis of this nature of the learning process. Rather – and this is the thrust of Watson's argument – they are rooted in the character and structure of our educational systems, particularly their shared meanings of *knowledge, learning,* and *teaching* and their shared perspectives on the social, cultural, behavioral, and emotional factors involved in the learning and teaching of mathematics. These meanings and perspectives dictate the responsibility and school-related behaviors of students, teachers, school administrators, and parents. They entail the kinds of tools for measuring what students know and sets expectations for what they ought to know. And they orient the educational systems as to what, why, and how to carry out school-related actions.

Rather than calling for actions to study and accordingly reform our educational systems, Watson concludes that "school mathematics is *necessarily* not a subset of the discipline of mathematics, *whatever the nature of mathematics being taught, whatever the way it is taught*" (emphasis added). Research findings unequivocally support Watson's argument that current school curricula *do not* maintain the mathematical integrity of their content. On the other hand, her conclusion that school mathematics *cannot*, in principle, be mathematics is unfounded and surprising. It is surprising because her paper is a guide as to the nature of the reform needed. Specifically, Watson lists four reasons for her conclusion, each of which is a reason for change rather than for acceptance of impossibility of change. The competencies for authentic mathematical practice that Watson lists at the end of her paper as her first reason are required from any professional mathematician. The question for research in mathematics education is how to help students develop these competencies not through *authoritative need* but through *intellectual need.* The second reason about teachers' knowledge is not a justification for impossibility of change but rather a justification that change is imperative. The third reason – about time constraints, curricular pressures, and assessment regimes constraints – is a call for a fundamental reform of perspective; that, for example, teaching should be about the *quality* of the ideas students *uncover* rather than the *quantity* of the material being *covered* in a lesson. The fourth, and last, reason that "authority in school mathematics lies with teachers, textbooks and assessment regimes, not with mathematical argument," is an acknowledgement that a dash of authority is not completely harmful and perhaps unavoidable, but it is essential that teachers recognize that students must not be helpless without an authority at hand and that students should never regard a mathematical justification of a result as valueless and unnecessary.

Reference

Harel, G. (2008) 'What is mathematics? A pedagogical answer to a philosophical question', in Gold, R. B. and Simons, R. (eds.), *Current issues in the philosophy of mathematics from the perspective of mathematicians,* Washington, DC, Mathematical Association of America, pp. 265-290.

School and academic mathematics

ALF COLES, NICK PEATFIELD

(Alf Coles is Head of Mathematics at Kingsfield School, UK, where Nick Peatfield spent four months on teaching practice.)

Alf: You have been a professional mathematician and are now near the end of a one-year teacher education course. From these experiences, where would you locate the differences between mathematics as an academic discipline and school mathematics?

Nick: Although I would agree with Watson's conclusions about the differences between academic mathematics and the mathematics of many schools, at Kingsfield, I have seen most of them contradicted. Here the main goal and emphasis, throughout the mathematics curriculum, is on the learners being encouraged to behave as mathematicians. Still, it is the case that at school the students aren't pushing the boundaries of knowledge, and so are correct when they tend to assume that the teacher "knows the answer".

Alf: I'm interested in the issue of the teacher knowing the answer, and the students, in your words, who 'tend to assume' this. One of the things I have become aware of in my own teaching practice is when I ask questions that are basically "guess what's in my head"; questions; and I have been trying to avoid themse as much as possible. I like to place my attention in the learning and sense making of the students. When I am most "in flow" as a teacher it really is the case that "the answer" is not in my head – not because I don't know the answer, but because I am not engaging that aspect of myself. I am genuinely interested in what students say or do in mathematical contexts, and enjoy being exposed to perspectives I had never considered. There is an inherent ambiguity, for me, in mathematics – answers are always provisional on assumptions, and part of my work with our first year classes (aged 11) is getting them to become aware of their assumptions, or the conditions under which their generalizations do and do not apply. A counter-example does not so much prove a conjecture wrong as lead to a modification (to borrow John Mason's language).

Nick: Your description of ambiguity and becoming aware of assumptions chimes with what I would call, in a university context, a good mathematical conversation. One mathematician (often more of an expert in the particular topic under discussion) will have a conjecture, or method of attacking a problem, which both then try to test, and then amend. 'Being exposed to perspectives [they] had never considered,' which the other mathematician can provide, will often result in a breakthrough of some kind. 'Answers always being provisional on assumptions' is important here, as it can be hidden assumptions in the "expert" view whose removal can be key in a solution. The idea of counter-examples being openings rather than endings is also important – they often lead to more interesting questions and almost certainly to a deeper understanding.

What I have seen at Kingsfield makes me think there is no necessity for school mathematics to be anything distinct, for example if students are actively pushing the boundaries of their own knowledge. It feels like you have set up structures in the department with the aim of making this as close an approximation to "the real thing" as is possible; any differences lie in the short-comings of the approximation (and apply just as much, if not more, to most undergraduate courses). Of course, as teachers you frame the context for any enquiry, but that is not so different from university. The context there is set by the lecturer, or even sometimes at PhD level by the supervisor. Arguably, in almost any mathematical question the context has been set by previous mathematicians (the one who first asked the question, and others who have thought about it). It is often breaking out of this context and re-framing the problem that might lead to its solution.

