MAKING CONNECTIONS: MATHEMATICAL UNDERSTANDING AND NETWORK THEORY

ELIZABETH MOWAT

In the late-1990s, researchers began to explore the underlying structure of complex systems by viewing elements of a system as *nodes* in a network and their interactions as *links* among nodes (Barabási, 2003). Using this technique, Watts and Strogatz (1998) and Barabási and Albert (1999) identified patterns not previously seen in complex phenomena and formulated simple, yet comprehensive, laws that describe network structure and evolution. Scientists in many disciplines, ranging from physics to sociology, have found these principles invaluable in explaining the behavior of complex systems.

Although the use of network theory is rapidly expanding, it has received relatively little attention in the field of mathematics education. Some researchers have looked at different types of structures in mathematical understanding. For example, Vergnaud (1988) investigates the multiplicative 'conceptual field', which comprises all situations related to proportion, multiplication and division. He notes that "[s]everal kinds of mathematical concepts are tied to those situations and the thinking needed to master them" (p. 141). Skemp (1989) describes the need to analyze a mathematical idea before introducing it to students – to "'take it to pieces' … to see what are the contributory concepts" (p. 67). He represents the results of such analysis in a 'conceptual map'. These works emphasize that concepts do not develop in isolation, but through relationships to other ideas; complicated edifices of connections are proposed and explored.

Yet, mathematics is more than merely a structure of concepts with links to related situations and topics. It is a dynamic, evolving form with emergent properties – a complex system "constituted by a large number of … units forming nodes in a network with a high degree of non-linear connection" (Cilliers, 1998, p. 91). Thus, the techniques of network theory offer appropriate and fruitful ways to analyze mathematics and mathematical learning.

In this article, I explore the suggestion that network theory may present a novel and productive way to interpret mathematical understanding. More specifically, I suggest that a possible network structure for mathematical understanding may be found in the *conceptual domains* and *conceptual metaphors* presented in the theory of embodied mathematics, as put forth by Lakoff and Núñez (2000). After exploring certain important aspects of these cognitive mechanisms, I consider how they constitute a network, which I refer to as the *metaphoric network of mathematics.* The topology and dynamic behavior of this structure are then examined. Finally, implications for students' learning and understanding of mathematics are discussed.

In developing this argument, I acknowledge that network theory and the theory of embodied mathematics offer only one possible construal of what mathematics is or how mathematics is structured. They offer rich sources of images and insights, with which I attempt the beginnings of a scientific explanation for mathematical understanding and learning. In other words, I work from the assumption that the discussion should be treated in terms of a useful – not an objectively real – interpretation of the structure of mathematics.

The theory of embodied mathematics

Lakoff and Núñez (2000) conceive of mathematics as being extended from a rather limited set of inborn skills through bodily experiences to an ever-growing web of conceptual domains. These are connected by conceptual metaphors – cognitive mechanisms used automatically and unconsciously – that carry inferential structure from one domain to another. *Grounding metaphors* make basic arithmetic possible by forming correlations between innate abilities and physical actions like "forming collections, putting objects together, using measuring sticks, and moving through space" (p. 102). Other conceptual metaphors link arithmetic to more abstract mathematical concepts, each of these in turn being metaphorically connected to even more sophisticated ideas. [1] New concepts develop as domains fuse and create *conceptual blends;* new metaphors involving these blends are subsequently formed. This "network of ideas" is the basis of mathematical learning and knowledge (p. 376). Although it is not clear that Lakoff and Núñez mean to evoke the mathematical discipline of network theory with this phrase, I contend that closer examination of the nature of the metaphoric network of mathematics formed by these cognitive mechanisms should shed light on subjective understanding of mathematics and on ways in which the learning of mathematics may be supported.

The metaphoric network of mathematics

Nodes

I propose to take, as the nodes of the metaphoric network of mathematics, conceptual domains like the ROTATION IMAGE SCHEMA, [2] ARITHMETIC, or SET. [3] Even the simplest of

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these domains possesses considerable internal structure (Johnson, 1987). Each domain contains interconnected elements forming a complex of sensory experiences, language, and related concepts. Thus, each conceptual domain is a subnetwork of the larger network that forms the cognitive system (Kövacses, 2002; Lamb, 1999). Concepts are not just basic units, but can be understood as networks in themselves.

While most work in this area has been done in other fields, I believe that the principles apply to mathematical domains. For example, considered as a network structure, the conceptual domain of CIRCLE contains many nodes representing a person's knowledge of and experiences with circles (Lamb, 1999). This network is dynamic, changing as an individual learns from new experiences, and differs from person to person. Figure 1 illustrates only part of a potential network for CIRCLE: it is not possible to provide a complete map, for any concept may contain thousands of nodes (Lamb, 1999). When enough nodes in the domain labeled CIRCLE are activated, they operate together in gestalt and form an individual's perception of circles (Lakoff, 1987; Lakoff & Núñez, 2000; Lamb, 1999). A node, like the visual node labeled 'seeing a pie' in figure 1, is itself a subnetwork of the conceptual domain (Lamb, 1999). It includes the many different visual features (color, size, type) that might be involved in a person 'seeing a pie'.

