Inventions and Conventions: A Story about Capital Numbers[1]

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> Ana: Capital letters and capital numbers.
> Barbara: What are capital numbers?
> Ana: Thirty-three. So thirty is a capital number of three. And that's the other way to write the three (pointing to the 3 in the tens place).

In mathematics education, conventions and inventions are oftentimes considered to be unrelated and unconnected aspects of knowledge; the first are "discovered or learned by transmission from the environment [Kamii, 1985, p. xii], while the second are *created* by subjects. This position seems to postulate a dichotomy between conventions (such as place value, and the numerical notation system) and inventions. Moreover, many times little value is given to what learners invent in the process of knowledge construction. Through the "story of capital numbers" I wish to provide an illustration of the co-operation, collaboration, and interaction that takes place between conventions and inventions; indicating that both aspects are necessarily complementary. In addition, most conventions started out as inventions at some point in time, so that their relationship should be considered as a continuous rather than a discontinuous one. My argument will be two-fold: regarding inventions, that they are of utmost importance in knowledge development; regarding conventions, that they play an important role in subjects' inventions, and provide a support for their development, but that at the same time they are subordinate to inventions and to the assimilatory aspects of thought. This position borrows from Piaget's perspective regarding the figurative and operative aspects of thought, which he considered to be complementary [1972; Gruber & Voneche, 1977; Piaget & Inhelder, 1966/1971; Piaget & Inhelder; 1967/1972][2].

In the analysis of the protocols from extended clinical interviews[3] which were carried out with Ana, a 5 year old child, I will focus on her making-sense of conventions, and on the roles that conventions and her own inventions played in her knowledge construction.

Ana's making-sense of conventions

Ana is a 5-year-old child who attends a public kindergarten in the Boston area. She is the only child of a middle class family. She is bright and relates easily to both children and adults. She is extremely lively, and tends to adopt leading roles in different environments: with neighbors, at kindergarten, and at her after-school program.

A series of four interviews was carried out with Ana. For three months, interviews were held every three weeks, and each lasted between 30 and 45 minutes. Each interview was videotaped, and later transcribed verbatim. In each interview, Ana was presented with different kinds of materials (coins, pencil and paper, dice, tags with numbers printed on them); and with questions related to the number system and the notational aspects of place value. The questions presented were not designed *a priori*; the areas which they intended to explore were. These questions developed into conversations when Ana manifested interest in them.

On our first interview, Ana showed that she could write numbers from one to twelve, and count from one to twenty-eight. At first, when I asked her to write numerals beyond twelve, she would say that she did not know how to do that, and there seemed to be no pattern in the way that she named numerals beyond twelve. She could give a numeral a certain name (usually not conventional), and then name it as something different a few minutes later.

In this first interview, Ana provided various examples of her knowledge of certain mathematical conventions. For example, she told me that could write the numbers from one to twelve, and that she knew them because the clock in her house had those numerals.

During our second interview I showed Ana nine paper tags with a number from one to nine printed on each one, and I asked her:

> B: Of all these numbers, Ana, which one is more?
> A: The nine.
> B: Why?
> A: *Because it goes* one, two, three, four, five, six, seven, eight, nine. So this one (pointing to tag 9) is more. (Italics added.)

Later, I gave Ana three more tags, with numbers ten, eleven, and twelve printed on them. Ana took the tags and spontaneously placed them, in a row, in ascending order. When I asked her what she had done, she said.

> A: I made the numbers *like in the real world.*
> B: Like where?
> A: *Like real counting.* (Italics added.)

In these excerpts, we can gather that Ana has developed the idea that there is a certain order to numbers, which must be followed, and which is already determined. During this same interview, I wrote the numbers 48 and 100, and asked Ana:

> B: Can you think of what numbers they (pointing to 100 and 48) could be?
> A: No.
> B: Try to imagine, which number is this one (pointing to 100)?
> A: One hundred.
> B: This one is one hundred. How do you know?
> A: *Because I have a book that has that, and it says "one hundred".* (Italics added.)

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In this example, Ana showed some other sources from the "real world" that she had used to develop an understanding of the number system. On our following interview, I asked her to write thirty-four, and told her:

B: Try to think of it before you write it. What would it look like?
A: I know.
B: You know? Did you already think of it, in your mind?[4]
A: Yeah (picking up the pencil and starting to write). Yep. You know what (she stops writing), I know, I didn't even think in my mind, but every time I watch TV in my house, thir ... , I have to put kid's channel, I have to put three four. And then if it ... every time they say I have to do thirty four I do like ... (and she writes 34).

Ana took information from different sources that she was in contact with every day: a clock, books, television. Through these sources she "taught herself" (as she argued) how to write the numerals from one to twelve and she constructed the idea that there is a certain order to numbers, which is the way "numbers go in the real world". In addition, she developed the notion that in that order, the last numbers are "more": when I asked her which was more from a series of numbers from one to nine, she named the whole set and finished off with nine, and she claimed that therefore nine had to be more.

These examples illustrate how Ana co-ordinated information, which she assimilated from her environment, with her previous knowledge, in the process of solving certain problems: how to write thirty-four, what number 100 is. Faced with what was problematic [Confrey, 1991], Ana used information from the environment (the tags with which she was faced, for example), co-ordinated it with previous knowledge, such as the number for the kid's channel, and constructed something new: how to write thirty-four. The numerals on the clock, the number one hundred, the numeral thirty-four, were not simply socially transmitted, or taken as they previously existed. Each "piece of information" was integrated with the rest, and was transformed. At the same time, it became integrated with her existing mental structure, together with her previous knowledge: Ana used the information when appropriate; it had not been copied but assimilated and reconstructed.

Capital numbers: Ana's invented tool

In the following stage, Ana was asked to deal mainly with larger numbers (beyond the twelve which she could write). Faced with this new problem, Ana's inventions played an important role in her trying to solve it. On our second interview, Ana wrote 310, and told me that the number said "twenty one". Then, during our third meeting, I asked Ana to write "two hundred", and she wrote 08. She then wrote 38, and said "I don't know" when I asked her what number that was. A few minutes later, when I asked her what number 48 was, the following dialogue took place:

A: Thirty one, thirt ... (pause)
B: Which one could it be?
A: Forty eight.
B: You're right Ana, that number is forty eight. How did you know?
A: Because, I like ... I was doing like this (putting her hands at the sides of her head), I was thinking in my mind and I was doing like this ... (pause) what if, what if you write another number here, do another number there (pointing to the paper).
B: OK. (I write 46).
A: I was thinking like this (puts her hands at the sides of her head and looks at the numeral). Forty six (speaking slowly).
B: But how do you think like that?
A: Because I just know in my mind.
B: Can you teach me how to do that, so I know how to do that also?
A: Yeah, just ...
B: How do you know what sound it is, like, what to say?
A: Because I just know that it's, first I say a four and then I say a six, and then I say: Ah! Forty six!!!