Alf: In talking about Kingsfield, I am wary of Watson's comment that 'it is always possible to … argue to exclude your particular patch of the world'. I read her, in essence, as saying that 'whatever the way it is taught' the phase of learning mathematics is different from the phase of mastery. I am reminded of Varela (1999), in the context of ethics, discussesing a mode of reflection on life that 'takes the middle way between spontaneity and rational calculation' (p. 31), whereby 'after acting spontaneously' we 'reconstruct the intelligent awareness that justifies the action … as a stepping-stone for continued learning' (p. 32). Varela states 'even the beginner can use this sort of deliberate analysis to acquire sufficient intelligent awareness to bypass deliberateness altogether and become an expert' (p. 32). I have always takening this quotation to imply that learning and mastery do not have to be distinct phases, and that I can use myself in such a way to learn through mastery from the beginning. I try to work with my students so that they use such 'deliberate analysis' in learning mathematics.

This point takes me to the issue of the role of practice in learning mathematics. In Watson's image of the cafeteria at the mathematics faculty in Cambridge, '[N]o one is doing exercises or practiscing techniques'. She also states 'Mathematics as a discipline … does not include … practising core techniques'. Would you say that in your post-doc research, when you were presumably pushing boundaries of knowledge presumably, that practice and drill played any part in what you did? (If so, would you consciously choose to do this? I am reminded of my brother who is a professional musician and yet continues to practice his scales.)

Nick: I definitely did "drill" myself in various different techniques, even as a post-doc, and I believe that other mathematicians also do this, especially when embarking on a new problem, in an area slightly unfamiliar to them. And it certainly was a conscious choice. When exploring a new structure in mathematics, one has to play with it. You pose a lot of questions, and then try to solve them. Asking and working on many similar problems can lead to conjectures and more questions. When you think you have a method that will solve the problem you are really interested in, it is important to practise it a lot so that you are familiar with how it works and can be applied in different contexts. I think that this is probably very similar to what mathematics teachers are trying to get their students to do in many schools.

Reference

Varela, F. (1999) *Ethical know-how: action, wisdom and cognition,* Stanford, CA, Stanford University Press.

There can be, should be, and sometimes is a connection

VICKI ZACK

Watson opens by asserting "that school mathematics is not, and perhaps never can be, a subset of the recognised discipline of mathematics", and re-emphasizes the point in her concluding section by omitting the 'perhaps' when stating that school mathematics "necessarily is not a subset of the discipline of mathematics". I differ, and insist that, when conditions are propitious, school mathematics can be – and in some cases *is* – a foundation for a mathematician's work. There must be a connection.

I do agree with much of what Watson says in regard to the differences between what happens in schools and what is entailed in the work of mathematicians. However, the foundation mathematicians' work is often laid in school, if the settings are such as the school Watson mentions at the outset (*i.e.,* the one using the marvelous reference to Burton's work). Was Watson herself a teacher at that school, so "well-known to her"? I think also of the progressive school (Phoenix Park) in which Boaler (1997) did her study and the rich questions those students tackled.

I was inspired by the Phoenix Park students' work. I was also distressed by Boaler's postscript that the school board decided to shift back to a traditional curriculum for Phoenix Park. The authentic mathematical activity that Watson describes and the project-based curriculum that Boaler described represent exactly what we should strive for. Hence my disappointment when Watson seems to be saying that the worlds of schools and of mathematicians are separate. How can that be? It is for the most part in challenging and creative classroom settings that students can apprentice.

Watson distinguishes between situations in which children work by rote and ones in which the focus is making meaning. Her life's work has been around the latter. However, I do not, even when it is done for the sake of making a point, wish to hear of the two lumped together ("whatever the nature of the mathematics, whatever the way it is taught"). My own school experience leads me to decry situations in which children learn by rote. I was one of a multitude who was "cognitively bullied". As I came to know about the beauty of mathematics as a mature adult, my anger grew at being cheated. No one offered me a glimpse of the creativity of mathematics when I was young – in elementary school or in high school or out of school. I spent half a lifetime feeling inferior about my mathematics ability. We have to look critically at what is not working in traditional classrooms.

To that end, Lockhart (2008) looks at the situation in schools, deploring the fact that what could be a "rich … adventure of the imagination has been reduced to a sterile set of "facts" to be memorised and procedures to be followed" (p. 5). He describes mathematics as "the art of explanation":

> If you deny the students the opportunity to engage in this activity – to pose their own problems, make their own conjectures and discoveries, to be wrong, to be creatively frustrated, to have an inspiration, and to cobble together their own explanations and proofs – you deny them mathematics itself. (p. 5)

The work in which Lockhart, an elementary and secondary teacher, engages with the students is nothing less than a full-fledged mathematical endeavor.

