Mathematics, Semiotics, and the Growth of Social Knowledge

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Any epistemology, any methodology, and any learning theory as well, is based on a metaphysics, on an idea of what makes up the world and what brings about the things which exist in the world. In his 1898 Cambridge lecture on the logic of continuity Peirce said, for instance: "Every attempt to understand anything — every research — supposes, or at least hopes, that the very objects of study themselves are subject to a logic more or less identical with that which we employ" [CP 6.189]. And vice-versa any logic or methodology is to be grounded in some subject matter. There is no thought without an object of thought, thinking always means thinking about something.

Prevailing epistemology and methodology considers, in contrast to the above, method to be independent of subject matter. It gained its plausibility from Newton's great achievement, making mechanics the basic science, and it conceives of the world in terms of particles in motion. It also introduced a distinction between the things themselves and the manner in which they come about or how they come to be known.

This fundamental idea of explaining the world in terms of particular definite bodies in motion and interaction looks back on a long history and it has its merits and its strengths. As everyone knows, the problem of mechanical motion concentrated the efforts of many brilliant minds from the days of Zeno and it has been the focus of mathematical sciences since Galileo and the Scientific Revolution. Now, great achievements are also the source of great errors or biases. It is a common prejudice to try to explain everything in mechanical terms. There are areas or questions where the world might perhaps better be conceived of in terms of signs and interpretations, rather than in terms of objects and interactions. Bateson has given the following description of these two options. "The difference between the Newtonian world and the world of communication is simply this: that the Newtonian world ascribes reality to objects and achieves its simplicity by excluding the content of the context-excluding indeed all meta-relationships — *a fortiori* excluding an infinite regress of such relations. In contrast, the theorist of communication insists upon examining the meta-relationships while achieving its simplicity by excluding all objects. This world, of communication, is a Berkeleyan world" [Bateson, 1973, 221].

On the pages that follow I shall try to outline some consequences of these different views with respect to mathematics.

Descartes vs. Desargues on geometry

During the seventeenth century geometrical perception became separated, so to say, into two relatively distinct forms of geometry, into two different geometrical styles. One of these is represented by the work of Descartes (1596-1650): the geometry of mechanical-metric activity. The straight line in Cartesian geometry corresponds to an axis of rotation or to the stiffness of a measuring rod. The other geometrical style is represented in the work of Desargues (1591-1661). The straight line of Desarguesian geometry is the ray of light or the line of sight. It is a geometry on which, among other things, perspective in painting is based. Each system is based on different basic concepts and fundamental objects. In Cartesian geometry, these include the concept of "distance", or the circle as a geometrical object. Other structures or concepts are defined in terms of these. The infinite straight line in a plane or the infinite plane within space appear as geometrical loci of all points that have the same distance from two fixed, given points. For Desargues' geometry, however, the infinite straight line and the infinite plane are basic elements. For nearly two centuries these two sides of geometrical thinking developed relatively independently from each other into two virtually exclusive theoretical systems. There were no aporias, but mutual exclusion. Each system completed itself as a closed, consistent theory.

By means of the new instrument of arithmetic and algebra developed by the early modern society of the European cities, predominantly in Italy and in the Netherlands, which were oriented towards commerce and the market, Descartes, as expounded in his 1637 *Géométrie*, wished to achieve something the Greeks had not attained, that is to introduce a common bound into the totality of mathematical knowledge, and to create the basis for further generalization by this systematic. Descartes uses the structure of arithmetic, and in particular the fact that "the entire arithmetic contains but four types of calculations", to classify all the problems of geometry and to present them consistently. After having explained the parallelity of geometric construction and of arithmetic operation by means of some diagrams, and after having stressed that "all problems of ordinary geometry can be constructed by exclusive application of the few things contained in the figures explained", he proceeds to criticize the "Ancients" for obviously not having realized this, "as they would otherwise have shied away from the effort of writing so many voluminous books on it, books in which the order of their theorems alone shows that they were not in possession of the true method which supplies all these theorems, but merely had picked up those which they had accidentally come across" (Descartes' *Géométrie*, our translation).

In this role, algebra functions like a logic, an idea which Leibniz, rather than Descartes, because of his own emphasis on the priority of form over concrete content, was to develop. "Leibniz thought that Descartes had stopped short, and did not see his way through to a completely general abstract Universal Characteristic" [Hacking, 1984, 213].

Now Leibniz wanted his characteristic because he thought that truth is constituted by proof. Descartes did not believe this. Both, however, shared the view that science and mathematics aim at a characterization of individual objects or substances, rather than being concerned with the general and the implicitly and incompletely defined.

Formal proof is related to form rather than to content, and to subjugate content to form, or the particular to the general, one had to geometrize algebra. This idea occurred to Leibniz when he dealt with systems of linear equations, trying to design an algorithm for solving them. In a series of letters to l'Hôpital, written in 1693, Leibniz stresses that the variables occurring in the matrix derived from a system of linear equations are nothing but place holders within a certain spatial arrangement, so therefore algebra should be understood as a branch of combinatorics rather than as generalized arithmetic. But Leibniz' views remained ambiguous. On the one hand, the ultimate goal of scientific enquiry, which is in general only to be accomplished by God through an infinite analysis, lies in the determination of individual substances. On the other hand Leibniz introduced relational thinking and the idea of general proofs into mathematical thinking.