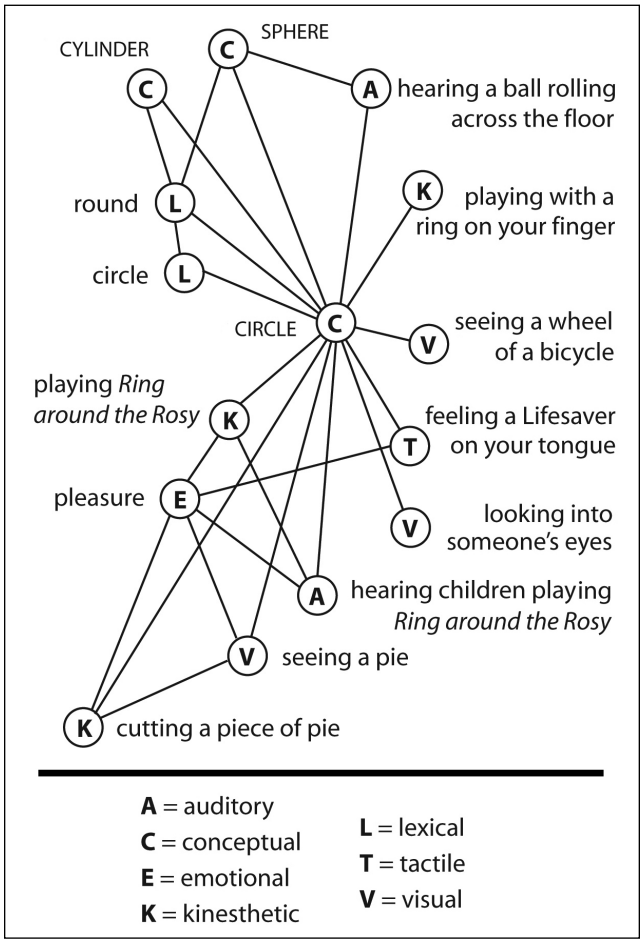

Figure 1: Part of the concept CIRCLE, following Lamb's (1999) example of CAT

Links

Following the work of Lakoff and Núñez, I propose that links in the metaphoric network of mathematics are conceptual metaphors. Particular features from one domain (the *source*) are mapped onto corresponding aspects of another domain (the *target*) by these cognitive mechanisms. It is typical of a metaphor that it has 'unused parts' (Lakoff & Johnson, 1980); if aspects of a source domain do not correspond to some portion of the target domain, they will simply not be mapped on to it (Kövacses, 2002).

Consequently, a single metaphor may not be sufficient to provide a foundation for a complex concept (Lakoff & Johnson, 1999). For such domains, more than one source is necessary to supply inferential structure for different sections of the target Kövacses, 2002). Together, metaphors from these different source domains can construct a coherent understanding of the common target domain (Lakoff & Johnson, 1980), each metaphor enabling an alternative interpretation of the target in a different context. [4] For example, Lakoff and Núñez suggest that four grounding metaphors are needed to fully capture the many characteristics of ADDITION (see fig. 2). [5]

Just as target domains may be metaphorically linked to more than one source, some source domains provide a framework for a variety of targets (Kövacses, 2002). For example, consider the many metaphors (some of which are listed in fig. 3) that can be based on the conceptual domain of SET. These metaphors all project significant characteristics from the source of SET onto various targets, thus developing a common inferential structure for these disparate domains. Thus, the metaphors, AN ORDERED PAIR IS A SET and A NUMBER IS A SET, can provide a foundation for understanding all of mathematics using set theory (Lakoff & Núñez, 2000).

ARITHMETIC IS OBJECT COLLECTION

ARITHMETIC IS OBJECT CONSTRUCTION

ARITHMETIC IS MEASURING WITH PHYSICAL SEGMENTS (THE MEASURING STICK METAPHOR)

ARITHMETIC IS MOTION ALONG A PATH

Figure 2: Grounding metaphors for the target domain ARITHMETIC

AN ORDERED PAIR IS A SET.

A FUNCTION IS A SET.

A LOGICIAL PROPOSITION IS A SET.

A NUMBER IS A SET.

A LINE IS A SET.

A GRAPH IS A SET.