I returned to the classroom in 1989, at age 43, to teach full time and do research, looking at what 5th-grade children could do with non-routine problems. It was then that I began to perceive the complexity and beauty of mathematics (*cf.* Zack & Reid, 2003, 2004). I have gloried in the children's, and my own, investigations and discoveries. They were able to do work as mathematicians do: discovering patterns, making conjectures, constructing counterexamples, and presenting powerful arguments. The work done by adults in a community of mathematicians is a mature version of these kinds of endeavors.

The joy that I continue to feel in regard to the children's endeavors and accomplishments in mathematics has led me to think about the joy I feel as a learner and as a teacher. I have always looked for ideas, books, art that would touch me, make me laugh, make me cry, put me in awe of their power. In my work with students I have always shared my excitement about diverse topics – but have encouraged them to pursue answers to questions that they found fascinating. That was true for me in regard to literature and social studies prior to 1980 and, although a bit intimidating, was as true about mathematics when I came to be intrigued by it. I know that the children will far outdistance me, both as 10-year-olds, and later on.

It might seem trite to speak about instilling a lifelong love of learning and of seeing the beauty in various disciplines – literature, mathematics, and history – but that has been my goal. I am the lover of literature who can write critiques but will never write a literary work worthy of publication. But I know that some of my students will. Our investigations of questions in mathematics will inspire some of those students to do something of significance in the mathematics. A number of them will go on to do great things. What Watson describes as the discipline of mathematics has its roots in school-based endeavors.

References

Boaler, J. (1997) *Experiencing school mathematics: teaching styles, sex and setting,* Buckingham, UK, Open University Press.

Lockhart, P. (2008). *Lockhart's lament,* available (last accessed, 2008 August 05) at: http://www.maa.org/devlin/devlin_03_08.html

Zack, V. and Reid, D. (2003). Good-enough understanding: theorizing about the learning of complex ideas (pt. 1), *For the Learning of Mathematics* **23**(3), 43-50.

Zack, V. and Reid, D. (2004) Good-enough understanding: theorizing about the learning of complex ideas (pt. 2), *For the Learning of Mathematics* **24**(1), 25-28.

Teacher as model learner

HILARY POVEY

In her paper, Watson reminds us of the distinction between "school maths" and the activities of mathematicians and she writes compellingly and convincingly about the latter. She describes in detail and with care the processes in which mathematicians engage, the processes that define what it is to do mathematics. She has had as an aim for a long time that "school students should be introduced to authentic mathematical activity such as is practised by professional mathematicians"; but, she fears, "school mathematics is *necessarily* not a subset of the discipline of mathematics" (my emphasis). I share Watson's concern that the difficult shifts of reasoning demanded by mathematics are being "edited out" of the mathematics experienced in school and appreciate that the alternative is rarely seen. However, despite the fact that she argues forcefully and accurately that deep institutional structures militate against (unfortunately, very effectively) students "acting like mathematicians" in schools, I remain, finally, unconvinced of this necessity.

She has called up a vivid image of undergraduate and graduate students of mathematics working mathematically together in their Cambridge coffee room. Watson is right that, for these highly privileged and highly successful students of mathematics, there is a pretty good chance that no-one explicitly told them to behave like this; but this does not seem to me to cause a fundamental breach between *their* learning and doing mathematics and the learning and doing mathematics that might happen in school classrooms if we had the will. Watson does not, but I do, see them practising techniques - for a well defined and self-chosen purpose, of course; I see them even, in the same spirit, doing exercises or practise problems; and certainly, I see them wandering over from time to time, and very pleased when the opportunity presents itself, to check out results, ideas, muddled thinking and so on with their teachers who have also popped in for a coffee and a bun.

I agree with Watson that, even in the mathematics classrooms in school that take most seriously the injunction "Together we can learn to be mathematicians", other things will happen than doing mathematics. But so will they in the Cambridge coffee room. People will gossip, will drink coffee, will have to leave to take their laundry out of the washing machine and so on. Also, more importantly, they will also struggle to understand the already existing mathematics of others; will practise routines, perhaps to gain deeper algorithmic understanding, perhaps to become more proficient and fluent; will welcome instruction; will sometimes just accept closure on uncompleted problems; will accept, at times, that a model "works" without fully understanding why and will not pursue the matter further; will use inductive reasoning, pattern spotting, experimental results, sometimes even as worthwhile ends in themselves. And, whilst Watson is right to draw attention to the fact that a vital and significant underlying purpose of their activity is to sustain and develop mathematical habits of mind, those undergraduate and graduate students in the coffee room will sometimes experience their studies as 'a "performance" route' (Mann, 2003, p. 20) to success and will, from time to time, have their mathematising undermined by high stakes external assessment.