Only during the 19th century were were these insights pursued further. Peirce states: "Logical algebra ought to be entirely self-developed. Quantitative algebra, on the contrary, ought to be developed as a special case of logical algebra" [CP 4.134]. Algebra gains such a position only if linked to geometry, from the concentration on relational thinking made possible by this connection of structure and space. One can say therefore that Descartes has arithmetized geometry. But he should rather have geometrized algebra.

Peirce praised his own logical characteristic for "its being veridically iconic, naturally analogous to the thing represented, and not a creation of conventions. It represents logic because it is governed by the same law. ... Still more closely, it resembles the application of geometry to algebra. By this I mean what is commonly called the application of algebra to geometry, but surely quite preposterously and contrarily to the spirit of the study. ...The habitual neglect by students of analytical geometry of the real properties of loci, of which very little is known, and their almost exclusive interest in the imaginary properties, which are non-geometrical, sufficiently show that it is geometry that is the means, algebra the end ... For no doubt it was geometry that suggested the importance of the linear transformation, that of invariance, and in short almost all the profounder conceptions" [CP 4.368]. As soon as one shifts to the realm of the negative, the irrational, or the imaginary, a relational understanding becomes indispensable [Gauss, 1831]. Relations and relata are linked by a principle of continuity (Leibniz, Poncelet) or permanence of forms (Peacock, De Morgan). This evolution, however, came about only during the 19th century, when the notion of space attained a new status.

Peirce learned his own principle of continuity partly from people like L. Carnot and Poncelet, who in turn had studied the dilemma of analytical geometry, namely the "rule of signs"— something which became fundamental also to Grassmann's invention of vector algebra. When Carnot or Poncelet speak of the *êtres de raison* of algebra they do not mean that algebra is just a syntactical game but intend to say that geometry should imitate and even surpass algebra in its capacity to deal with objects that are not strictly individual and are not completely and individually describable according to Leibniz' principle of indiscernibility. From the point of view of philosophical realism (as opposed to nominalism) the relational generality of the continuum is prior to the particulars of the discrete. This was, with respect to mathematics (not logic), Desargues' idea. Granger for instance, writes about Desargues: "A "conic", for the Ancients or for Descartes is but the generic name of a *virtual* and incomplete object, which becomes specific — a circle, a hyperbola, a parabola — when the parameters are determined. For Desargues it is a real object, completely determined in its projective nature; specification occurs only for accidental reasons" [Granger, 1968, 69].

Desargues' work grew out of the practical tradition of applied geometry, as exemplified by the Renaissance treatises on linear perspective. These treatises in the beginning concentrated mainly on recipes for foreshortening such key elements as cubes, column capitals, and the like. Gradually they became concerned also with the manner in which regularities in natural space become transformed into regularities within the image space of paintings. Therefore artists occupied themselves with perspective constructions of accurate patterns (like tiled floors, squared pavements) which could be employed to give a sense of spatial depth as part of the design of the picture. This loosened the fixation on geometrical semantics, shifting emphasis to the rules of transformation. Still, emphasis remained more with what remained invariant under the projection. "Desargues was to concentrate instead upon the elements that remained unchanged. His important original contribution seems to have been the concept of invariance" [Field & Gray, 1987, 28]. In this manner he paved the way towards a mathematical methodology capable of handling "general" objects: that means, types or kinds that are not directly given but given by means of a "continuum" of species, of particular manifestations, and which therefore are general and as such are not specified in every detail. Descartes' approach, in contrast, tries to solve the foundational problem by admitting only objects that can be defined in very specific, arithmetical terms. Descartes' approach operates with very narrowly specified objects, clouding our intuition of the semantics of mathematics, but giving it a rich and well elaborated syntax. Desargues, on the other hand, operates with an intuitively rich semantics but with very difficult syntactical rules. One style is metonymical, the other metaphorical. Gilles-Gaston Granger has indeed marked the differences of style in Descartes' and Desargues' work by referring to the topics of "metonymy" and "metaphor", respectively [see Granger, 1968, chap. 3].

The metaphorical style requires a type of "metaperspective" — that is, a perspective that appears alongside the actual, context-dependent activity. Thus the metaphor requires a distancing from the immediate situation or from the personal perspective that is attained by using a meta-language (which is frequently identified with the object language). This is one side of a metaphor. On the other hand metaphors must have something compelling about themselves; they are not merely comparisons. A metaphor is more like making an equation $A = B$ instead of merely repeating $A = A$! Thus metaphors represent relational thinking. This is their most important aspect since theoretical thinking in general, and mathematical thought in particular, is essentially relational. The contents of theoretical concepts and their extensions are relations between objects rather than objects.