Ana had developed a system for "reading" large numbers. She was successful, but was still not able to explain how she was able to do it. Then, Ana wrote 34, and I asked her:

B: How do you read that? How do you say that number?
A: You say ... first you think of a three, and then you do like a capital letter but instead of a capital, a capital number, so it's (pronouncing slowly) thirty-four.
B: So this one (I point to 31, which I had written previously), what number could this one be?
A: Thirty three ... Thirty one!
B: Thirty one. Yeah. (Pause.) Could it be, do you think? Is that right? Is that thirty one?
A: Hey, now I know, because you made two threes in each one (pointing to 34 and 31) and that's a three (pointing to the 3 and 31), I remember how to do a three now, and now I know how to do thirty three!
B: Do you know how to write thirty three? Which one was this one (I point to 31)? I got confused.
A: Thirty three ... what is the capital of three, in this number (31)?
B: You said ... how much was this one (pointing to 31)?
A: What was the capital of one again?
B: Why do you call them capitals?
A: Thirty!!! ... one.
B: Why do you call them capitals?
A: Capital *letters* and capital *numbers*.
B: What are capital numbers?
A: Like, if I write a little, little number (see Figure 1),

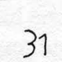


Figure 1
A little, little number

it could be a capital one, it could be a little number ... it's not really like that ... It's really like ... capital is another way ... this is one way how to write an "e", right? (see Figure 2)

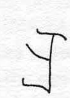


Figure 2
One way to write an "e"

B: Yeah.

A: And then this another way how to write an "e" (see Figure 3). That's capital.

Figure 3
Another way to write an "e"

B: Which one is capital?

A: This one (pointing to Figure 3).

B: So which one of these is capital (pointing to Figure 1)? Of the numbers?

A: I say this one (points to Figure 3) ... Of the numbers?

B: Yeah.

A: Thirty three. So *thirty* is a *capital* number of three. And that's the other way to write the three (pointing to the 3 in the tens place in Figure 1).

Inventions

Learning and constructing knowledge involve inventions, which are novel productions we create, using our present cognitive structures, while trying to make sense of a situation, or phenomenon. Certain features of the situation are assimilated, and as a result of the interaction between what previously existed and what is assimilated, through the reciprocal assimilation [Piaget, 1952] of the existing and the novel schemas, the learner *invents*. But the knowledge which results from these interactions is "richer than what the objects can provide by themselves" [Piaget, 1970, p. 713]. The notion of "capital numbers" which Ana creates is richer and goes beyond her prior knowledge (for example, the number for the kid's channel, her mother's age, that there is a certain order to numbers and that the "last ones" are more) and the situations and information she is faced with (the tags with numerals on them, the number one hundred, the numeral thirty-four).

Inventions must be analyzed in the context of the situation which is being assimilated, and of the problem which is being faced, in order to be understood by those who are not their creators. At the same time, children's inventions should be fostered and respected [Bamberger, 1991; Confrey, 1991; Ferreiro, 1986; Sinclair, 1982]. For example, if we considered Ana's "capital numbers" in isolation from the problem she was facing (trying to read and write numerals beyond twelve), we might think she had "confused" letters and numbers, and not that she had created a tool that would enable her to name and write multi-digit numbers.

Piaget [1970] has considered inventions as central to knowledge construction, and as characterizing "all living thought" [p. 714]. He has stated, for example, that:

> *The problem we must solve, in order to explain cognitive development, is that of invention and not of mere copying.* And neither stimulus-response generalization nor the introduction of transformational responses can explain novelty or invention. By contrast, the concepts of assimilation and accommodation and of operational structures (which are *created*, not merely discovered, as a result of the subject's activities), are oriented toward *this inventive construction which characterizes all living thought.* [Piaget, 1970, p. 713-714. Italics added.]
>
> Genuine optimism would consist of believing in the child's capacities for invention. Remember also that each time that one prematurely teaches a child something he could have discovered for himself, that child is kept from inventing it and consequently from understanding it completely. [Piaget, 1970, p. 715]

The history of mathematics also provides us with examples of the role that innovation and creation have played in the development of the field. For example, although the Babylonians of ca. 2100 B.C. used a blank space to mean zero, and eventually a special symbol during the Seleucid period (300 B.C.-0 B.C.), it was not until ca. 900 A.D. that the Hindus used a sign for zero which is comparable to the way we use zero today — whereas the Babylonian symbol only occurred between digits, the Hindu zero also appeared at the end of a number [Struik, 1987]. Mathematics, as it exists today, owes much to creative and inventive individuals throughout history. In addition, individuals' inventions form part of a coherent whole in the context of their mental structures; and they can be understood as individuals' attempts to make sense of situations. Again in Ana's case, capital numbers form part of her attempt to *write and name* numbers beyond twelve.

We are constantly in contact with different kinds of conventions: conventions in reading, writing, mathematics, music, science. At some point in history, we can think of a convention as being someone else's invention: "a unique achievement ... a form of organization within a domain that has never before been accomplished in quite the same way" [Feldman, 1994, p. 11]. This invention developed into a convention once its use became widespread because of its utility, because it facilitated tasks in some way. Mathematical conventions, for example, make it easier to keep track of things, to carry out computations, and to deal with large numbers.[5] To the learner who is faced with having to use certain conventions, without being given a chance to make sense of them, conventions seem totally arbitrary. Historically, however, there is a reason for their adoption.

Discussion

During our conversations, Ana was faced with trying to read and write numerals beyond twelve. She had a clock to help her with numbers from one to twelve, but she had no "ready-made" tool to help her beyond that. She thus constructed a tool (capital numbers), which allowed her to offer a reading of numerals which was conventional (she read 48 — forty eight — and 46 — forty six). This tool was useful to her because it helped her to read and write multi-digit numerals, to make sense of conventional numerical notations, and to find a pattern in the way that numbers are written. How did she construct this tool?

First, in capital numbers, Ana co-ordinated knowledge from apparently distinct realms: language and mathematics.[6] From language: the existence of capital letters and some of their characteristics (that they precede certain words, that they are important in reading, and that they are "another way" of writing letters). She then co-ordinated this knowledge with knowledge from mathematics: that the same digits have different names according to the place they occupy, that there is certain order to numbers, that "in the real world" there is a "given" way to read numbers. She transformed the knowledge as it existed before, and co-ordinated it with the problematic that she was faced with (*how can I read those notations*?), in order to solve it.

As a result of the activities of comparing, transforming, and co-ordinating, and of the interaction between her invention and the conventions, Ana made up her own theory regarding capital numbers. In her theory, there is a pattern to numerical notation: each digit has a capital number which corresponds to it and there is a linguistic connection between the digits and their capitals.