There is one aspect of the injunction to which Watson refers at the beginning of her paper that I should like to explore a little further: the suggestion is that *together* we should learn to be mathematicians. In a personal communication, one teacher (Corinne Angier) who attempts to model her practice on this wrote, "I was thinking about the students as being apprenticed as learners of maths and of me being a model learner not a teacher". Watson draws attention to the nature of authority that is appealed to in school classrooms; as Alrø and Skovsmose have it (1996, p. 4), the teacher, the textbook and the answer book make up a united authority there which needs no specification or justification. But it is possible for classrooms, even school classrooms, to be different from this - to have school students, occasionally, successfully challenging what a teacher, a textbook, an answer book says. This is going to happen in a classroom where there is a strong sense of the learner and the teacher, from time to time, being engaged in the *same,* not a different, activity. In the teaching of art, the model of apprenticeship resonates much more closely than is usual in mathematics classrooms. Why might this be so? I see two reasons that are relevant here. First, whilst the students are, of course, practising techniques, doing exercises, acquiring knowledge of the art already completed by others, there is also an expectation that they are throughout, from time to time, themselves working as artists. Second, almost uniquely among subject teachers (in England, at least), art teachers will themselves be practitioners of the subject they are teaching. They will be artists, and will describe themselves as such, as well as being teachers of art: indeed, they will often practise their craft side by side with their students in the same classroom. Some mathematics teachers, albeit a few, also behave like this; that suggests to me that the fact that the mathematics of school classrooms is "coordinated and aligned" differently from the discipline of mathematics is a contingent rather than a necessary phenomenon.

One cannot think of what happens in mathematics classrooms in school as being *only* modelled on what mathematicians do. Watson is right to say that there are alternative (and some probably necessary and even desirable) purposes to which attention is paid. But, and I am aware of the risk that I am being utopian, she has not successfully shown me that "learning to be a mathematician" *cannot* be the organising principle for teaching mathematics in school.

References

Alrø, H. and Skovsmose, O. (1996) 'On the right track', *For the Learning of Mathematics* **16**(1), 2-40.

Mann, C. (2003) *Indicators of academic performance,* available (last accessed 2003 December 17) at: htp://www.admin.cam.ac.uk/reporter/2003-03/weekly/5913/6.html

Mathématiques scolaires et Mathématiques

VIVIANE DURAND-GUERRIER

Dans son article, Anne Watson écrit: « Authority in school mathematics lies with teachers, textbooks and assessment regimes, not with mathematical argument ». Ceci conduit à se poser la question de ce qu'est un argument mathématique.

En accord avec Barallobres (2007), je soutiens que la validation intellectuelle (mathématique) ne se limite pas à la démonstration, mais engage « ce processus complexe d'allers-retours, d'intuitions, de doutes et de méfiances qui caractérisent le développement de la pensée mathématique » (op.cité, p. 41). Dans la perspective du modèle théorique initié par Tarski (1969), ceci relève de l'articulation entre le « domaine de réalité » (lieu de l'action des sujets avec les objets) et la théorie mathématique émergente. Ceci renvoie également à la dialectique action/formulation/validation qui est au cœur de la Théorie des Situations Didactiques (Brousseau, 1997, 1998).

Prenons l'exemple classique de la situation d'agrandissement du Puzzle dans l'ouvrage cité en référence. Les élèves de la fin de l'école primaire (10–11 ans) sont invités à agrandir le puzzle de sorte qu'une pièce qui mesure 4 cm sur le modèle devra mesurer 7 cm sur le Puzzle agrandi. Chaque élève d'une équipe de 4 ou 5 agrandit sa pièce, avant de reconstituer le Puzzle. La mise en défaut de la procédure additive se fait par l'impossibilité de reconstituer le puzzle avec les pièces obtenues. Ce sont ici les rétroactions du milieu qui vont permettre aux élèves de remettre en cause la procédure utilisée, ce qui permettra au professeur de relancer la recherche, et de faire, peut-être, émerger une procédure de type multiplicatif.

Dans le modèle proposé par Brousseau, le travail des élèves se fait dans un va-et-vient entre les actions sur les objets et la formulation de conjectures. À ce stade du curriculum, le théorème de Thalès ou l'homothétie ne sont pas encore disponibles. Les élèves peuvent alors chercher différentes manières de « passer de 4 à 7 » et les tester en réalisant un nouvel agrandissement. Si le résultat obtenu est probant, on peut voir apparaître une sorte d' « axiomatisation locale » qui consiste à poser comme « axiome » un énoncé qui, réinterprété dans la situation, donne le résultat attendu. [1] Ce que le professeur ou les manuels doivent alors garantir, c'est que cette « axiomatique locale » peut s'insérer dans les théories mathématiques de référence dans lesquelles la situation s'inscrit. Nous avons également rencontré ce type d'élaboration « d'axiomatique locale » chez des enseignants non experts en mathématiques à qui nous avions proposé de déterminer tous les polyèdres réguliers convexes, en mettant à disposition, outre le matériel classique de géométrie, des pièces en plastiques permettant de faire et défaire facilement des assemblages (Dias & Durand-Guerrier, 2005). Un « axiome local » est apparu très rapidement : « *on peut construire un polyèdre régulier pour chaque polygone régulier* ». À l'issue d'un important travail sur les objets et après de nombreux échanges contradictoires, les membres du groupe sont finalement tombés d'accord pour poser que « *si la somme des angles au sommet est égale à 360°, alors on ne peut pas faire un angle (sous-entendu, un angle trièdre)* », ce qui détruit l' « axiome local » précédent. Cette même situation proposée à des étudiants mathématiciens de Master première année (4e année d'université) donne des résultats similaires, illustrant le fait que les cours classiques de géométrie ne permettent pas d'expliciter ce type de résultat.