It can be said that the general is directly perceived or cognized in the metaphorical style, but that metaphorical thinking at a distance from the flow of reality becomes analytic. The metaphor is, as we have seen, a way of speaking beyond contexts, and it thereby relativizes each and every perspective that has once been taken. It conceives of an object as a unity within an endless continuous series of presentations. So the unending straight line with its points at infinity is a fundamental element of projective geometry or, for example, the imaginary in geometry. Such conceptions are ideal objects, hypostatic abstractions, and they function as "ground" for a series of representations. The expression of a special perspective, through appropriate change, becomes the real. "If a figure is given in which there are imaginary parts, one can always conceive of another construction, as general as the first, in which those parts that were initially imaginary are now real" [Chasles [1839], 1982, 395].

metaphor

With this metaphorical concept formation, a separation of theory from empirical practice and the establishment of the theory as a *sui generis* reality is introduced. This may provoke difficulties of understanding. If you want, writes Descartes in a letter to Desargues in July 1639, "to write for people who are interested but not learned, and make this subject ... which is very useful for Perspective, Architecture, etc., accessible to the common people and easily understood by anyone who studies it from your book", you must "employ the terminology and style of calculation of Arithmetic, as I did in my Geometry" [Field & Gray, 1987, pp. 176-177]. Descartes repeats arguments to be encountered already in the writings of Petrus Ramus (1515-1572), in particular in Ramus's book on arithmetic and geometry of 1569.

The principle of continuity

The lecture, which Felix Klein had given to the *Königl. Gesellschaft der Wissenschaften* in Göttingen on November 2, 1895, has become famous and has been translated into many languages. In this lecture, entitled "Über Arithmetisierung der Mathematik", Klein wished to describe the perspective under which he saw the development of the mathematics of his time, and in particular he wanted to characterize his own position "towards that important mathematical trend whose major representative is Weierstrass ..., the *arithmetization* of mathematics". Klein began by pointing to the exemplary character of Euclid's *Elements* which represented the indubitability and exactness of geometrical truths as something unique among man's scientific insights. "The spirit in which modern mathematics was born, however, is quite a different one. Starting from the observation of nature, and aimed at explaining nature, it placed uppermost a certain philosophical principle, the *principle of continuity*. This applies to the whole of the 18th century, which, in regard to the development of mathematics, was really a century of discoveries. Only gradually did rigorous criticism emerge, which inquired into the consistency of these bold developments — something like the re-establishment of an orderly administration after a long campaign of conquest. This is the age of Gauss and Abel, of Cauchy and Dirichlet ... hence the demand for an exclusively arithmetical proof" [Klein, 1895, 143/144].

This summary by Klein represents a view of the development of mathematics in the 19th century current far and wide to this day. It entails, however, some difficulties and problems.

The first problem is of immanent order and concerns the opposition established by Klein between mathematical discovery in the 18th century and the foundational codification of mathematics in the 19th century. Is it true that this codification has nothing to do with the development of new knowledge, that the new foundation of mathematics has no productive function at all? Does this separation between development and foundation really apply to the mathematics of the 19th century, which showed all the marks of a historically unprecedented productivity? And did he not himself call the principle of continuity a "philosophical principle"? One might better start from other assumptions: first, that the history of philosophy from the seventeenth to the end of the eighteenth century is largely identical with the history of the elaboration of the calculus and its foundations. Second, with the French Revolution there begins a separation between pure and applied mathematics that brought forward a certain interest in geometry and vector algebra. It is clear that this social change brought new importance to the foundational debate.

Klein's views with respect to the problem of mathematical rigor and the principle of continuity were in agreement with a tradition of purely foundational concerns. From Berkeley's rejection of the principle of continuity, and Hume's complaints that demonstrations which make use of it seem "as unexceptionable as that which proves the three angles of a triangle to be equal to two right ones, though the latter opinion be natural and easy, and the former big with contradiction and absurdity", to Cauchy's criticism of Poncelet's work on projective geometry for its lack of rigor, and Comte's comparison of Lagrange's algebraic version of the calculus with that of Leibniz — stating that the former is "philosophically more satisfying" whereas the latter is better suited to applications of the calculus — all opinions seemed to point in the very same direction. Nevertheless, since the days of Descartes and Desargues, there have always been complementary tendencies. Taking this into account, it seems to me that the opposition between the principle of arithmetisation and the continuity principle is better judged in ontological terms rather than methodological ones, although the latter play a certain role, for social reasons primarily. One movement tries to progressively replace mathematical objects, which naturally cannot be described in detail, by ever more sharp definitions of these objects. The other trend, exemplified in the work of Poncelet, Grassmann, Schröder, and Peirce, among others, on the contrary, tries to enable mathematical reasoning to cope with "general" — that is, not fully specified — objects. (Think of the terms every schoolboy knows: "general triangle", "general conic", etc.) Making a similar judgment Bos and co-authors write (although they do not see so clearly the ontological problem): "The program of rigorization through arithmetization found the solution of the foundational problems in precise definition of objects of mathematical study; complex numbers were interpreted as pairs of real numbers, and real numbers themselves were explicitly constructed from the natural numbers ... In Poncelet's approach we see an attempt to solve the foundational problems in an entirely different way. Rather then extending or embedding the objects of mathematics, Poncelet wanted to extend the mathematical rules of inference. For that purpose he ... applied the principle of continuity" [Bos *et al.*, 1987, 304]. Poncelet in fact tried to extend the rules of inference in order to cope with "general objects", with types or kinds, rather than with minutely defined concepts. Grassmann and Poncelet accepted, for instance, complex numbers as mathematical objects in their own right and having a meaning which did not require them to be reduced to real numbers. Grassmann's revolutionary innovation of what is nowadays called linear algebra in fact depended on his introduction of new types of non-Cartesian quantities. Mathematical meaning is not to be treated, Poncelet, Grassmann, Peirce, and others believed, in a reductionist manner, because mathematics progresses by introducing ever new (ideal) objects into its cognitive practice. These objects as a rule can only be described axiomatically, and that means incompletely. Skolem has shown that even the natural numbers cannot be completely characterized, admitting infinitely many axioms in terms of number variables. A. Robinson used these results by Skolem to elaborate his "Nonstandard Analysis", which contains "ideal objects" that were formerly treated (by Leibniz, for instance) by means of the principle of continuity. In fact, "Skolem's works on non-standard models of Arithmetic was the greatest single factor in the creation of Nonstandard Analysis" [Robinson, 1996, 278].

