

Some Ideas on the Use of History in the Teaching of Mathematics

VICTOR J. KATZ

The teaching of mathematics is difficult and it is not necessarily easier when one uses the history of mathematics as part of this teaching. But our aim is to discover the ways in which the use of the history of mathematics makes *learning* better for the student. Professor Radford has discussed what he calls the Simple Teaching Model, in which one (paradoxically) assumes that mathematical knowledge is essentially unhistorical and that therefore one can always link the mathematics accomplished in some particular historical period with the mathematics of today without worrying at all about the historical or cultural context in which the mathematics has developed. This leads to the question of whether “pure” mathematical knowledge exists, that is, whether our knowledge of a mathematical topic is really the same as the knowledge of the topic by our mathematical ancestors of centuries ago. If so, perhaps the “unhistorical” historical treatment of mathematics in class will be useful. But if not, then any use of history must indeed be problematic because today’s students will not be able to place themselves in the context of their ancestors of centuries or millennia earlier.

Professors Bkouche and Waldegg have discussed the theoretical underpinnings of the use of history, including the question of the sense in which it is true that ontogeny recapitulates phylogeny. Both agree that one cannot take this idea literally, but Professor Waldegg does cite several studies which indicate how one can understand this idea; namely, work on epistemological obstacles, on the mechanisms of passage from one stage of understanding a mathematical idea to a following stage, and the approach of “didactic transposition” by which modern teaching can in fact utilize old mathematics. The studies seem to indicate specific ways one can use history to help students understand particular points and even how to use historical methods in teaching a modern course.

Now it is certainly clear that to have success in using history, one must understand something about how students learn mathematical concepts. And these theoretical studies will undoubtedly help us do this. But to accomplish these studies, we need to keep experimenting with various ways of using history and keep sharing these with each other. One of the central points to keep in mind as we experiment is the need to consider larger pieces of mathematics than just a single idea or two in the use of history. One needs to think about how one can set an entire series of ideas or even an entire course in some historical context.

To illustrate this, let me begin by trying to answer Professor’s Radford’s question whether “pure” mathematical knowledge exists. My immediate answer to this question is “yes”, although I know that will be disputed by many. Certainly, one can argue that there are many aspects of mathematical thinking closely tied to the culture in which that thinking occurs. Nevertheless, I would insist that, for example, the Pythagorean theorem meant the same thing to

the ancient Chinese as it did to the ancient Greeks. And the rule calculating the number of permutations of k objects in a set of n meant the same thing for Levi ben Gerson in the fourteenth century as it did for Marin Mersenne in the seventeenth and as it means for us today. What does differ is the approach to discovering the relationships and to “proving” them.

What a teacher always needs to determine is what approach will work best in helping his or her own students grapple with and understand the ideas, and this is certainly one place where history can help. It can even help with the eternal struggle to teach the idea of “proof”. But one can and should use history in a more sophisticated way than the Simple Teaching Model. Even though there is such a thing as “pure” mathematics, the approaches leading to the discovery of such ideas are imbedded in the context of the civilizations. It is certainly not a “bad thing” for our students to learn about various cultures and why they were often interested in the same “pure” mathematical ideas. But if, as in some cases, the cultural and philosophical context is really beyond the ken of the average student of today, it can be left out. Thus, to explain that one is not a “number” for the Greeks is probably not worth the effort. And to give the full context of the Greek approach to proof would also be too demanding of many of our students. But the notion of proof is still central to the mathematical enterprise, so it is absolutely necessary that ideas about proof be shared with students whenever possible.

How can history help with the Pythagorean theorem and its proof? The basic historical question is how the result was originally discovered, and we really don’t know the answer. It just appears in many different civilizations. Therefore, one can use an imaginative reconstruction of the discovery by leading the students to discover it for themselves, first for isosceles right triangles — where the theorem probably appears already in the tiled floor of an ordinary classroom — and then to arbitrary right triangles. If we are dealing with the Pythagorean theorem in the context of a geometry course, we certainly need to “prove” the result. What does this mean? In the Chinese context, where there is a lovely picture illustrating this result, the proof was by rearranging various geometric shapes in an appropriate diagram.¹ Will our students understand this as a proof? Do we? So what is a proof of the theorem?

Euclid’s own proof of the Pythagorean theorem is somewhat more difficult than the Chinese version. Can we help our students discover it? There are probably several ways to accomplish this, but even if we just present the proof, this theorem makes an ideal starting point in developing the notion of proof from axioms. Thus, with a good deal of work on the teacher’s part, students can be lead through Euclid’s proof and then work backwards, confirming each piece of the proof by proving an earlier result — until one is forced to assume certain axioms. Unlike many first

attempts at teaching “proof” to students, where they are asked to prove “obvious” results, this method enables them to deal with results that are not obvious. Those are the kind of results for which students will eventually understand the necessity of proof. And is this not a “historical” way of introducing the notion of proof by means of logical arguments from explicitly stated axioms? In fact, one could argue that Euclid organized Book I precisely in this fashion — to provide a proof of the Pythagorean theorem.