Dans ces deux exemples, le travail proposé aux élèves ou aux professeurs en formation nous ramène à la question des arguments mathématiques. De fait, une part importante de l'activité du mathématicien est d'élaborer des nouveaux objets et des axiomatiques permettant de rendre compte de phénomènes observés, que ce soit à l'intérieur des mathématiques elles-mêmes, ou dans d'autres domaines de l'activité humaine. Or, selon Chevallard, « dans une mathématisation complète, on s'assure qu'un fait est vrai sur une base expérimentale, puis on s'assure que dans la théorie qu'on a bâtie par ailleurs, on peut déduire le fait en question. » (Chevallard, 2004, p.35), ce qui correspond à une dialectique de la vérité (dans les domaines d'interprétation) et de la validité (dans les théories) (Durand-Guerrier, 2008). Les deux exemples précédents et ce qui est développé dans Barallobres (*op. cité*) montrent selon moi que l'on pourrait considérer qu'un processus analogue de mathématisation est à l'œuvre dans l'apprentissage des mathématiques. J'ajouterais en outre que ceci vaut dès le plus jeune âge : il suffit en effet de penser à l'apprentissage du nombre comme *mesure de la taille des collections discrètes* chez les jeunes enfants, notion abstraite s'il en est qui, quelques années plus tard, sera devenu suffisamment familière pour les élèves pour pouvoir soutenir la construction de l'algèbre élémentaire.

Note

[1] Ici : « Pour agrandir, il faut multiplier toutes les dimensions (longueurs) par un même nombre ».

Références

Barallobres, G. (2007) 'Introduction à l'algèbre par la généralisation : problèmes didactiques soulevés', *For the Learning of Mathematics* **27**(1), 39–44.

Brousseau, G. (1997) *Theory of didactical situations in mathematics,* Dordrecht, NL, Kluwer.

Brousseau, G. (1998) *Théorie des situations didactiques,* Grenoble, FR, La Pensée Sauvage.

Chevallard Y. (2004) 'Pour une nouvelle épistémologie scolaire', *Cahiers Pédagogiques* **427**, 34–36.

Dias T. et Durand-Guerrier V. (2005) 'Expérimenter pour apprendre en mathématiques', *Repères-IREM* **60**, 61–78.

Durand-Guerrier, V. (2008) 'Truth versus validity in proving in mathematics', *Zentralblatt für Didaktik der Mathematik* **40**(3), 373–384.

Tarski, A. (1969) *Introduction à la logique,* Paris-Louvain, FR, Gauthier-Villard.

The purpose of having mathematics in schools tells what school mathematics should be

ROMULO LINS

The core of Watson's argument is, I think, that the purpose of school mathematics is quite different from that of disciplinary mathematics. All the other differences somehow derive from this. For instance, it *could* be the case that the purpose of school mathematics is to teach (or induce) pupils (all) to act as mathematicians; even in this case, there would be a crucial missing component, namely that in disciplinary mathematics one is operating on the boundaries of the known, while in (such) school mathematics pupils would be only *acting as if* they were there. This is the reason why a Lakatosian (fallibilist) 'foundation' for mathematics education does not make sense.

On the other hand, is it a fair (or even interesting) goal for mathematics education in schools that all pupils come to act as mathematicians? Perhaps it would be fair to aim at enabling all pupils to act as mathematicians whenever they wanted to. But that does not seem reasonable, and the reason for this is in the huge difficulty in getting mathematicians not to act as mathematicians when talking about mathematics (for instance, when discussing school mathematics and the learning of mathematics). It seems the drive is too strong for them to keep acting as mathematicians. In other words, maybe as one truly crosses the border and begins acting as a mathematician, it is too late to come back, even if eventually.

Watson correctly states that the "discipline of mathematics has different warrants for truth, different forms of reasoning, different core activities, different purposes" in relation to school mathematics, and it seems wise to extend this observation to everything 'mathematical' normal people also do outside school. And in doing so we are naturally led to the question 'why is mathematics in school?' While Watson spends quite a while on the question "what is 'mathematics' in school?" it is less visible a concern with "why is 'mathematics' in school?" If mathematics as a discipline disappeared altogether at the disciplinary (research) level (production of new mathematics) would school mathematics also disappear? Not necessarily. The other way around? Yes, possibly in a couple of generations.

Watson also says that, in her experience, "those who have the opportunity to do this [act like mathematicians] become better mathematical learners than those drilled in more traditional methods" – but that could well be because the traditional approach is (with respect to any subject) boring to *most* (but not all) people, not because of any peculiarity of mathematics. In school I loved to read and write, but hated formal grammar; I loved to play (real) football, but hated drilling passes or even tactics. Maybe normal people only like to (drill-like) practice the things they enjoy enough as to want to become a pro in them. And that choice sounds perfectly healthy. That points to a fundamental and irreconcilable difference between school and disciplinary mathematics: people who do the latter do it *voluntarily*. And I sense that fits quite well with the fact, mentioned above, that these people also do it *for real*.