Joan Richards in her study of the pursuit of geometry in Victorian England very aptly speaks of a contrast between a "definitional" and a "descriptive" view of mathematical rigor. But the methodological question is, I believe, only secondary to the ontological one. Do we live in a Newtonian or a Berkeleyan universe? Bateson might ask.

During the 18th century, numbers, in their inseparable linkage to the quantity concept, represented the actual object field of mathematics, while algebra and the symbolic calculi of mathematics were regarded merely as a language permitting an easy and suggestive manner of representing relationships between numbers and quantities. These distinctions became precisely reversed in the 19th century. Algebra was not to include directly actual mathematical relationships, which constitute the subject matter under study, while arithmetic, for its part, became the language of algebra and hence of the entirety of mathematics, by means of which, and in which, all mathematical facts must ultimately be expressible. This process of arithmetization finally culminated, towards the close of the century, in the reduction of the consistency of mathematics to the consistency of arithmetic, raising arithmetic to the position of the proper foundational science of mathematics. Hilbert's program, which attempted to reduce the whole of mathematics to finitist combinations of signs and/or numbers, is but a pinpointedly formulated version of these efforts which, eventually, led to Gödel's arithmetization of the logical system of "Principia Mathematica".

arith/algebra

Let us try and summarily state a possible conclusion from the last two paragraphs. Pascal's (1623-1662) distinction between geometrical intelligence and subtle intelligence, a distinction Pascal drew in order to criticize the epistemic claims of Cartesianism, has remained famous ever since, at least in France [Ullmo, 1971]. Cartesianism, as well as classical positivism, which followed it and which is a branch of it, assumed that all things could be explained by mathematically necessary deductions from self-evident initial observations and clear definitions. This view contains at least two confusions. First it confuses the map with the territory: it assumes that the world is already known in principle and that scientific deduction will in the course of time fill all lacunae, attaining all particular truths. Pascal's subtle intelligence, in contrast, understands that the world can be appropriated cognitively only by the constructive incorporation of ever new types of objects into thought. There is no thought without an object. Thinking is an intentional activity, and, moreover, the new can never be completely reduced to the already known. With respect to mathematics one could say, employing Peirce' terminology, that Descartes misunderstood the hypothetical character of mathematics and, secondly, that he did not see that mathematical deduction in its more important applications contains the introduction of hypostatic abstractions, i.e. general ideas.

Mathematics "studies the substance of hypotheses" and in so far as a hypothesis is mathematical it "is merely a matter of deductive reasoning" [NEM IV, 267-268]. "Hence to say that mathematics busies itself in drawing necessary conclusions, and to say that it busies itself with hypotheses, are two statements which the logician perceives come to the same thing" [CP 3.558]. In addition it should be mentioned that, according to Peirce, "all necessary reasoning is of the nature of mathematical reasoning" [CP 5.147]. The hypotheses which necessary reasoning will unfold with respect to their consequences are not the product of deduction but of abduction. "Abduction is the process of forming an explanatory hypothesis. It is the only logical operation which introduces any new idea; for induction does nothing but determine a value, and deduction merely evolves the necessary consequences of a pure hypothesis. Deduction proves that something must be; Induction shows that something actually is operative; Abduction merely suggests that something may be" [CP 5.170-71].