What next? One can deal with Pythagorean triples. Again, we need imaginatively to reconstruct, for example, the ideas behind the Babylonian teacher’s writing down his results on Plimpton 322.² But this is again an ideal way to have students work together in groups and come up with these results. Many teachers have, in fact, reported success in using Plimpton 322 in this way. But one might also consider the Indian use of Pythagorean triples in their altar constructions and enlighten the students to an entirely different culture.³

The Pythagorean theorem is, of course, the central theorem behind the development of trigonometry, another “pure” piece of mathematics, but one whose development stemmed not from the necessity for indirect measurement on earth, but from the necessity for measurement of the heavens. Can one develop trigonometry historically? The answer is a qualified yes, because in most contexts it would not be useful to deal with the chords of the originators of the subject. But assuming that one starts with the modern notion of a sine as a ratio, one can in fact adapt Ptolemy’s treatment into a very successful method for helping students understand the basics of the subject. And — given the context of trigonometry as a means for understanding the motions of the heavens — this study provides an ideal place for discussing elementary astronomy — in fact astronomy based on the fixed earth hypothesis. Our students in general know very little about the heavens; since this knowledge was at the root of so much science over the centuries, it is well for us to introduce it whenever possible. I have also always had fun getting the students to debate the question of whether the earth moves (using only arguments derived from evidence available to the ancients). Naturally, trigonometry in the heavens is spherical trigonometry. But the problems connected with the motion of the sun, at least, are not difficult and the spherical trigonometry provides another bit of contextual knowledge to the students, knowledge that is not generally part of the curriculum but which is even of current use. And if one wants to introduce more context to the study of spherical trigonometry, one can consider the problem of the *qibla*, the direction that Moslems must face in prayer.⁴

How about the idea of permutations? How does one develop the counting procedures for elementary combinatorics in a class historically? These ideas are “pure” mathematics, but again, one can begin with the ancient contexts for these ideas. In both the Jewish and Islamic traditions, the context was to count the number of words one can form out of certain collections of letters. Why was this of interest? Because God created everything in the world; the things were indicated by words spelled with the letters of the alphabet. It was thus of interest to determine the possible things God could have created. One can then develop lessons, using the history of the counting laws in the Islamic

and Jewish worlds. One can help the students discover the “Pascal” triangle of combinatorial numbers, when they count combinations. One can help them discover some of the relationships in that triangle, as the Islamic authors did. One can relate the triangle to the binomial theorem, and develop a proof of that theorem by induction. One can help the students discover the multiplicative rule for determining combinatorial numbers, following the methods of ibn Munim and ibn al-Banna, two thirteenth century Islamic mathematicians from North Africa. And one can even demonstrate induction in giving proofs related to the combinatorial numbers as was done by Levi ben Gerson and later by Pascal.⁵ To use Professor Bkouche’s terms, this kind of series of lessons will illustrate not only the question of *demonstration* but also the ideal of *simplicity*. The aim of the authors cited was in general to enable a calculation of an arbitrary counting problem to be made without actually doing the counting — and in the simplest possible way.

In all of these cases, there is a lot of work for the teacher to set up lessons that use the history. The teacher must know the history in sufficient detail to be able to pull out the details relevant to the particular class and to arrange them in the best way. One cannot follow the history blindly if one is to use it for pedagogy. One must massage it a bit and interpret it somewhat — and in the cases where the history is not known, one must make plausible recreations. Nevertheless, the teacher will find the exercise worthwhile. Using history to develop the concepts does work. This is no longer the Simple Teaching Model, but a rather sophisticated teaching model. It shows that, yes there is “pure” mathematics, but that it is often imbedded in a cultural context which can be made real to the students. And the students will not only learn the mathematics, but will also learn about other cultures, and that is surely one of our goals as well.

I have found that a useful exercise is to attempt to look at every course one teaches from a historical point of view. One may well find that it is not always possible to organize an entire course historically, nor even use the history of every topic in its teaching, but one will always find historical ideas which can be successfully incorporated into the course. And, of course, not every use of history can be used by every teacher in every class. But mathematics teachers who believe that history can be useful must continue to share their ideas with each other and continue to use all available forums to persuade others to try the use of history. Our students will surely benefit.

References

1. J. Needham, *Science and civilization in China*, vol. 3 (Cambridge: Cambridge University Press, 1959), p. 22
2. O. Neugebauer, *The exact sciences in antiquity* (Princeton: Princeton University Press, 1951), pp. 36-40.
3. B. L. Van der Waerden, *Geometry and algebra in ancient civilizations* (New York: Springer-Verlag, 1983), pp. 8-9.
4. J. L. Berggren, *Episodes in the mathematics of medieval Islam* (New York: Springer-Verlag, 1986), pp. 182-186.
5. V. J. Katz, “Combinatorics and induction in medieval Hebrew and Islamic mathematics,” in R. Calinger, ed., *Vita mathematica: Historical research and integration with teaching* (Washington: MAA, 1996) pp. 99-106