Let me push the analogy further. In Brazil, youngsters who go to football clubs to be trained act exactly like the pros: they drill passes, shooting, tactics, carrying the ball and so on. And not only they do not complain of this traditional drilling, they will actually tell their peers who only play for fun that these are not doing it the right way (if they ever bother to comment, much as the mathematician does not usually comment with non-mathematicians on what he does). And, quite naturally in this case, *in the discipline the ways of acting are transparent, but in school mathematics they are a teaching goal,* as Watson and others correctly point out.

It could be that "the institutionalization of mathematical knowledge for novices … is not, and perhaps never can be, a subset of the recognised discipline of mathematics" not just because of different warrants for truth but, ultimately, because we are looking at an institutionalization *for all* and that means that school mathematics should *not* be an incubator of future mathematicians, as much as physical education in school should not be (although it similarly could) be an incubator of Olympic athletes.

Of course, I agree that the public image of mathematics is, in too many countries, quite negative, and there is plenty of room for improvement in that area. But surely that is not going to change simply by making people eat all their maths.

Watson's argument sufficiently supports the point she wants to make. If anything, I would just add that perhaps *as it stands* the effort to understand the relationship between school and disciplinary mathematics is better framed as a struggle between mathematics educators and mathematicians, possibly representing professional interests ideologically dressed as 'truths'.

And maybe, paradoxically, that struggle means that the relationship will not be best understood by remaining only within our specialist fields.

What's so great about doing mathematics like a mathematician?

HEATHER MENDICK

Watson argues that school mathematics should try to produce people who act like mathematicians. *And* that this is not possible because of the different conditions in which the practices of school and disciplinary mathematics are embedded. I agree with the second part but not the first part of her argument. I start with the part where we agree.

Watson mentions several reasons why there will always be a mismatch between school and disciplinary mathematics:

- schoolteachers do not have the experience of doing mathematics over time;

- schools need to prepare students for future study and/or employment;
- and, within schools, authority lies with the textbooks and timetable and curriculum and assessment systems constrain what can be done.

Except for the constraints of assessment and curriculum, these were not the first reasons that came to mind. Instead, I came up with the following list:

- school students do not get paid for doing mathematics;
- they do not apply for opportunities to do it;
- and, even when they have an identity that is invested in being good at it, mathematics never defines them in the way that one's employment does as an adult.

The differences between these two lists are indicative of different emphases and approaches. If I had more space I might have chosen to dissect these differences in detail and to argue that, while socio-cultural approaches may not be able to put "detailed flesh on [their] stories [of identity development] in terms of the development of specific mathematical ideas", sociological approaches can do this. Hopefully some of this will come through as I talk through our main point of disagreement over whether school mathematics should try to produce people who act like mathematicians.

About her image of the Cambridge cafeteria, Watson writes: "One might also say that Cambridge is not typical, or question the dominance of men and ask who is excluded, or simply laugh at how anyone could want to replicate such behaviour". In this way Anne tantalisingly offers, and then forecloses, four interesting points of departure. I will take-up one of them: Why would anyone want to replicate this behaviour? Should disciplinary mathematics be our model for school mathematics? Katrina Miller (personal communication) has suggested that school science should be taught by sociologists so that we no longer attempt to separate science from what it does in the world. Perhaps we should want at least some people to do mathematics like sociologists rather than like mathematicians – or like historians or geographers …. This questioning of mathematicians as our model for school mathematics is coming from a range of directions, including the mathematical literacy curriculum in South Africa [1] and Appelbaum's (2007) suggestion that mathematics teachers be 'professional amateurs' using the popular culture practices of young people and other school subjects as models for their practice.

We could take this argument further and ask why professional mathematicians would want to replicate this behaviour? In fact, many do not. They feel excluded by it (another of Watson's foreclosed points of departure) and find other ways through, as far as the discipline allows, as Henrion's (1997) biographical studies of female mathematicians show. However, Watson seems to give a universal status to the ways of being a mathematician, for example, through discussing shifts that "ha[ve] to be made to become a mathematician", without acknowledging either the cultural and historical specificity of these shifts or the variation in practices among contemporary mathematicians. Taking a sideways step into the popular, this part of her argument reminds me of *The Rules* (Fein & Schneider, 2000) – a self-help book which tells women how to act in ways that get a man to propose. These rules include: never accept a Saturday date after Wednesday, never call a man and never go out without make-up. It appears to be very successful but at the expense of leaving in place the status quo of sexual double standards. We need to teach people both to be mathematicians (the rules of the game) and to teach them to be critical of the practices of mathematicians and of mathematics (to want to change the rules).

Note

[1] Graven, M. and Venkat, H. (2008) *Mathematical literacy: issues for engagement from the South African experience of curriculum implementation,* presented at the International Congress of Mathematics Education, Monterrey, MX, 2008 July 6–13.

References

Appelbaum, P. (2007) 'Afterword: bootleg mathematics', in de Freitas, E. & Nolan, K. (eds.), *Opening the research text: critical insights and in(ter)ventions into mathematics education,* New York, NY, Springer, pp. 221–248.

Fein, E. and Schneider, S. (2000) *The complete book of rules: time-tested secrets for capturing the heart of Mr Right,* London, UK, Thorsons.