Now obviously deduction must, at least in its more prominent cases, imply some abduction. Deductions, says Peirce, "are of two kinds, which I call corollarial and theorematic. The corollarial are those reasonings by which all corollaries and the majority of what are called theorems are deduced; the theorematic are those by which the major theorems are deduced. ... when it comes to proving a major theorem, you will very often find you have need of a lemma, which is a demonstrable proposition about something outside the subject of inquiry; and even if a lemma does not have to be demonstrated, it is necessary to introduce the definition of something which the thesis of the theorem does not contemplate" [CP 7.204]. In the most remarkable cases, this is some form of hypostatic abstraction. Hypostatic abstractions "are particularly congenial to mathematics" [CP 4.235]. In general a hypostatic abstraction will be nothing but the hypostatization of a hypothesis. Deduction is about "if...then" statements. If people take opium, then they will fall asleep. The hypothesis here is thus the general statement that "Opium puts people to sleep". By means of abstraction this is converted into "Opium is a soporific". This transformation turns out to be "so essential to the greater strides of mathematical demonstrations that ... most practically important results of mathematics could not in any way be attained without this operation of abstraction" [NEM IV, 49]. Thus mathematical reasoning proceeds, contrary to the tenets of positivism, by introducing ever new mathematical objects or general ideas. This provides the continuity principle with a role of extraordinary importance since the process of generalization, as well as the inseparable interactions of general and particular, are governed by it.

> "Once you have embraced the principle of continuity no kind of explanation of things will satisfy you except that they grew. The infallibilist naturally thinks that everything always was substantially as it is now. Laws at any rate being absolute could not grow. They either always were, or they sprang instantaneously into being by a sudden fiat like the drill of a company of soldiers. This makes the laws of nature absolutely blind and inexplicable. Their why and wherefore can't be asked. This absolutely blocks the road of inquiry. The fallibilist won't do this. He asks may these forces of nature not be somehow amenable to reason? May they not have naturally grown up? After all, there is no reason to think they are absolute. If all things are continuous, the universe must be undergoing a continuous growth from non-existence to existence. There is no difficulty in conceiving existence as a matter of degree. The reality of things consists in their persistent forcing themselves upon our recognition. If a thing has no such persistence, it is a mere dream. Reality, then, is persistence, is regularity" [CP 1.175].

This regularity, however, comes last and it presupposes that the sudden presence of the unexpected in the mind indicates that our cognition does not consist in merely dreaming. Thus Newton reminded us that reality is not what we believe it to be and that it is more than experience tells. But science in the spirit of Newton, Peirce might argue, does not take sufficiently into account that the knowing subject takes part in the object to be known, that everything belongs to one and the same reality in evolution.

Mathematics and generalization

Generalization is the goal and destination of mathematics. The compulsion to generalize is seen already in the way that a more general problem is often more easily treated in mathematics than a special case of the same problem. Generalization makes things movable and ascribes meaning to them. But there is never a fundamental or ultimate level of generality that can be reached. We can never, in fact, make exhaustively explicit the foundations of our own activities or of the practices by which we render the world and our own lives intelligible. Mathematics provides us with privileged access to such insights, and from this stems the importance of mathematics with respect to epistemological and metaphysical considerations.

The measure of mathematical realism, Bachelard says, "lies in the extension of concepts and rather less in their content". This universalization makes mathematics "poor", "abstract", it becomes formal, for its wealth lies concealed in its functional meaning. And the latter only becomes visible in a complex social context. Robinson Crusoe can have no use for non-Euclidean geometries, imaginary numbers, and the like. Up to the point we have reached, our thesis seems quite suggestive. It is not satisfactory, however, to ground it only in a kind of mathematical generality which remains close to Piaget's general structures, or to Hilbert's conception of mathematics. Without relating mathematics to individuals, it will not develop. If we assume that general and particular are inseparable, and if we believe that the general mediates between diverse particulars, we will be more interested in the specificity and the conditions of generalization than in the status of ready-made knowledge.

general/particular

For what follows we should like to assume that generalization consists in introducing abstract objects, ideas or, as Peirce says, *hypostatic abstractions*, and that the latter possess the mode of being of signs or continua, or alternatively constitute signs. This implies that a general idea or an abstract object does not mean anything apart from its functioning as the ground for a sign. This does not imply a subjectivist epistemology as a sign is something beyond an individual person's will and decision. We cannot at will prescribe the effects of signs. Signs are real. A sign is to be regarded as real because its character as a sign does not depend on the will or arbitrary determination of any interpreter. Realism holds that anything is, or can be, intrinsically a sign, not only because we use it as a sign. The difference between a mere thing and a sign consists in the fact that a sign has an effect, that it has the function, as Peirce says, "of making ineffective relations effective" [CP 8.332].

Imaginary numbers, for instance, sometimes play the role of mediators between the premises (a problem) and the result without themselves appearing in either. "The curious thing," young Törless observes in Musil's novel, "is that in spite of that [their nonexistence], one can quite readily calculate with such imaginary or somehow impossible values, and there is a tangible result in the end". This became understandable even to mathematicians only after Gauss, in 1830, had imbedded all numbers in a continuum, the so-called complex number plane, making perceptible how paths may lead from the real through the imaginary and back again to the real. With fractional and irrational numbers, we had already encountered similar problems [Peirce 1988, 129, MS 956]. From Leibniz through Bolzano to Peirce, it has been stressed again and again how the continuity to be found in the ideal serves us in our cognition of the real. Thus Leibniz writes, for instance: "If, however, the idea of the circle is not quite equal to the circle, we may nevertheless conclude from the idea to truths which experience with regard to the real circle would indubitably confirm" [*ibid.*, 65]. Bolzano, in his analysis of the equation concept, offers an example identical to Peirce's in justifying the use of fractional and imaginary quantities. In the context of what an equation says, such representations attain an objective character as soon as we switch from constant to variable quantities [cf. Otte 1990, 239 f]. Mathematical objects are relations.