Figure 3: Metaphors with a source domain of SET

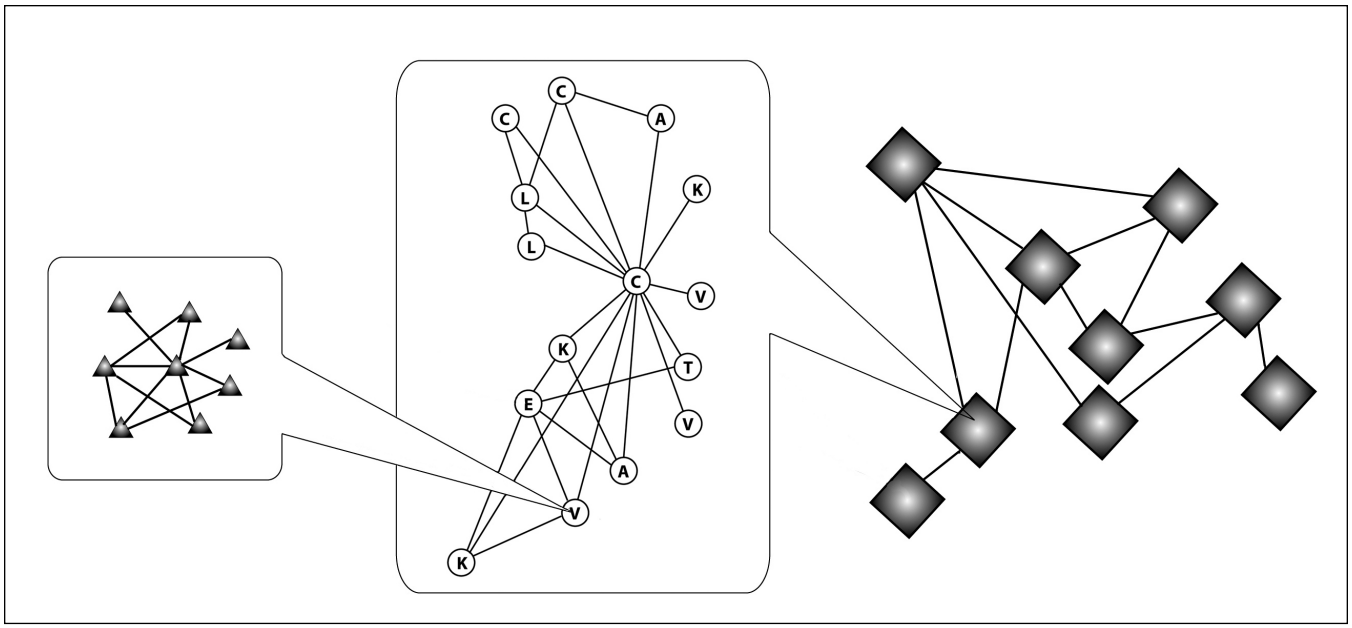

Figure 4: A subnetwork (perhaps of the many visual features involved in 'seeing a pie') nested within the subnetwork of the CIRCLE *domain (see fig. 1), which is itself nested within the metaphoric network of mathematics*

While typically considered to be directional mappings from source domain to target domain, metaphors do not always project inferential structure in just one direction (Danziger, 1990; Meisner, 1992). Over time, conceptual metaphors may become bidirectional. The target domain gradually comes to be viewed as a source, [4] thus changing understanding of the original source domain or domains that shaped it (English, 1997; Sfard, 1997).

This transformation is most likely to occur when the target is structured by more than one source (Meisner, 1992). To illustrate, the metaphor ARITHMETIC IS OBJECT CONSTRUCTION is revealed in expressions like '2 plus 3 *makes* 5' and '6 can be *broken up* into 3 groups of 2'. Experiences constructing objects affect our thinking about arithmetic and numbers. In turn, these imported modes of reasoning influence perceptions in other source domains for ARITHMETIC, such as MOTION ALONG A PATH; we speak of '*breaking* a journey up into three easy stages'. Something of the structure of numbers that originates in experiences constructing objects – not part of the MOTION ALONG A PATH domain – has been added. A conceptual metaphor is not static, but involves "ceaseless two-way interaction between the old and the new … [in a] process of coemergence" (Sfard, 1997, p. 355).

The topology of the network

Cognitive mechanisms offer a possible network structure for mathematics with conceptual domains (nodes) connected to each other in complex ways by conceptual metaphors (links). Each node is not a 'basic unit', but is a network itself, representing a multitude of sensory perceptions, linguistic forms, and relationships that involve the concept. A closer look at the domain's subnetwork reveals yet more deeply embedded structures. Figure 4 offers an image of this nested organization, which is typical of complex systems (Davis & Simmt, 2006).

Networks representing complex systems tend to display a *scale-free* topology or structure (Barabási, 2003; Watts, 2002). [6] In this pattern, clusters are formed within which every vertex is connected to a key node; these *hubs* are in turn linked to more central nodes, and so on (see fig. 5). Two conditions – *growth* and *preferential attachment* – are necessary and sufficient conditions for a network to be organized in this way. I argue that the metaphoric network of mathematics possesses these attributes and, thus, can be modeled by a scale-free structure.