Henrion, C. (1997) *Women in mathematics: the addition of difference,* Bloomington, IN, Indiana University Press.

Reflections on mathematics as practiced in schools and in academia

UBIRATAN D'AMBROSIO

In asking whether the bases of School mathematics and Academic mathematics are the same, Watson touches a sort of taboo. Are good mathematics students potentially good mathematicians? The main questions are "Why is mathematics a school subject?" and "What justifies the inclusion of mathematics in the curriculum?"

Watson's reasons for distinguishing between School mathematics and good Academic mathematics are significant and well justified. They also point to some interesting ideas about ways of doing mathematics, the nature of mathematics as a discipline, and of its organization as a field of knowledge.

It is important to understand why societies, all over the world, maintain some form of schooling. I see two main reasons:

- to prepare new generations for citizenship, and
- to enhance creativity.

These reasons are two sides of the same coin. One, the conservative facet, draws on established practices and values; the other calls for innovation, for the new. How does School mathematics fit in these two objectives?

Let me be very clear about my position. In what follows I am considering School mathematics and Academic mathematics as practiced, since late-18th century, by the "included" population. I refer to the world that, since then, has come under western influence (or dominance). My reflections do not refer to other periods, or to "excluded" populations and to other cultures, or to different or "alternative" school systems and academic mathematics (which may be considered within the general framework of Ethnomathematics).

I see School mathematics and Academic mathematics as having different methods and, above all, different objectives. They may have a few very basic elements in common, but not all of these are essential and might be overcome. In Watson's words, "most school students are taught and examined on mathematics of a kind that is done, both in the academy and in other workplaces, by machines."

The usual practice of School mathematics is, essentially, catechetic. More emphasis is given to the conservative or citizenship facet. Students are required to accept and to perform in a valid way. According to Watson "validation and authority are highly structured through textbooks, examinations, and curriculum systems rather than by mathematicians working as a community".

We have to recognize that some Academic mathematics is practiced in very much the same way, with strict limitations. This is what Pierre Samuel (2002) called the style "âne qui trotte", and brings to my memory what J.D. Salinger (1961) says in *Franny and Zooey*: "they're not *real* poets. They are just people that write poems that get published and anthologized all over the place, but they're not *poets*". Indeed, we have a very large academic population recognized as mathematicians, whose contributions are hundreds of printed pages (discussing details of sameness) – read by practically no one and soon forgotten. These writings serve the purposes of earning degrees and securing promotions. Such activity is accepted by the academy as a whole, with the argument – which I believe valid – of the necessity of a critical mass of workers. It is, metaphorically, the concept of citizenship in the academic world.

Something similar occurs in School mathematics. Low achievers are part of the system, and considerable resources are aimed at forcing them to achieve a mediocre passing status. Some are low achievers in mathematics, but bright in other subjects. Regrettably, mathematics educators have given practically no attention to low achievers, which can seriously damage their creative capacities. I will not pursue this argument in this short commentary.

I feel it important to address creative mathematics – which is, essentially, open-ended. The paragraph in which Watson describes the cafeteria of the Cambridge mathematics faculty is a very good one. Such a scene is typical where creative mathematics is going on. Sometimes a seminar initiates with a challenging question. After three or four hours of discussion, the question remains unanswered. These are very creative hours! We all know that after working for weeks on a mathematical question, often the only results are scribbled and discarded pages and a still-unanswered question. Normally, creative mathematicians "adjust" the questions, modify the hypotheses, create new situations, and reach new results. This is a major characteristic of the humbleness that is intrinsic to creative mathematics. It has much to do with the ethical facet of mathematics and with the intrinsic character of mathematics as the essential language of Science, which led Wigner (1960) to refer to the "unreasonable effectiveness of mathematics". He describes mathematical creation in a most interesting way:

> The great mathematician fully, almost ruthlessly, exploits the domain of permissible reasoning and skirts the impermissible. That his recklessness does not lead him into a morass of contradictions is a miracle in itself: certainly it is hard to believe that our reasoning power was brought, by Darwin's process of natural selection, to the perfection which it seems to possess. (p. 2)

As Watson discusses and exemplifies, neither this open-ended style nor the essentiality of mathematics to Science and the ethics associated with it belong to current School mathematics. Although School mathematics does not aim to prepare future mathematicians, an appreciation and the possibility of acquiring the essence of mathematics reasoning should be achieved – for example, through exposure to abstract thinking and symbolic reasoning, or through practices leading to inquiries about major issues. These higher capacities of the human mind are, in a sense, discouraged in School mathematics, which emphasizes an answer or result to a question. Elsewhere I have discussed a proposal of curriculum aiming at the development of these higher capacities of the human mind in our civilization dominated by technology (D'Ambrosio, 1999).

Watson provocatively, although briefly, raises the issue of "authority". This point may lead to a very interesting discussion, absolutely pertinent to this "era of standardized testing". This opportune article gives also some hints on which contents might be brought into School mathematics.

References

D'Ambrosio, U. (1999) 'Literacy, matheracy and technoracy: a *trivium* for today', *Mathematical Thinking and Learning* **1**(2), 131-153.