This same pattern and construction will, probably, lead her to further learning and knowledge construction. For example, she states that she "does not know" which are the capitals of one, two, and five. The linguistic relationship between these numbers and their "capitals" is not directly obvious: the relationship between thirteen or fourteen and one, for example, is not as obvious or direct as that of four to forty or six to sixty. The cases of two and twenty, and five and fifty, are similar.[7] In the case of three, Ana again uses previous knowledge (both her mother's age and the kid's channel are thirty-four), to construct the capital for this number (thirty, whose relationship to three is not obvious either). I argue that once she finds it problematic, when her invented tool, as it presently exists, cannot help her to find the "capital numbers" for all digits, this will probably lead to cognitive conflict and to further developing her invention and understanding of the convention.

In summary, while conventions are important in learning, learners co-ordinate and assimilate [Piaget, 1970] them into an existing mental structure. They are integrated with the existing schemata and transformed (in addition, the assimilating mental structures are pivots in the integration of those conventions into a coherent whole. In the field of music, Bamberger [1991; Bamberger & Ziporyn, 1992] adopts a similar perspective, and argues that "rules", or conventions, should not be thought of as static entities that have a "life and meaning of their own" because each person has a different interpretation and use of them. What is important is for learners to develop those multiple interpretations and representations of them, and become owners of those rules:

> Each individual displays ... his or her own *version* of the rules, and this version will always differ from person to person ... The rules themselves are only interesting in that they allow for so many different ways to get them wrong. And since nobody is getting them right, the rules themselves are abstracted out of existence. [Bamberger & Ziporyn, 1992, p. 38)[8]

Both the conventions and the individual's creations play a part in the re-creation of socially accepted knowledge and in making sense of mathematical conventions. Knowledge about the conventional system, such as that of place value, is constructed through the interaction between what the individual brings into the situation (the inventions) *and* what the greater social order presents to the learner (the conventions). But the emphasis must be placed on the *importance of children's inventions* in the processes of learning and knowledge construction, for it is through their constructions and assimilatory structures that individuals will be able to make sense of that which is presented, and which is otherwise foreign: conventions.

Conclusions

"A learner is an active, intelligent subject who thinks. A subject who assimilates to understand, who *must create* to be able to assimilate, who transforms what he knows, who constructs his own knowledge to appropriate others' knowledge" [Ferreiro, 1986, p. 130. Italics added]. Inventing and creating are of utmost importance for knowledge construction. Why, then, should we reject children's inventions? Why not create the most appropriate situations for them to take place? According to Piaget [1973], "... to understand is to discover, or reconstruct by rediscovery, and such conditions *must be complied with* if in the future individuals are to be formed who are capable of production and creativity and not simply repetition" [p. 20. Italics added].

"Knowledge, then, at its origin, neither arises from objects nor from the subject, but from interactions — at first inextricable — between the subject and those objects" [Piaget, 1970, p. 704]. The evidence that Ana provides, through her inventions, supports the claim that both inventions and conventions are fundamental in the construction of mathematical knowledge, and that although conventions depend on inventions, they also provide a support for their development. Through the interactions between conventions and inventions, inventions become richer and conventions are embedded with personal meaning for the learner. Thus, conventions can become tools for making sense of mathematics, instead of being simply arbitrary symbolizations. Conventions, then, can be re-constructed by children, through the interactions and co-ordinations between what they invent and what society provides to them.

Notes

1 The author wishes to thank Analúcia Schliemann and David Wheeler for helpful comments on earlier drafts of this paper.

2 Piaget [1972] defines figurative aspects of thought as "an imitation of states taken as momentary and static", while the operative aspect "deals not with states but with transformations from one state to another" [p. 14]. Figurative aspects of thought are essentially accommodative, while operative aspects are characterized by the assimilation process [Piaget & Inhelder, 1966/1971; 1968/1973].

3 See Duckworth [1987]. The reference is to Piaget's clinical interviews, and in the case of the research being described in this paper, they are extended in time. They could also be extended by involving more than one interviewee at once.

4 My use of this expression, "in your mind", reflects Ana's own words and way of verbalizing her thought process. Whenever she wanted to explain how she knew something, she would say "I know it in my mind". See Piaget [1976] for an in-depth analysis of children's notions of thought.

5 For example, "the invention of Arabic numerals provided a powerful mnemonic system, as did the abacus, slide rule, calculator, and computer" [Kilpatrick, 1985, p. 66].

6 Historians such as Neugebauer [1945] and Struik [1987] have also made connections between these realms. They compare the alphabet and the place value system because both inventions came to replace a complex symbolism by a method easily understood by a large number of people. Positional notation is "one of the most fertile inventions of humanity", comparable to "the invention of the alphabet" [Neugebauer, 1962, p. 5].

7 This is the case for English. In Japanese and Chinese, number words are more regular and systematic: e.g., in English we say "twelve, thirteen", whereas the Japanese say "ten-two, ten-three". See Kilpatrick [1985].

premises, definitions and logical deductions and the acceptable forms of presentation of proofs are made apparent. In others, definitions and criteria for proving are either implicit or negotiated during the activity. Informal discussions with teachers in one country [the U.K.] reveal a multitude of opinions about how proof should and would be introduced and judged. Some teachers declare that they would be comfortable with an informal explanation while others would judge this to be hopelessly inadequate and would require a logical argument with each step explicitly justified and formally presented — rather like what is known as a two-column proof.

These considerations lead me to question the existence of a universal hierarchy of "proving competencies". I have argued elsewhere [see Noss & Hoyles, 1996] that hierarchies of this sort (e.g. concrete/abstract or formal/informal) are largely artifacts of methodology — if we restrict our terms of reference simply to the interaction of epistemology and psychology, and ignore the social dimension, then it is inevitable that mathematical learning will be perceived as the acquisition of context-independent knowledge within a hierarchical framework. Thus starting from such a position of epistemology/psychology locks research and its findings into a tautological loop.

There seem to be two ways out of this dilemma. One is to search for patterns of reasons for differences in student response that stretch beyond the purely cognitive — encompassing considerations of feelings, teaching, school and home. Another is to ensure that the goals for including proof in the curriculum and how these are operationalised are clarified and taken into account. Clearly proof has the purpose of verification — confirming the truth of an assertion by checking the correctness of the logic behind a mathematical argument. But at the same time, if proof simply follows conviction of truth rather than contributing to its construction and is only experienced as a demonstration of something already known to be true, it is likely to remain meaningless and purposeless in the eyes of students [see, for example, de Villiers, 1990; Tall, 1992; Hanna & Jahnke, 1993]. Hanna has argued for an alternative approach based upon what she calls explanatory proofs — proofs that are acceptable from a mathematical point of view but whose focus is on understanding rather than on syntax requirements and formal deductive methods [Hanna, 1990, p. 12]. Maybe school proofs, where the content is "given", should aim to provide insight as to why a statement is true and throw light upon the mathematical structures under study rather than seek only to verify correctness. One way to operationalise this approach that also encourages student engagement and ownership of the proving activity has been to add a social dimension to explanatory proving; that is, to insist that students explain their arguments to a peer or a teacher as well as to convince themselves of their truth. It is this sense that has been taken up in U.K. and it is to this innovation that I now turn.