Growth

The network representing an individual's mathematical understanding is not static, but changes as he or she continues to learn. Metaphors have the power to constitute previously unknown concepts; metaphoric mappings do not merely highlight domains that already exist, but bring new concepts into existence by transplanting inferential structure from one

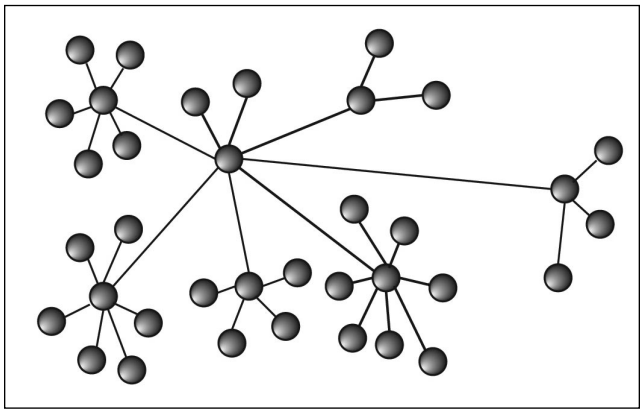

Figure 5: A simple network displaying a scale-free topology

context into another (Sfard, 1997). For example, complex numbers are given geometric meaning through the metaphor linking rotation by 90˚ to *i* (Lakoff & Núñez, 2000). [7]

Entailments of metaphors can lead to new understandings of previously encountered conceptual domains and to the development of new concepts. For example, the MEASURING STICK METAPHOR portrays numbers as physical segments. Using this metaphor, anything that can be measured linearly – not just using a ruler or other rigid item, but with any device, perhaps a measuring tape – can be considered a number; this provides one metaphoric foundation for a previously unknown domain, the irrational numbers (see fig. 6).

Nodes are also added to the metaphoric network of mathematics through conceptual blends. These cognitive mechanisms construct a partial correspondence between two unrelated sources and project this onto a novel blended target; the new concept possesses these alignments, but is augmented (Fauconnier & Turner, 1998; Lakoff & Núñez, 2000). To illustrate, the UNIT CIRCLE is the conceptual blend of a circle in the Euclidean plane and the Cartesian plane with coordinate axes (see fig. 7). [8] It possesses characteristics of both of these domains, but also has emergent properties related to trigonometry that are not found in either of the original concepts.

Preferential attachment

The metaphoric network of mathematics does not grow in an evenly distributed manner; "if a source domain is used to shed light on one or more salient target domains … this increases its likelihood to be chosen as a source domain in the future". [9] Thus, through recurrent use, a source domain becomes an attractor of meaning – a hub in the metaphoric network of mathematics.

Several factors may determine whether a source becomes an attractor of meaning. Domains added to the network early in its development have more time to acquire links (Barabási, 2003; Barabási, Albert, Jeong & Bianconi, 2000). Thus, sensori-motor image schemas learned in early childhood may become hubs in the metaphoric network of mathematics. For example, ROTATION serves as the source domain for many metaphors (Lakoff & Núñez, 2000). Concepts that have a greater degree of 'fitness' also tend to have more connections than other nodes (Bianconi & Barabási, 2001). Certain domains, like SET, are repeatedly employed