In their effect, signs are real and general. With this assertion we arrive at the problem of generalization. A generalization, it can be said, is the extension or spreading of the effect of a sign. Now just as concepts and natural laws do not apply themselves, as works and theories of art do not produce their own recipients, so generalization has its own conditions relatively independent of the general. Generalization always contains elements of a factitious "here and now" whose precise existence cannot be anticipated. "When a stone falls to the ground, the law of gravitation does not act to make it fall. ... The stone's actually falling is purely the affair of the stone and the earth at the time. This is a case of reaction. So is existence which is the mode of being of that which reacts with other things" [CP 8.330]. Generalization thus always has to do with the relationship between the particular and the general or, to say it in different terms, a generalization or a sign depends on a particular being or a thing. Now generality is *Thirdness* or Mediation or Continuity, and Law and Existence is *Secondness*. Generalization thus depends on *Secondness* and *Secondness* in turn cannot exist without *Firstness*.

Thirdness is Peirce's central category, although there is no *Thirdness* without *Firstness* and *Secondness*. *Firstness* is the basis of it all as it is just the continuity of being or of life, which is interrupted by the springing up of a Second, which than has to be mediated by an interpretation or representation, that is, by a Third. As Peirce states it: "The immediate present, could we seize it, would have no character but its Firstness. Not that I mean to say that immediate consciousness (a pure fiction, by the way), would be Firstness, but that the quality of what we are immediately conscious of, which is no fiction, is Firstness" [CP 1.343]. The essential point here is in understanding Firstness as the basis for future development, Firstness involved with Secondness, and that mean. It is contingent whether a future event will happen or not, although it exists as a real possibility. Normal empirical science in searching for indications or symptoms favors Secondness. Speculative theory concentrates on Thirdness.

But *Firstness* is the most difficult of the three for us to understand. Our rationalist traditions in philosophy have accustomed us to *Thirdness* which can be conceived of in terms of conceptual thinking. The recurrent empiricist cast of mind makes *Secondness*, as represented by existing interactions, plausible. *Firstness,* however, which means quality or feeling or possibility or chance, is very difficult because our nominalist tradition makes an unperceived unity or an undifferentiated continuity hard to swallow. We are used to conceiving of the world as a set of unrelated "atoms", things, and the like, and we tend to believe with Kant that continuity is the product of a synthesis by the mind. This seems one-sided and erroneous [see CP 1.384]: continuity is the most important prerequisite of knowing. What turns an individual observation or mental event into thought, what provides a particular statement with meaning, is its connection with other such events and statements. This continuity of reasoning cannot be established by language, conceptual thinking, and logical syllogisms alone, but must rather find some foundation in reality itself. *Firstness* is important as it gives rise to *Secondness*, to unexpected existence and reaction. Even if we were to insist that "no matter of fact can be stated without the use of some sign serving as an index" [CP 2.305], we would also need to be aware of the fact that there is no fruitful Secondness without Firstness.

Firstness is something which only the various phenomenological traditions have been able to take into account. Heidegger's work, for example, grew out of questions of phenomenology posed in the context of the foundational crisis of mathematics by his teacher Husserl. Heidegger argues that the separation of subject and object denies the more fundamental unity of being-in-the-world (Dasein), which is *Firstness* in Peirce's terminology. By drawing the distinction that I (the subject) am perceiving something else (the object, a cat, for instance), I have stepped back from the primacy of experience and feeling that operates unconsciously and without reflection. This stepping back, which may lead to a perceptual judgment, like: "This is a cat", arises in an event which Heidegger calls *breaking down*.

One simple example Heidegger repeatedly gives is that of a hammer being used by someone engaged in driving in a nail. To the person doing the hammering, the hammer as such does not really exist. It is part of the background of readiness-to-hand that is taken for granted without explicit recognition or identification as an object. The hammer presents itself as a hammer only when there is some "*breaking down*". Its "hammerness" emerges if it splits or slips from the grasp or mars the wood, or if a nail needs to be driven in and the hammer cannot be found. I would like at this point to add several comments. First, because the individuals in our social surroundings are much more active than the objects of nature, such breaking down occurs much more frequently within a social context. From this observation stems the theory that the evolution of larger brains and greater intelligence in mammals has something to do with their particularly complex social behaviors. For instance, "the evolutionary pressure selecting for large brain size and super-intelligence in primates did seem to have something to do with the need to weld large groups together" [Dunbar, 1996, 64]. And Dunbar identified the function of language, in establishing and servicing social relationships, as the cause of its development. Mathematics now adds immensely to our semiotic capacities and offers possibilities for world creation that verbal language can never afford.