Proof in the U.K. National Curriculum

In the U.K., the main response to evidence of children's poor grasp of formal proof in the 60's and 70's was the development of a process-oriented approach to proof. Following Polya [1962] many argued [for example, Bell, 1976; Mason, Burton, & Stacey, 1982; Cockcroft, 1982], that students should have opportunities to test and refine their own conjectures, thus gaining personal conviction of their truth alongside the experience of presenting generalisations and evidence of their validity.

Clearly, there are potentially considerable advantages in this approach in terms of motivation and the active involvement of students in problem solving and proving. Indeed many prominent researchers at the present time [see, for example, de Villiers, 1990] are arguing for just such a shift in emphasis, suggesting that students develop an inner compulsion to understand *why* a conjecture is true if they have first been engaged in experimental activity where they have "*seen*" it to be true. But before other countries follow this route it would be useful to learn some lessons from what has happened in the U.K. What the mathematics education reform documents failed to predict was how teachers, schools, and the curriculum would act upon and re-shape this "*process*" innovation: in fact, the deliverers of the innovation ignored just the same potential influences on student response as alluded to earlier in my description of a fictitious mathematics education research study. How will the goals and purposes of the different functions of proof be conceived and how will these functions be organised when they are systematised and arranged into a curriculum? What will be the implications of this choice of organisation? How will the changes be appropriated and moulded by teachers and students?

Answers to these questions can be sought by an analysis of the present situation in the U.K. following the imposition of the National Curriculum. The National Curriculum in Mathematics for children aged 5-16 years is organised into four attainment targets [Department for Education and Employment Education, 1995][2].

AT1 Using and applying mathematics
AT2 Number and algebra
AT3 Shape, space, and measures
AT4 Handling data

Rather oddly, proving and proof are to be found in the target named "Using and applying mathematics" (AT1). This may well have implications for their meanings but has had an immediate consequence in that almost all of the functions of proof are separated from other mathematical content[3]. Many textbooks written for the National Curriculum are now divided into sections according to attainment targets. Rather than construction, justification, and proof[4] working together as different windows on to mathematical relationships, students are expected to *use* results of theorems in, for example, "Shape, space, and measures" — Pythagoras' theorem will be stated and students asked to apply it to calculate a length of a side of a triangle — while proofs may be encountered elsewhere in AT1. What will be the status of these proofs and what is likely to be the reaction of students when they have already used the results as facts?

A second consequence of this fragmentation of the curriculum has been that work under the banner of AT1 has become strangely transformed into an "investigations curriculum" dominated by data-driven activity during which students are expected to spot patterns, talk about, and justify them. Rarely if ever are students required to think about the structures their justifications might illuminate — it is the process that counts.

The third consequence of the imposition of the National Curriculum is the division of all of its attainment targets into 8 levels of supposed increasing difficulty. In AT1, the sequence in the proving process is given below.

The Mathematics National Curriculum of England and Wales Attainment Target 1: Using and applying mathematics[5]

Level 3	Students show that they understand a general statement by finding particular examples that match it.
Level 4	They search for a pattern by trying out ideas of their own.
Level 5	They make general statements of their own, based on evidence they have produced, and give an explanation of their reasoning.
Level 6	Students are beginning to give a mathematical justification for their generalisations; they test them by checking particular cases.
Level 7	Students justify their generalisations or solutions showing some insight into the mathematical structure of the situation being investigated. They appreciate the difference between mathematical explanation and experimental evidence.
Level 8	They examine generalisations or solutions reached in an activity, commenting constructively on the reasoning and logic employed, and make further progress in the activity as a result.
Exceptional performance	Students use mathematical language and symbols effectively in presenting a convincing reasoned argument. Their reports include mathematical justifications, explaining their solutions to problems involving a number of features or variables.

First, it is worth noting that the division into levels and the stipulation of eight as the number of levels applies to all subjects in the National Curriculum. This decision was not based on any analysis of stages of progression in subject areas but rather arose from the need to impose uniformity over the whole curriculum — in order that levels could serve as a mechanism to measure and compare the achievement of students, teachers, and schools. What was not anticipated, however, were the far-reaching implications of this levelled classification on the subject disciplines themselves and on how they would be experienced by students. For example, the levels of AT1 mean that the majority of students will engage in data generation, pattern recognition, and inductive methods while only a minority, at levels 7 or 8, are expected to prove their conjectures in any formal sense. The imposition of this strictly prescribed hierarchical organisation has therefore meant that most students have little chance to gain any appreciation of the importance of logical argument in whatever form and few opportunities to engage in formal discourse requiring any linguistic precision.[6] In a nutshell, it is now official that proof is very hard and only for the most able.

Clearly, the shift in emphasis to a process-oriented perspective is an understandable attempt to move away from the meaningless routines that characterised what was largely geometrical proof in an earlier period. While some students managed to undertake the routines of Euclid correctly, far fewer understood more about geometry as a result. But in trying to remedy one problem, others have come to the surface: the meaning of "to prove" has been replaced by social argumentation (which could mean simply giving some examples); justifying is largely confined to an archaic "investigations curriculum" separated from the body of mathematics content; and proof is labelled as inaccessible to the majority.

But what are the consequences for student attitudes to and understanding of proof following this massive change in the treatment of proof? What are the consequences for student learning of a curriculum that now contrasts sharply with that adopted elsewhere — in the U.S., France, Germany, and countries on the Pacific Rim to name but a few. Some recent research by Coe & Ruthven [1994] into the proof practices of students who have followed this curriculum suggests, rather unexpectedly, that nothing appears to have changed and students remain locked in a world of empirical validation. Even more surprisingly, given the emphasis in the curriculum reforms, the researchers also report that students show little attempt to explain why rules or patterns occur, or to locate them within a wider mathematical system.

How far are these findings generalisable? As yet this is not known, but far more influential than any research study is the pervasive belief amongst influential groups in the U.K., that students' understanding of the notion of proving and proof in mathematics has deteriorated. There has been a huge outcry, mainly amongst mathematicians, engineers and scientists in our universities, complaining about the mathematical incompetence of entrants to their institutions. The argument is that the National Curriculum only pays lip-service to proof with the result that even the more able students who go on to study mathematics after 16 years have failed to grasp the essence of the subject. The debate culminated in 1995 in a publication spearheaded by the London Mathematical Society, a powerful group of mathematicians in the U.K. Points 4 and 5 of the summary of their report, known as the LMS Report, are reproduced below:

> 4. Recent changes in school mathematics may well have had advantages for some students, but they have not laid the necessary foundations to maintain the quantity and quality of mathematically competent school leavers and have greatly disadvantaged those who need to continue their mathematical training beyond school level.
>
> 5. The serious problems perceived by those in higher education are:
>
> (i) serious lack of essential technical facility — the ability to undertake numerical and algebraic calculation with fluency and accuracy;
>
> (ii) a marked decline in analytical powers when faced with simple problems requiring more than one step;
>
> (iii) a changed perception of what mathematics is — in particular of the essential place within it of precision and proof.
>
> (London Mathematical Society, 1995, p. 2)