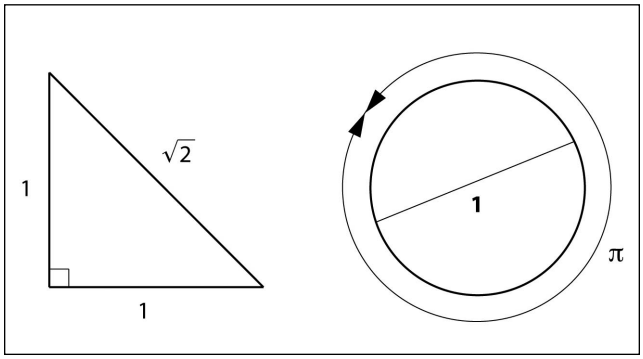

Figure 6: Grounding √2 and p using the MEASURING STICK METAPHOR

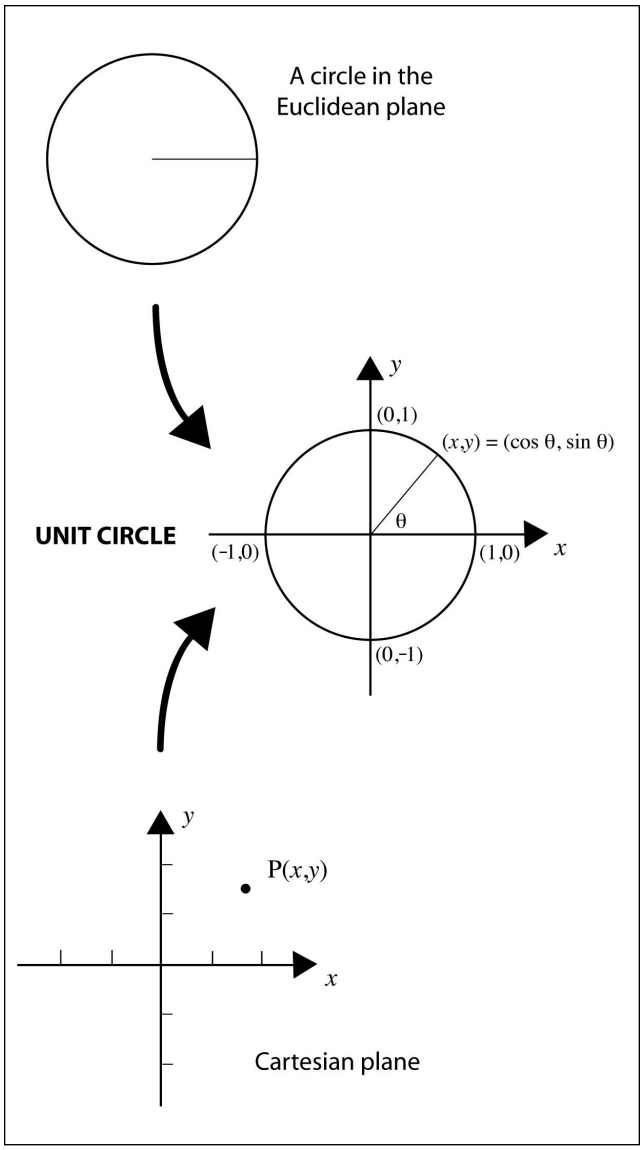

Figure 7: Features of Euclidean and Cartesian geometry combined into the UNIT CIRCLE

as sources because of the power of their particular inferential structures. As well, people tend to use sources that they are already familiar with; "in the end, we all follow an unconscious bias, linking with higher probability to the nodes we know which are inevitably the more connected nodes of the [network]" (Barabási, 2003, p. 85). Whatever the reason, source domains with numerous links are likely to become even more well-connected, causing the evolving network of metaphors in mathematics to exhibit the property of preferential attachment.

These two features of growth and preferential attachment are the defining characteristics of a scale-free network topology (Barabási & Albert, 1999). As the network of conceptual metaphors in mathematics exhibits these characteristics, it shares the "common blueprint … [that governs] the structure and evolution of all the complex networks that surround us" (Barabási, 2003, p. 6).

The dynamics of the network of metaphors

To understand a complex system, it is necessary to consider more than network topology; one must focus on the dynamics of the structure (Barabási, 2003). The most significant feature of a scale-free network is the distribution of connectivity; most nodes have relatively few links to the rest of the network, but a few nodes have a great many connections (Barabási, 2003; Watts, 2002; see fig. 5). In the metaphoric network of mathematics, these hubs are the concepts to which other domains tend to be connected – that is, sources that are attractors of meaning and linked to many targets.

These extremely well connected nodes have a critical effect on the overall behavior of the network. Since not all metaphors between nodes in this system are directional, the likelihood that the network is a single, unified structure rather than a collection of fragments is increased (Barabási, 2003). One might thus expect that any conceptual domain in mathematics can be linked to any other by a sequence of relatively few conceptual metaphors (Barabási, 2003), a point that might be supported through reference to recent examinations of the figurative underpinnings of some mathematical concepts (*cf.,* Lakoff & Núñez, 2000; Mazur, 2003).

These scale-free networks are generally very robust. Since the majority of nodes have only a few links, a significant number can be removed from the system with little or no effect. However, the weakening of a key concept or hub may reverberate throughout the network; nodes directly connected to the hub fail first, nodes linked to these fall next, and so on. While such a *cascading failure* can go unnoticed for a long time, the collapse of one highly connected hub may eventually cause a large part of the network to disintegrate (Albert, Jeong & Barabási, 2000; Barabási, 2003; Watts, 2002).

Thus, the scale-free pattern of metaphoric links joining conceptual domains into a network determines dynamic behavior within its structure. Lines of communication go through powerful, commonly used sources. These domains serve to connect clusters of related concepts and, ultimately, all nodes in the network. At the same time, dependence on these key ideas puts the network at risk. For if a hub fails, large portions of the network may break down. Those parts of the network that remain are isolated, fragmented, and consequently more vulnerable. The scale-free topology of the proposed metaphoric network of mathematics proves to be both the strength and the weakness of its structure.

Implications for the learning of mathematics

I argue that awareness of the scale-free topology of the network formed by conceptual domains and conceptual metaphors offers a way for educators to gain some understanding of mathematical learning. In particular, recognizing events and changes that are characteristic of a scale-free structure can shed light on students' difficulties in learning mathematics and, thus, can suggest useful ways for assisting students to avoid and overcome these problems.

A scale-free structure is inherently vulnerable because of the crucial role that certain conceptual domains play in ensuring the connectivity of the system. In the metaphoric network of mathematics, some nodes are simply more important than others. An inadequate understanding of one of these hubs has the potential to handicap severely a student's comprehension of ideas that are connected to it.

To illustrate, consider the many concepts in mathematics that are based on the image schema of ROTATION. Figure 8 illustrates just some of the conceptual domains where a student's perceptions might be shaken if his or her cognizance of this schema falters.