Second, what emerges from a breaking down is not just an object but an idea or some hypostatic abstraction like "hammerness". At least, this is the case if there is a recurrent pattern of breakdown in the activity. Thus a conscious visual perception, for instance, contains two elements, a certain relatively stable breakdown in continuity and an interpretation, or representation, by means of some general idea like "cat", "redness", or "hammerness". We do not see white or green spots but rather cats or trees. The apparent mismatch between empirical perception and (mathematical) intuition, which is largely responsible for the neglect of the role of the latter, is due to a misunderstanding about empirical knowledge. The objects of ordinary perception are constituted or constructed rather than perceived in an entirely spontaneous and natural way. Visual perception is a highly complex phenomenon strongly influenced by sign systems and categorical schemata. Kant's idea of a schema of the imagination expresses these facts, and the philosophy of science speaks of the theory-ladenness of observation in this context. Physicists or mathematicians are making abstract models of the universe to which they "give a higher degree of reality than they accord the ordinary world of sensations" [Weinberg, 1976, 28; see also Bochner, 1974]. The philosophy of mathematics has put forward an even stronger claim, to the effect that all objects are essentially abstract objects, possibly even mathematical objects [Tymoczko 1991], and computer scientists like Flores and Winograd are led to a "radical recognition about language and existence: Nothing exists except through language", they claim. "We are not advocating", they continue to explain, "a linguistic solipsism that denies our embedding in a world outside of our speaking. What is crucial is the nature of existing. In saying that some thing exists (or that it has some property), we have brought it into a domain of articulated objects and qualities that exists in language and through the structure of language, constrained by our potential for action in the world" [T. Winograd & F. Flores, Understanding computers and cognition, Addison-Wesley, Reading, Mass. 1986, 68f.]. Thus Secondness has a lasting effect only if it leads to Thirdness. For example, no paradigm is abandoned for reasons of empirical contradiction alone and without an alternative paradigm being available.

Third, by his "semiotic transformation" of epistemology Peirce was able to take into account that looking from different perspectives on one and the same thing, and viewing different objects from one and the same point of view, become indistinguishable approaches, as in the fusion of analytical geometry with linear algebra. It is the relations between perspectives that may gain objectivity, as in the theoretical conception of geometry as concerned with invariants, or in the interpolation of measurements by means of mathematical functions. To mention one more example, the objects of probability theory, for instance, are not the individual events, nor collections of these events, but are the probability distributions. All these relations are, however, signs according to Peirce's semiotic theory of knowledge.

One further consequence is to be seen in semiotic theory's attempts to explain cognitive growth as a process in which the stages are indifferently members of a social community and the sequential states of a single person. Knowledge and cognition are objective only to the extent that they have to grow and be generalized. That is their essential nature. Man is himself a sign and the processes of objective and of communicative generalization become unified into a single process. Generalization has to be understood equivocally as the dissemination of certain concepts and representations among various social subjects, and as an extension of their intended application.

Every student of linear algebra experiences this "inseparability thesis" since in linear algebra one cannot distinguish between linear mappings (a change in the object under consideration) and coordinate transformations (a change in perspective on the same object). As Leibniz illustrated beautifully, we can break up this inseparability only if we introduce an absolute unit of measurement. In reality our body functions as such a unit. Thus in reality we take part in both of Bateson's worlds, the Newtonian and the Berkeleyan.

References

Bateson, G. [1973]. Steps to an ecology of mind. St. Albans: Paladin

Bateson, G. [1979]. Mind and nature. New York: Dutton

Bochner, S. [1966]. *The role of mathematics in the rise of science.* Princeton: Princeton University Press

Bochner, S. [1969]. *Eclosion and synthesis.* New York: W.A. Benjamin

Chasles, M. [1839/1982]. *Geschichte der Geometrie*, reprint. Wiesbaden: Sändig

Davis, R. B. [1984]. *Learning mathematics.* London, Sydney: Croom Helm

Descartes, R. [1954]. *The geometry of René Descartes.* New York: Dover

Dunbar, R. [1996]. *Grooming, gossip and the evolution of language.* London: Faber & Faber

Field, J.V. & Gray, J.J. [1987]. *The geometrical work of Girard Desargues.* New York: Springer

Gauss, C.F. [1831]. *Theoria Residuorum Biquadraticorum.* Commentatio secunda. In: Göttingische gelehrte Anzeigen. April 23, Werke II

Gaidenko, P.P. [1981]. Ontologic foundation of scientific knowledge in 17th and 18th-century rationalism; in: H.N. Jahnke/ M. Otte (eds.), *Epistemological and social problems in the sciences in the early 19th century*, X, 55- 63 Dordrecht: Reidel

Gödel, K. [1944]. Russell's Mathematical logic, in: P.A. Schilpp (Ed.) *The philosophy of Bertrand Russell*, Open Court, LaSalle/USA. (4. Aufl. 1971), S. 123-154) S. 138.