The message of the LMS report is clear. Students now going on to tertiary education in mathematics and related subjects are deficient in ways not observed before the reforms: students have little sense of mathematics; they think it is about measuring, estimating, induction from individual cases, rather than rational scientific process. Clearly we might argue that the evidence of "decline" is not sound or that its putative causes are hard to pinpoint given the complexity of the educational process — not least the massive expansion in the university population in the U.K. But this argument is difficult to sustain in the absence of systematic evidence. In fact, the conclusions concerning proof are eminently plausible: given that there are so few definitions in the curriculum, it would hardly be surprising if students are unable to distinguish premises and then reason from these to any conclusion. But rather than pointing out what students "lack", it would seem to be more fruitful and constructive to find out what students *can* now do and understand following the reforms of the curriculum and the different functions of proof they have experienced. What is needed is a comprehensive study of students' views of proving and proof and the major influences on them. Having followed the new curriculum, what do students judge to be the nature of mathematical proof? What do they see as its purposes? Do they see proving as verifying cases or as convincing and explaining? Do they forge connections between the different functions of proof or do these functions remain fragmented and isolated? What are their teachers' views? Although we have a national curriculum, are there variations in how it is delivered and experienced and, if so, why and what are the implications for student learning?

These questions take me to a discussion of a research project, Justifying and Proving in School Mathematics, which I have been undertaking with Lulu Healy at the Institute of Education in London since 1995. In this research, we aim to answer some of these questions by surveying student views of proof and trying to explain these against a landscape of variables and influences that extends beyond a simple description of students' mathematical competencies. In the next section, I will describe some aspects of the project in more detail.

A nationwide research project

The first phase of the research has been to conduct a nationwide survey of the conceptions of justification and proof in geometry and algebra amongst 15-year-old students[7]. Our aim is to open a multiplicity of windows on to students' conceptions of proof in order to find out what they think it involves, what they choose as proofs and how they read and construct proofs. We are only mildly interested in discovering what students cannot do. Rather, we seek to describe profiles of student responses in order to identify strengths as well as weaknesses. We also want to tease out how student conceptions might have been shaped — by the curriculum, teachers, and schools. Given that it is only high-attaining students who have any acquaintance with proving in our curriculum, the sample surveyed is drawn from this group. The findings from the survey will form the basis for thinking about how we might introduce students to proof in the future — to capitalise on the outcomes of the reforms in the curriculum that are positive while seeking to reduce those that are negative. In fact, the survey is only the first phase of our project. In the second phase, following our analysis of student and teacher responses, we will design and evaluate two computer-based microworlds for introducing students to a connected approach to proving and proof.

We spent many months reviewing existing literature and discussing with teachers, advisers, and inspectors in order to come up with two survey instruments: a student and a school questionnaire. For the former, we wanted the mathematical content to be sufficiently straightforward for the proofs to be accessible, familiar and in tune with the U.K. National Curriculum, yet sufficiently challenging so there would be differentiation amongst student responses. In our questionnaire, proofs and refutations were to be presented in a variety of forms — exhaustive, visual, narrative and symbolic — and set in two domains of mathematics — arithmetic/algebra and geometry[8].

The questionnaire was pre-piloted by interviews with 68 students in 4 different schools aiming to find out how far the questions were at an appropriate level and engaging for students. Following the pre-pilot, items were removed that were too easy or modified if too hard. We also wanted to be able to make comparisons between responses in algebra and geometry, so revised the format to make its presentation in each domain completely consistent.

Simultaneously with the development of the student questionnaire, we designed a school questionnaire to obtain information about the schools — the type of school, its organisation generally and the hours spent on mathematics, the textbooks adopted and examinations entered, and specifically the school's approach to justification and proof. We also sought teacher data to provide information on their background, qualifications, their reactions to the place of proof in the National Curriculum and the approaches they adopted to proof and the proving process in the classroom.

We piloted both questionnaires with 182 students in 8 schools after which we were able to iron out any remaining ambiguities and to specify the time required to complete the survey (70 minutes) and the instructions for its administration. The questionnaires were completed between May and July 1996 by 2459 students in 94 classes from 90 schools in clusters throughout England and Wales. We had originally planned to use 75 schools but more requested to take part in our survey — a reflection we believe of the interest teachers have in this topic, their recognition of its importance, and their concern about the changes that have taken place. The questionnaires were administered by members of the project team or mathematics educators in different parts of the country who had volunteered to help us. This process ensured consistency in administration procedures and a 100% return of questionnaires. While the students answered their questionnaires, their teacher filled in parts of the student questionnaire (see later) as well as completed the school questionnaire.

Schemes for coding the questionnaires were devised and all the coding undertaken and checked during July and August 1996. We are now producing descriptive statistics based on frequency tables and simple correlations as well as modelling student responses against all our teacher and school variables using a multilevel modelling technique [see Goldstein, 1987]. The purpose of this paper is not to report the findings of this statistical analysis but rather to provide a flavour of how students in the U.K. see proof through the presentation of a selected sample of questions together with some student responses.

Windows on students' approaches to proof

The first question of the student questionnaire asks students to write down everything they know about proof in mathematics. A rather typical answer is given below:

> "All that I know about proof is that when you get an answer in an investigation you may need some evidence to back it up and that is when it is proof. You have to prove that an equation always works.
>
> Another student wrote:
>
> "all I know is the proof in mathematics is that, if say you are doing an investigation, and you find a rule, you must prove that the rule works. So proof is having evidence to back up and justify something".

These responses clearly echo our curriculum structure. It is only a special type of mathematical activity — the "investigation" — that requires proof, where this means the provision of some sort of evidence.

Following this open-ended question, the questionnaire is divided into two sections, the first concerned with algebra and the second with geometry. The first question of each section is in a multiple-choice format, as illustrated in Figures 1 and 2.

The purpose of having a multiple-choice question at the beginning of each section is to introduce students who may not be acquainted with the meaning of "to prove" to a range of possible meanings — remember that our students are not introduced to definitions, nor generally required to produce logical deductions in mathematics[9]. Almost all the student responses used as options for these questions were derived from our pre-pilot and pilot studies or from school textbooks, so we could be fairly sure that some at least would be familiar. These questions (and others with a similar multiple-choice form) were designed to help us ascertain what students *recognised* as a proof. These responses could then be compared and contrasted with what students actually *produced* as proofs later in the questionnaire. Clearly these two processes are related but not identical — constructed proofs require specific knowledge to be accessible. As well as presenting different types of proof (here empirical or analytic), the choices in the questions ranged over different forms to enable us to analyse how far students might be influenced by the form as well as the content of a proof. The "proof types" shown in Figures 1 and 2 can be categorised as: empirical, enactive, narrative, visual[10] or formal, with two examples of formal proof, one correct and one incorrect.