To increase the robustness of the network of metaphors, attention might be focused on students' learning of major source domains used in mathematical metaphors. There are two difficulties with this approach. First, little research has been done examining which source domains might be hubs; such research is confounded by the fact that hubs are not necessarily the same for everyone. Second, teachers can assist learners to construct more stable conceptions of source domains that are seen as key, but this will not eliminate the vulnerability that is characteristic of a scale-free network. Hubs are still hubs; they retain their central position as cultural attractors of meaning in the network of mathematics.

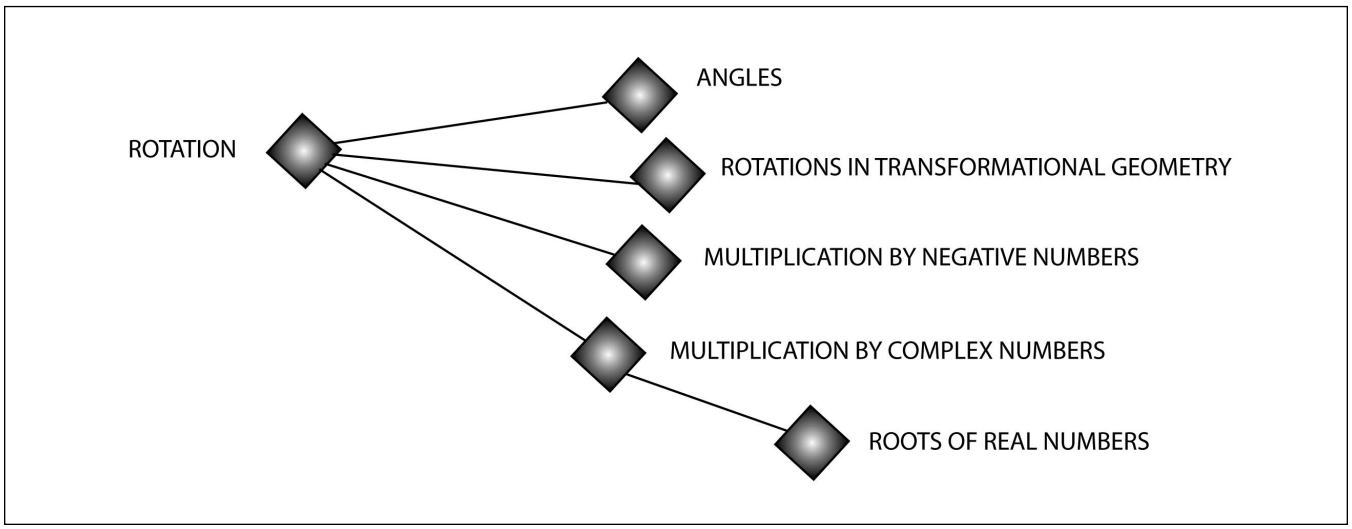

Figure 8: Some concepts metaphorically linked to ROTATION

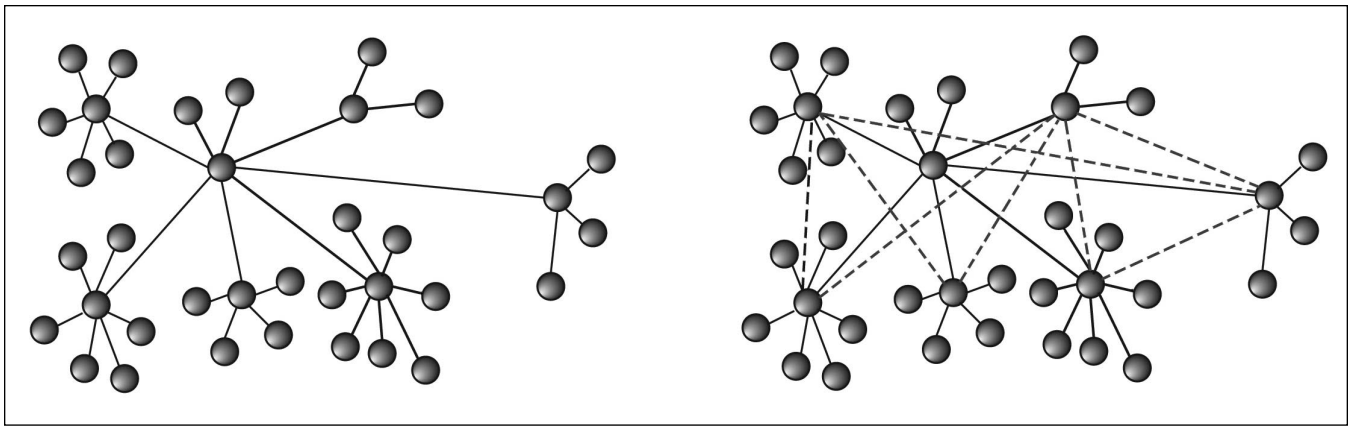

Figure 9: Decreased dependence on a central node as a result of adding a few weak links to a scale-free network

In order to improve the robustness of the network of metaphors, one must change its structure. There are several approaches that might be attempted. Watts (2002) suggests that reducing the number of connections to a hub would lessen the likelihood of network failure. He notes that "[e]ven in the event a hub did fail, fewer [nodes] would be affected, causing the system as a whole to suffer less" (p. 193).