Salinger, J. D. (1961) *Franny and Zooey*. Boston, MA, Little, Brown and Company.

Samuel, P. (2002) 'Résultats élémentaires sur certaines équations diophantiennes', *Journal de théorie des nombres de Bordeaux* **14**(2), 629-646.

Wigner, E. (1960) 'The unreasonable effectiveness of mathematics in the natural sciences', *Communications in Pure and Applied Mathematics* **13**(1), 1-14.

Just Yoking? on the Conjunction of 'School' and 'Mathematics'

WILLIAM HIGGINSON

Almost exactly 40 years ago the distinguished literary critic Northrop Frye (1968) addressed a national meeting of Canadian educators on the theme of "The Social Importance of Literature". Frye concluded his address with the following observation:

> The study of literature, as I define it, is not a panacea; it is not a cure; it does not solve social problems. What it does is to base education on the sense of a participating community which is constantly in process and constantly engaged in criticizing its own assumptions and clarifying the vision of what it might and could be. The teaching of literature in that sense, and in that context, seems to me to be one of the central activities of all teachers and educators in their continuous fight for the sanity of mankind. (p. 23)

This view of Frye's with its early emphasis on the educational importance of participating communities criticizing and clarifying their assumptions and visions has always seemed to me to resonate strongly with the FLM 'statement of purpose/epigraph' found on the inside front cover of every issue of this journal since it was first drafted by the founding editor more than a quarter of a century ago. One reason for raising the 'journal aims' at this point is because I think (high praise indeed) that Watson's contribution is very much the sort that David Wheeler would have been pleased to include in this journal when he was editor. Much more than most mathematics education papers, even in this location, it "stimulates reflection", "promotes study of practices and theories", and reveals that "the learning and teaching of mathematics are complex enterprises about which much remains to be revealed and understood". Before turning to some specific points I would like to make an observation about the 'solicited comment' approach. In my view the nature of the FLM community makes this style of dialogue a desirable and worthwhile one. Whether or not it is effective for individual readers may vary, but it is worth noting that there are journals in some academic areas that make this a major feature of their standard publishing model. One of the most prominent of these 'Open Peer Commentary' journals, *Behavioral and Brain Sciences,* on occasion co-publishes as many as twenty-five reactions to a given lead article. It is perhaps a model that the editors and advisory board might consider extending to some degree in future issues.

From the comments made above it will be clear that I find much of merit in this paper. Watson's 'feel' for school mathematics curricula, classrooms, children and teachers is nuanced and deep. Her ideas on necessary 'shifts of understanding' are provocative and interesting. Is it really the case that "most school students are taught and examined on mathematics of a kind that is done … by machines"? Does 'bullying' parse so neatly that we can really have something called "cognitive bullying"? All good food for thought.

Her conclusion that school mathematics is necessarily not 'junior' mathematics, namely a subset of the discipline seems uncontroversial to me. The conjoining of concepts (think a * b, for some entities a and b, and operator * : consider 'turtle' and 'geometry'; 'water' and 'board', 'Holmes' and 'Fortnum') is far from a straightforward process, as 20th-century scholars in computer science, botany, zoology, linguistics, philosophy, and mathematics have found. Bring together (yoke, merge, unite, concatenate, …) an institution (technology?) and a body of knowledge (discipline) and the result is influenced by both but isomorphic to neither. Much of the paper documents the very strong influence of the school side of the family tree of school mathematics in the form of resulting homogenization and routinization.

If I had to quibble (it comes with the commentator territory) it would be mainly with the paper's portrayal of the mathematical side of the dyad. In particular, the image of the Cambridge cafeteria/conjecture-palace doesn't work for me as an ideal for authentic mathematical activity. Yes, the conversations are animated, the cerebral leaps impressive, but this is still (unless it has changed from my last visit) largely a group of students mostly working on assigned questions with most of the time and configuration constraints that characterize school mathematics. In general, I see the discipline as being much more fragmented (fractalized?) than is suggested here. While there are family similarities (à la Wittgenstein) between and among, for example, 'experimental', 'applied' and 'recreational' mathematics, their differences are significant. While the author never says so explicitly, my sense is that when she says mathematics, she means research in pure mathematics.

While one might feel that a case could be made for this area of endeavour as a pearl of human achievement, it is less clear that its methods and procedures should form a central part of the 'common curriculum' for all learners. In concluding, let me go back to Frye's view, noted above, and suggest two things. First, that if one were to replace 'literature' by 'mathematics' in the statement, its validity would be weakened very little. Second, that in today's world it is not just mankind's "sanity" that is in doubt. Mathematics has clearly contributed much more than literature to the creation of today's global problématique. Does this not imply the need for a common mathematical curriculum more connected to issues of human survival?

Reference

Frye, N. (1968) 'The social importance of literature', *Educational Courier* **39** (10/11), 19-23.

Group Photos

Ten people in a team line up in two ranks for a group photograph. They are all of different heights, and each person in the back row stands directly behind a person in the front row. In how many ways can they be arranged so that every person in the back row can look over the corresponding person in the front row? (unknown origin; selected by Leo Rogers)