A1. Arthur, Bonnie, Ceri, Duncan and Eric were trying to prove whether the following statement is true or false:

When you add any 2 even numbers, your answer is always even.

Arthur's answer
a is any whole number
b is any whole number
$2a$ and $2b$ are any two even numbers
$2a + 2b = 2(a + b)$
So Arthur says it's true.

Bonnie's answer
$2 + 2 = 4 \quad 4 + 2 = 6$
$2 + 4 = 6 \quad 4 + 4 = 8$
$2 + 6 = 8 \quad 4 + 6 = 10$
So Bonnie says it's true.

Ceri's answer
Even numbers are numbers that can be divided by 2. When you add numbers with a common factor, 2 in this case, the answer will have the same common factor.
So Ceri says it's true.

Duncan's answer
Even numbers end in 0 2 4 6 or 8. When you add any two of these the answer will still end in 0 2 4 6 or 8.
So Duncan says it's true.

Eric's answer
Let x = any answer whole number, y = any whole number
$x + y = z$
$z - x = y$
$z - y = x$
$z + z - (x + y) = x + y + 2z$
So Eric says it's true.

From the above answers , choose **one** which would be closest to what you would do if you were asked to answer this question.

From the above answers, choose the **one** to which your teacher would give the best mark.

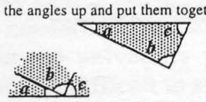
Figure 1
The first algebra question

G1. Amanda, Barry, Cynthia, Dylan, and Ewan were trying to prove whether the following statement is true or false:

When you add the interior angles of any triangle, your answer is always 180°.

Amanda's answer

I tore the angles up and put them together.

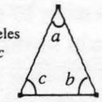


It came to a straight line which is 180°. I tried for an equilateral and an isosceles as well and the same thing happened.

So Amanda says it's true.

Barry's answer

I drew an isosceles triangle, with c equal to 65°.

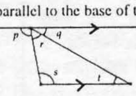


Statements	Reasons
$a = 180° - 2c$	Base angles in isosceles triangle equal
$a = 50°$	$180° - 130°$
$b = 65°$	$180° - (a + c)$
$c = b$	Base angles in isosceles triangle equal
$\therefore a + b + c = 180°$	

So Barry says it's true.

Cynthia's answer

I drew a line parallel to the base of the triangle



Statements	Reasons
$p = s$	Alternate angles between two parallel lines are equal
$q = t$	Alternate angles between two parallel lines are equal
$p + q + r = 180°$	Angles on a straight line
$\therefore s + t + r = 180°$	

So Cynthia says it's true.

Dylan's answer

I measured the angles of all sorts of triangles accurately and made a table.

a	*b*	*c*	*total*
110	34	36	180
95	43	42	180
35	72	73	180
10	27	143	180

They all added up to 180°.

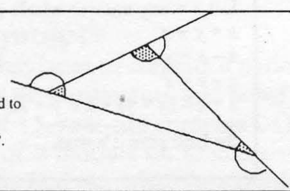
So Dylan says it's true.

Ewan's Answer

If you walk all the way around the edge of the triangle, you end up facing the way you began. You must have turned a total of 360°.

You can see that each exterior angle when added to the interior angle must give 180° because they make a straight line. This makes a total of 540°.
540° - 360° = 180°.

So Ewan says it's true.



From the above answers, choose **one** which would be closest to what you would do if you were asked to answer this question.

From the above answers, choose the **one** to which your teacher would give the best mark.

Figure 2
The first geometry question

We are seeking to investigate the influence of the teacher in various ways — through their responses to the school questionnaire but also through the eyes of the student in the student questionnaire. In the last part of each multiple-choice question, the student is required to choose the proof to which they think their teacher would give the best mark. Responses here will help us to see how students interpret what is rewarded by their teacher. The teachers are also asked to complete these same questions — to write down what *they* would choose as a proof, as well as what they think their students will choose as the one given the best mark. The analysis to date reveals a picture that is by no means simple, although in both domains, formal presentation (correct or incorrect) is highly favoured for the best mark, while narrative in algebra is the favourite for individual choice. What is interesting too is the sizeable minority of students whose personal choice bears no resemblance to the one they believe will receive the best mark (only 21% overall made the same choice for both[11]) and the small group who choose, for the latter, a formal proof that is incorrect!

Following each multiple-choice question in both algebra and geometry are questions seeking to find out how students evaluate each of the choices previously presented. Do they think it is correct? Do they believe that the proof holds for all cases or simply for a specific case or cases? Do they judge it to be explanatory or convincing? An example of the format used as it applies to Bonnie's "proof" is shown in Figure 3 below.

Bonnie's answer

2 + 2 = 4	4 + 2 = 6
2 + 4 = 6	4 + 4 = 8
2 + 6 = 8	4 + 6 = 10

So Bonnie says it's true.

Bonnie's answer:			
Has a **mistake** in it	1	2	3
Shows that the statement is **always true**	1	2	3
Only shows that the statement is true for **some even numbers**	1	2	3
Shows you **why** the statement is true	1	2	3
Is an easy way to **explain** to someone in your class who is unsure	1	2	3

Figure 3
Student evaluations of Bonnie's answer

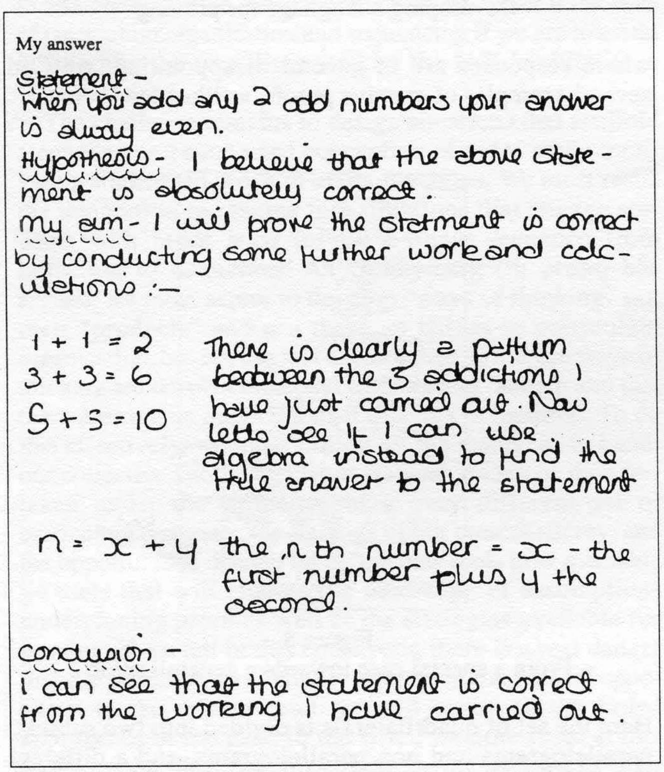
By analysing all the responses to this question, we will find out if students are convinced by a list of empirical examples. Do they judge that these examples help them to explain the result? Do they recognise that "the proof" only shows the conjecture is true for the given examples even though they might have chosen it as their response or as the one to which the teacher would give the best mark[12]? Does it make a difference if the proof evaluated was the one chosen by the student? Preliminary analysis in algebra suggests that students tend to choose for themselves "proofs" that they evaluate as general *and* explanatory, while the proofs they think will be assigned the best mark are evaluated as general but not necessarily explanatory.