This course is not one that a mathematics teacher can readily choose; particular concepts are repeatedly selected as source domains because of their usefulness and because the mathematics community has traditionally employed them to develop new ideas. It is not likely that a teacher would deliberately refuse to use domains that do provide a coherent structure for learning mathematical concepts, nor would this be responsible.

It would seem that another approach is required. Increasing the number of connections among conceptual domains would have the desired effect of reducing the network's dependence on its hubs. Adding even a few links – called *weak links* – between clusters of nodes decreases the network's vulnerability (Barabási, 2003; Buchanan, 2002; see fig. 9). The more distributed structure that results has sufficient redundancy to ensure that "even if some nodes [go] down, alternative paths [maintain] the connections between the rest of the nodes" (Barabási, 2003, p. 144). Thus, weak links reduce the possibility that a cascading failure will occur and cause the network to become fragmented.

In mathematics education, some metaphors are traditionally employed to make sense of certain concepts (*e.g.,* EQUATIONS ARE BALANCES); such connections are strong because they are widely used and constantly reinforced. However, relying on a single link to provide structure for an idea is dangerous. Failure to comprehend the source it is based on can jeopardize understanding of many connected mathematical domains.

It is desirable, therefore, that students be introduced to a variety of conceptual metaphors when learning about a new idea. Then, if a learner's comprehension of one source breaks down, he or she can rely instead on metaphors projecting inferential structure from other domains. When concepts are not dependent on the strength of a single key source domain, then the network of metaphors that constitutes a student's understanding of mathematics should become more robust and not subject to the cascading failures and fragmentation that are characteristic of scale-free networks. "If our mathematical conceptions are to be sound and stable, they must stand on more than one metaphorical leg" (Sfard, 1997, p. 367).

Support for the view that concepts should be based on more than one source can be found in recent studies in mathematics education (although the works do not refer explicitly to network theory). Tall (2003) stated that the automatic use of previously learned metaphors for arithmetic could cause confusion for students. He described how young children believed that adding two numbers should always yield a larger sum. This perception is justified if one is using the ARITHMETIC IS OBJECT COLLECTION metaphor, but the OBJECT COLLECTION source domain cannot deal with situations where the addition of integers leads to smaller sums ($2 + -7$ makes -5). Moses and Cobb (2001) found the same reliance on the ARITHMETIC IS OBJECT COLLECTION metaphor. They felt that ARITHMETIC IS MOTION ALONG A PATH would be more useful in this context and developed activities using experiences familiar to students, like riding on the subway, to strengthen this metaphor. With such techniques, they were successful in improving children's comprehension of integer arithmetic.

Davis and Simmt (2006) recently completed a study of teachers' understanding of the concept of MULTIPLICATION. While the most common definition for MULTIPLICATION given in primary textbooks is that of repeated addition ($3 \times 4 = 4 + 4 + 4$), participants were able to identify a large cluster of different representations that made sense of the operation. Teachers in the study realized that MULTIPLICATION did not have a single meaning, but many, as revealed by the images, actions, and analogies they listed. Moreover, they concurred that if learners were helped to appreciate these multiple descriptions, they would be able to understand why MULTIPLICATION is used in so many diverse contexts. These varied interpretations link MULTIPLICATION to the image schemas upon which the four grounding metaphors of ARITHMETIC are based (see fig. 10).

Following these examples, it is advantageous for students to utilize more than one metaphor when exploring a new concept or extending a previously encountered one. If a learner's understanding of one source falters, other domains

provide inferential structure to be used in reasoning about the mathematical idea. Thus, weak links are provided, increasing the robustness of the metaphoric network of mathematics. Comprehension of related concepts is not jeopardized and cascading failures are less likely to occur. Using a variety of metaphors to make sense of each concept should be of great benefit to students in their understanding and learning of mathematics.

Further thoughts

If one begins with the assumption that subjective mathematical understanding can be considered a complex form, then it is appropriate to consider this system as a network. Arguably, the theory of embodied mathematics provides a possible structure for a network of mathematical knowledge. Network analysis provides evidence that this interpretation is suitable, as the metaphoric network of mathematics displays the nested forms and scale-free topology that are typical of complex systems.

While the network described in this work is based on conceptual domains and conceptual metaphors, other network structures may exist that prove useful in describing the cognitive dynamics that are an intrinsic part of the learning of mathematics. Any study of a specific complex unity inevitably entails an imposition of some sort of artificial boundaries on the part of the observer. I should also comment that, unlike the work done by complexivists in many fields, this paper does not include the mathematical analysis of a network structure. A rigorous mathematical model for subjective understanding of mathematics has not been developed, nor has empirical data been presented. Instead, I offer and explore a possible representation for the structure of mathematical knowledge.