Granger G.-G. [1968]. *Essai d'une philosophie du style.* Paris: Armand Colin

Hacking, I. [1984]. Leibniz and Descartes: Proof and eternal truths. In T. Honderich (Ed.), *Philosophy through its past* (pp. 207-224). Harmondsworth: Penguin

Hume, D. [1975]. *Enquiries concerning human understanding and concerning the principles of morals* (3rd. ed.). Oxford: Clarendon Press

Jakobson, R. and M. Halle. [1980]. *Fundamentals of language.* The Hague: Mouton

D. Hofstadter. [1979] *Gödel, Escher, Bach.* The Harvester Press: Hassocks, UK

Klein, F. [1895]. *Über Arithmetisierung der Mathematik.* In: Pädagogische Zeitung

Meyerson, E. [1930]. *Identity and reality.* N.Y.: Dover Publications

Otte, M. [1989], The ideas of Hermann Grassmann in the context of the mathematical and philosophical tradition since Leibniz, *Historia Mathematica* 16: 1-35

Otte, M. [1990], Arithmetic and geometry: Some remarks on the concept of complementarity, *Studies in Philosophy and Education* 10: 37-62

Peirce CCL = The Cambridge Conferences Lectures of 1898, in: Ch. S. Peirce, *Reasoning and the logic of things*, ed. by K.L. Ketner, with an Introduction by K. L. Ketner and H. Putnam, Harvard UP, Cambridge/London 1992

Peirce CP = *Collected papers of Charles Sanders Peirce*, Volumes I-VI, ed. by Charles Hartshorne and Paul Weiß, Cambridge, Mass. (Harvard UP) 1931-1935, Volumes VII-VIII, ed. by Arthur W. Burks; Cambridge, Mass. (Harvard UP) 1958 (vol. and paragraph nr.)

Peirce MS = Manuscript, according to Richard S. Robin, Annotated catalogue of the papers of Charles S. Peirce, The University of Massachusetts Press 1967

Peirce NEM = Carolyn Eisele (ed.), *The new elements of mathematics* by Charles S. Peirce, Vol. I-IV, The Hague-Paris/Atlantic Highlands, N.J. 1976 (Mouton / Humanities Press)

Peirce W = Writings of Charles S. Peirce. A chronological edition, Vol. 1-5, Bloomington (Indiana University Press) 1982 ff.

Poncelet, J.V. [1822-1862]; Applications d'analyse et de géométrie, Vol. 1. Mallet-Bachelier; Paris

Richards, J. [1986] *Existential epistemology,* Oxford: Clarendon Press, 48
Robinson, A. [1996] *Non-standard analysis.* Second edition. Princeton University-Press
Tymoczko, T. [1991]. Mathematics, science and ontology. *Synthese,* 88, 201-228
Ullmo, J. [1971]. The geometric intelligence and the subtle intelligence. In F. LeLionnais (Eds.), *Great currents of mathematical thought* N.Y.: Dover Publications
Winograd, T., & Flores, F. [1986]. *Understanding, computers and cognition.* Reading, Mass.: Ablex Publ. Co.

Correction

We very much regret the omission of two paragraphs from Candia Morgan's article in the last issue ("The language of mathematics: Towards a critical analysis of mathematics texts", FLM 16(3): 2-10). We offer her, and all our readers, sincere apologies.

The corrected version of sections 3.2 and 3.3 of her article are printed alongside.

3.2 Interpersonal aspects

Specialist vocabulary The use by No. 3 of *times* rather than *multiply* is less formally "mathematical" and the use of such vocabulary may be read as a remnant of the early years of mathematics schooling and hence a sign of immaturity.
modality The modality of the phrase *you will end up with* suggests an authority over the reader's activity rather than over the mathematics, whereas *you get,* being in the present tense, focuses on the universal generality of the mathematical statement. The contrasting modality of the two statements introducing the final formulae, *You can write this as...* and *This therefore is the formulae,* also suggests that the two students differ in their levels of confidence.

3.2 Textual aspects

expressions of reasoning No. 2's use of *therefore* clearly signals that her text is presenting a logical argument whereas No. 3 merely juxtaposes statements and does not, therefore, force the reader to read her text in this way. Such juxtaposition, being characteristic of spoken rather than written languge, may also be taken as a sign of immaturity.
cohesion No. 2 presents the variables and operations in the same order in both procedure and formula, whereas No. 3 changes her order from $(T + B) \times S$ in the verbal procedure to $S(T + B)$ in the symbolic formula. This disjunction further reinforces the lack of logical structure in No. 3's text and may even be taken by a teacher assessor as a suggestion that the symbolic formula was copied from another source rather than "belonging" to the student herself.

Any of these features might have contributed to Dan's impression that No. 3 is less competent mathematically. While it is not possible to say precisely which aspects contributed to his assessment, there is clearly a mismatch between her text and Dan's expectations which appears to have affected his evaluation of the whole of No. 3's performance and even of her general level of intellectual "ability". Unfortunately, Dan himself was unable to identify the features of the two texts which gave rise to his impressions. As a consequence, it seems unlikely that he would be able to provide advice to a student on how to produce an acceptable text.