The second multiple-choice question in each section of the survey concerns an incorrect conjecture which is proved by some choices and refuted by others. Again, preliminary results show the same strong preference for formal language in the choices for the best mark, even in these circumstances where a simple empirical counter-example would suffice. It seems that despite our investigations culture, the students have picked up what the game *should* be!

A rather different picture emerges from students' own proofs where the influence of the investigations curriculum is again very evident, as illustrated in Fig. 4 below. It must be emphasised that great care was taken to choose proofs that would be familiar to or at least accessible to most students who had followed our curriculum.

A4. Prove whether the following statement is true or false. Write down your answer in the way that would get you the best mark you can.

When you add any 2 odd numbers, your answer is always even.


My answer

Statement –
when you add any 2 odd numbers your answer is alway even.
Hypothesis – I believe that the above statement is absolutely correct.
my aim – I will prove the statment is correct by conducting some further work and calculations :–

1 + 1 = 2
3 + 3 = 6
5 + 5 = 10

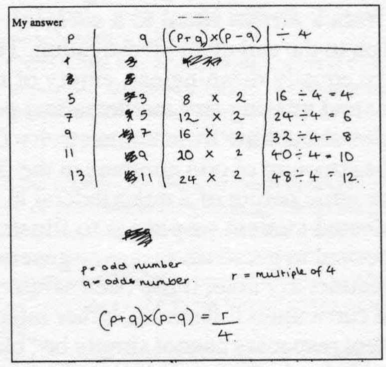
There is clearly a pattern between the 3 addictions I have just carried out. Now letts see If I can use algebra instead to find the true answer to the statement

n = x + 4 the n th number = x – the first number plus 4 the second.

Conclusion –
I can see that the statement is correct from the working I have carried out.

Figure 4
An "investigations" response to adding 2 odd numbers

Note the careful presentation of data in tabular form — the heart of an investigation —, the spotting of a pattern and the adding-on of letters to add status to "the proof". Similar phenomena are illustrated by another student's response to the second, rather harder, algebra proof construction:


My answer

p	q	(p+q)×(p−q)	÷ 4
~~1~~	~~3~~		
~~3~~	~~5~~		
5	3	8 × 2	16 ÷ 4 = 4
7	5	12 × 2	24 ÷ 4 = 6
9	7	16 × 2	32 ÷ 4 = 8
11	9	20 × 2	40 ÷ 4 = 10
13	11	24 × 2	48 ÷ 4 = 12.

p = odd number
q = odd number.
r = multiple of 4

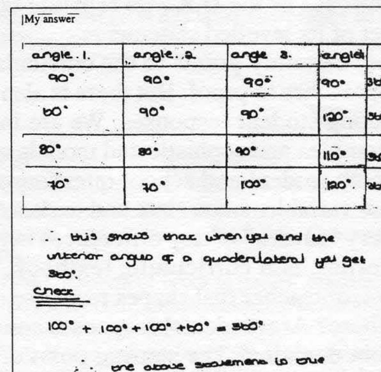
$(p+q)\times(p-q) = \frac{r}{4}$

Figure 5
An "Investigations" response to a harder algebra question

Even in geometry, rarely the site for a school "investigation", the discourse of investigations is evident in the form of many of the student responses and the explanations given, as illustrated in Figure 6.

G4. Prove whether the following statement is true or false. Write your answer in a way that would get you the best mark you can.

If you add the interior angles of any quadrilateral, your answer is always 360°.


My answer

angle. 1.	angle. 2	angle 3.	angle4	
90°	90°	90°	90°	360
60°	90°	90°	120°	360
80°	80°	90°	110°	360
70°	70°	100°	120°.	360

∴ this shows that when you and the interior angles of a quaderlateral you get 360°.
Check
100° + 100° + 100° + 60° = 360°
∴ the above statement is true

Figure 6
An "investigations" response to a geometry question

Here data are invented to fit the given pattern. There is even a set of data given special status to fulfil the investigation's requirement of providing a check!

What is evident from these responses is that the students connect the requirement to prove with the investigations part of the curriculum where they have learned a format and a language of presentation. They have appropriated some structures to help them make sense of the situation and to assist in developing a language for proof. But, the limitations of this scaffolding are very apparent. Students appear to be imposing a "type" of proof on every question regardless of whether it is appropriate; for example, a proof must involve data. Additionally, all too easily students seem to have shifted their notion of proving from one ritual to another — from a *formal* ritual to a *social* ritual — something added on to the end of an investigation. This new ritual is likely to be equally meaningless, empty of mathematical illumination, and missing any mathematical point. My student examples show how little connectivity there appears to be with the structure of odd numbers in the first two cases or to the geometric nature of a quadrilateral in the last one.

I have selected student responses to illustrate my point and it is important to guard against over-generalisation. Yet, despite this cautionary note, it has been salutary to trace the extent of the curriculum influences, either intended or unintended. Student responses cannot simply be "blamed" on the student. Their behaviours cannot be ascribed to properties of age, ability, or even individual interactions with mathematics. The meanings they have appropriated for proof have been shaped and modified by the way the curriculum has been organised. For example in our survey, responses in geometry are very different — and much worse from a mathematical perspective — from those in algebra. This finding is hardly surprising if one is aware of the almost complete disappearance of geometrical reasoning in the curriculum. Nonetheless, it casts doubt on how far proof can be considered as a unitary mathematical "object" with its own hierachy separated from any domain of application. Many mathematics educators have shown how we must take seriously the influence of the teacher — and our teacher data sheds light on this. But, surely, we have now to look for interpretations which also take account of curriculum organisation; in the case of the U.K., its separate targets and the straightjacket of its levelled statements.

Our survey certainly points to the curricular shaping of students' approaches to proof. But there is also considerable variation among student responses. We are in the midst of generating complex and sophisticated models using the variables from both student and school questionnaires, to tease out how these variables interrelate and to describe the range of contributory "causes" of any differences between student response profiles. Is it curriculum, textbook, examination board, school, or teacher that shapes response or is it a combination of these? Are students' responses consistent across or even within domains? The starting point of our research was the belief that proving in mathematics need not be restricted to "the exceptional" and that the organisation of the U.K. National Curriculum seriously underestimates the potential of our students. Responses to the survey are proving to be rather promising in this respect. Alongside the ritualised responses described above and the all too numerous solutions that simply resort to empirical examples, there are some fascinating and ingenious proofs which provide a pointer to the wealth of resources that might be tapped and built upon in the process of building a proving culture — of starting conversations with students about proof.