While the principal focus in this paper is subjective mathematics, the descriptions offered and the implications developed are relevant not just to the learning of individuals, but also to other levels of mathematical understanding, in particular to the cultural knowledge of formal mathematics. Just as the metaphoric network of mathematics contains nested forms within its structure, it is itself embedded in other complex unities, one of which is the discipline of mathematics (Davis & Simmt, 2006). While the evolution of the social construct of mathematics is unarguably different from that of an individual's constructed knowledge, the phenomena are inextricably intertwined; "the same evolutionary dynamics and same complex emergent qualities seem to be at play in the cognitive processes of individual and collective" (Davis & Sumara, 2000, p. 832).

Findings also pertain to other phenomena of concern to the mathematics educator, including pedagogy, professional development, and programs of study. Changes in understanding of students' learning of mathematics should lead to changes in methods of instruction – specifically, introducing a multiplicity of metaphors for each concept. Thus, increased knowledge of the many metaphors that give meaning to particular mathematical concepts is needed. As well, curricula could be designed to encourage the acceptance of such alterations by arranging concepts not in an isolated, linear fashion, but by connecting them in a web of ideas. Many questions regarding the implementation and efficacy of incorporating the inferences of network theory into mathematics education are raised.

Network analysis offers fresh insight into the nature of mathematical learning and understanding. In particular, an individual's knowledge of mathematics consists of a large number of concepts whose meanings are determined by metaphoric interrelationships. These links tend to involve significant source domains, which – while they ensure that every concept in mathematics is ultimately linked to every other conceptual domain – also compromise the robustness of its structure. Making additional connections among concepts reduces dependence on these key sources and thereby promotes students' learning of mathematics and strengthens their mathematical understanding.

Notes

[1] Other types of reasoning, such as metonymy, also allow us to go from concrete to abstract thinking. In this paper, following the example of Lakoff and Núñez, I focus on the role conceptual metaphors play in this transition.
[2] Following Lakoff and Núñez, an image schema is a general pattern developed through bodily experience that represents the spatial logic inherent in many physical situations.
[3] Throughout the paper, I follow the convention of identifying conceptual domains, conceptual metaphors, and other cognitive mechanisms by using small capitals.
[4] Kimmel, M. (2002) *Metaphor, imagery, and culture: spatialized ontologies, mental tools, and multimedia in the making,* Unpublished doctoral dissertation, Vienna, Department of Philosophy, University of Vienna.
[5] Metaphors will be identified using the convention TARGET IS SOURCE.
[6] In a scale-free structure, while a few nodes are very highly connected, most are linked to only a few other vertices. Thus, it is not meaningful to determine a *scale,* or typical number of links, for nodes in the network.
[7] Much of the discussion in this paper is organized around examples from Lakoff and Núñez. I acknowledge that mathematicians throughout history originally developed many of these selections. For example, in 1806, Jean Robert Argand used the image of i as a rotation through 90° to

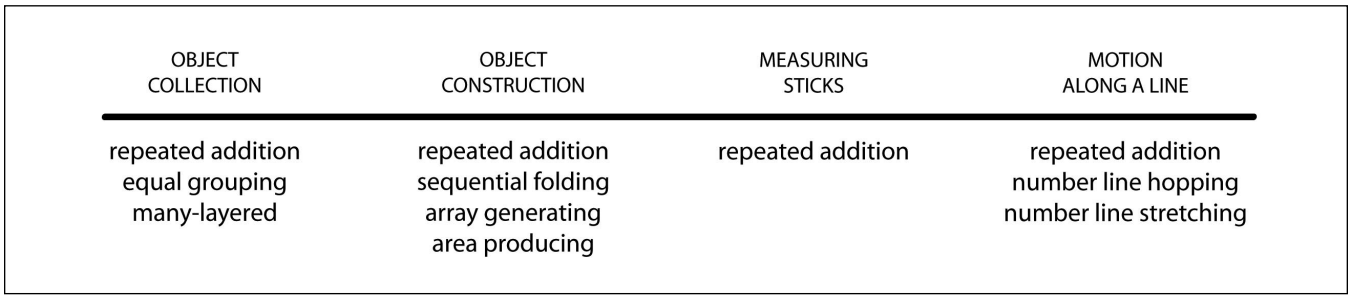

Figure 10: Interpretations of MULTIPLICATION *linked by the four grounding metaphors*

represent complex numbers geometrically (O'Connor & Robertson, 2000). I thank John Mason for bringing this to my attention.

[8] Figure 10 was my own attempt to illustrate the UNIT CIRCLE conceptual blend. Later, I discovered that it has a remarkable similarity to figures on pp. 390–392 in Lakoff and Núñez (2000). Independent development of the diagram illustrates how particular metaphors have entailments that compel certain interpretations. It is likely that any graphic representation of the UNIT CIRCLE conceptual blend would closely resemble Lakoff and Núñez's images.

[9] Kimmel (2002), p. 108. See note [4].

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What makes this a square?

Given any four points in a two dimensional plane, how do we show whether or not each point lies on a different side of a square? (originally proposed in the UK Journal, Mathematics Teaching some time ago; selected by Leo Rogers)