Take, for example, the following proof of the first algebra example which combines a visual approach with a narrative to indicate the generality of the argument.

There is a coherence in the explanation apparent in the diagram, the written text as well as the language describing the generalisation, which stems from the mathematics — here from the structure of odd numbers. Even in geometry

A4. Prove whether the following statement is true or false. Write down your answer in a way that would get you the best mark you can.

If you add any 2 odd numbers, your answer is always even.

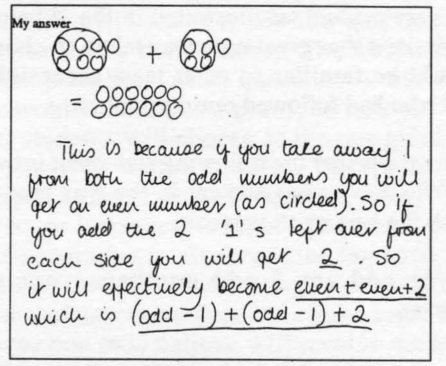


Figure 7
Developing a language for proving

where responses are in general disappointing, we find several examples of creative proofs, as illustrated below.

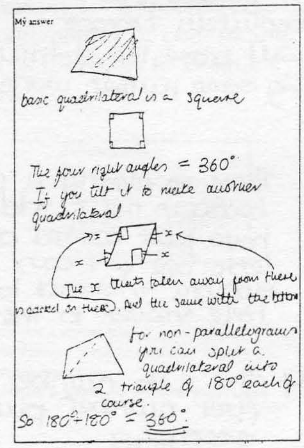


Figure 8
From a special case towards a generalisation

Here the set of quadrilaterals is divided into two subsets, parallelograms and non-parallelograms, and a different proof applied to each. The first "proof" hints at the invariance of the angle sum of parallelograms under carefully contrived parallel deformations, while the second (the obvious one — note the "of course") takes care of all other quadrilaterals by applying a known fact about the angle sum of a triangles. Both of these responses are likely to be influenced by school factors which we intend to investigate through follow-up interviews with teachers in the schools concerned. It may be, for example, that the geometry proof shown above has roots in prior experimentation with dynamic geometry software. Note how both of these proofs point to an approach to proof where the starting point is *not data* but rather a specific and special case where the conjecture is known to be true and from which the road to the general case is suggested — in the former case by language and image and the latter by means of "adding a bit in one place" and "taking the same from another". This calls into question the whole notion that students' development of mathematical justification *has* to proceed from inductive to deductive processes and maybe suggests a different route for proof in school. This will one of the many conjectures generated by the survey that we will investigate to explore in Phase 2 of our project.

Conclusions

We have a long way to go to unpick all the factors that together underpin student conceptions of proof in the new scenario we now face in the U.K. It is almost certain that many of the influences on student responses, interactions in classrooms and institutions were not anticipated. Nonetheless, we can learn from this experience. The main message of this paper is that mathematics educators can no longer afford simply to focus on student and teacher if we are to understand the teaching and learning of proof and if we seek to influence practice. We cannot ignore the wider influences of curriculum organisation and sequencing if we are to avoid falling into the trap illustrated by the U.K.'s ubiquitous "investigation".

The challenge remains to design situations that scaffold a coherent and connected conception of proof while motivating students to prove in *all* its functions. We must resist the temptation to assume that situations that engage students with proof *must* follow a linear sequence from induction to deduction. As Goldenberg (in press) has argued, we must aspire to develop "ways of thinking" not their "products" and use these as guides to curriculum organisation, but not neglect to recognise how these ways of thinking are deeply connected with content domain and that there are serious implications if this link is ruptured. To do this effectively, we must exploit all the resources to hand: our collective knowledge from research, much of it undertaken under the influence of a very different set of curriculum restraints; the findings of our present survey; and the opportunities opened up by the new tools now available — tools that will change the landscape of assumptions underpinning proof as well as the strategies available for proving. If we fail in this endeavour, there is a real danger that the pendulum will simply reverse in the face of opposition to the reforms and we will return to the failed approaches of the past. In the U.K., we now stand at this turning point. Will the curriculum "swing backwards"? Or will we seize the opportunity opened up by all the discussion around proof to find out how to shape student conceptions in new directions, so they come to see proof as generative not merely descriptive, negotiable but also mathematical.

Acknowledgement

I would like to thank the Canadian Mathematics Education Study Group for inviting me to give a talk at their annual meeting in Halifax, May 1996, which formed the basis of this paper. I thank Lulu Healy and Richard Noss for their helpful comments on an earlier draft of the paper and also wish to acknowledge that many of the ideas and design issues concerning the research project described here were developed together with Lulu Healy.

Notes

1 There have, for example, been special editions on proof in many of the major journals in mathematics education.

2 The National Curriculum has been through several changes, each time with a different number of attainment targets. Nonetheless, the basis of its organisation has remained the same. The structure described here was put in place in 1995 where some attainment targets only appear for children of certain ages.

3 Teasing out all the reasons for this compartmentalisation would make a fascinating story of the demise of geometry intertwined with political intrigue — but, unfortunately, this is beyond the scope of this paper.

4 In the remainder of this paper, I will take justifying to mean an explanation which convinces oneself and is communicated to others. I will leave the term "proving" to convey the more formal sense of logical argument based on premises.

5 Children age 11-14 years should be within the range of Levels 3 to 7. Level 8 is available for very able pupils.

6 A similar trend in North American has been noted by [Hanna, 1995] who has argued that the gradual decline of the position of proof in school mathematics and its relegation to heuristics can be attributed partly to the "process orientation of much of the reforms in mathematics education since 1960s". She also suggests that another contributing factor is the persuasiveness of constructivism — or at least the way it is operationalised in the classroom. Whilst I have some sympathy with Hanna's remarks, this paper suggests that she underestimates the effect of the curriculum.

7 The project is funded by the Economic and Social Research Council, Grant number R00236178.

8 We organised a small invited international conference on proof in order to share frameworks and present our first ideas for the student questionnaire [Healy and Hoyles, 1995].

9 Before the students started to respond to the questionnaire, it was pointed out to them that for this type of question several options could be "correct".

10 The visual option appears on another page of the questionnaire and is not shown here.

11 In fact, there is a significant relationship between these two choices but the correlation is low.

12 Preliminary analysis suggests this latter choice may be subject to interesting gender differences.

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It is sometimes unreasonably required by persons who do not even themselves attend to such a condition, that experimental information should be submitted without any theory to the reader or scholar, who is himself to form his conclusions as he may list. Surely the mere inspection of a subject can profit us but little. Every act of seeing leads to consideration, consideration to reflection, reflection to combination, and thus it may be said that in every attentive look on nature we already theorise. But in order to guard against the possible abuse of this abstract view, in order that the practical deductions we look to should be really useful, we should theorise without forgetting that we are so doing, we should theorise with mental self-possession, and, to use a bold word, with irony.

J. W. von Goethe
