The Multiplication Table: to be memorized or mastered?

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For more than a hundred years the multiplication table has had an important claim on what all children should learn Its conquest used to be considered almost a hallmark of successful elementary education, a kind of entrance certificate either to secondary school or to the world of work.

In the early 1980's there is again an increasing public voice that no matter what other activities, old or new, are experienced by students, the "times table" must continue to hold its place in a basic curriculum

However, its relevance, its content and its power can now be viewed quite differently Rather than the learning of the table being considered as an end, either to elementary mathematics or to the mastery of the multiplication facts themselves, it can — and indeed must — be seen as a source of discovery and wonder, as a compact amalgamation of patterns and inter-relationships which provide departure points to many other arithmetical and mathematical topics across the twelve years of school None of this need interfere with the learning by heart of the traditional facts. On the contrary, so much more practice and use of them is thereby encouraged that they become part of everyone's fluent vocabulary as they knit together mathematical insights which by tradition have been striven for amid separate chunks of subject matter

One reason why the times table seemed to go out of fashion during the 1960's, the era of new math, was because of a growing conviction that learning based on rote drill was stultifying With revelations, however, of what can be gleaned from and done with the traditional table it can be seen that simple memorization is clearly not the only way to master it

The recommendation here is that the multiplication table should be viewed, apparently for the first time by most people, as a dynamic synergetic combination of patterns, a veritable repository of mathematical relationships waiting as it were to gush forth from kindergarten through secondary grades. The table has been with us for a century, under our very noses, but stress has invariably been on "learning it", not studying it for all its riches.

Let us get some preliminaries out of the way first Assume that second graders, for example, have in various ways spread over a couple of years or as a concentrated unit, got to know a few of the multiplication facts in traditional form. Maybe there has been plenty of serious play under teacher's guidance, with manipulative materials so constructed, used and talked about that the students are familiar with statements like: "three times two is six" with its writing, $3 \times 2 = 6$ It is not necessary that every student has every one of these "facts", only that a collection of them can be made in the class. Those not known will be filled in as the study proceeds

Moreover, let us assume a certain vocabulary within the

X	0	1	2	3	4	5	6	7	8	9	10
0	0x0	0x1	0x2	0x3	0x4	0x5	0x6	0x7	0x8	0x9	0x10
1	1x0	1x1	1x2	1x3	1x4	1x5	1x6	1x7	1x8	1x9	1x10
2	2x0	2x1	2x2	2x3	2x4	2x5	2x6	2x7	2x8	2x9	2x10
3	3x0	3x1	3x2	3x3	3x4	3x5	3x6	3x7	3x8	3x9	3x10
4	4x0	4x1	4x2	4x3	4x4	4x5	4x6	4x7	4x8	4x9	4x10
5	5x0	5x1	5x2	5x3	5x4	5x5	5x6	5x7	5x8	5x9	5x10
6	6x0	6x1	6x2	6x3	6x4	6x5	6x6	6x7	6x8	6x9	6x10
7	7x0	7x1	7x2	7x3	7x4	7x5	7x6	7x7	7x8	7x9	7x10
8	8x0	8x1	8x2	8x3	8x4	8x5	8x6	8x7	8x8	8x9	8x10
9	9x0	9x1	9x2	9x3	9x4	9x5	9x6	9x7	9x8	9x9	9x10
10	10x0	10x1	10x2	10x3	10x4	10x5	10x6	10x7	10x8	10x9	10x10

Figure 1

X	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10
2	0	2	4	6	8	10	12	14	16	18	20
3	0	3	6	9	12	15	18	21	24	27	30
4	0	4	8	12	16	20	24	28	32	36	40
5	0	5	10	15	20	25	30	35	40	45	50
6	0	6	12	18	24	30	36	42	48	54	60
7	0	7	14	21	28	35	42	49	56	63	70
8	0	8	16	24	32	40	48	56	64	72	80
9	0	9	18	27	36	45	54	63	72	81	90
10	0	10	20	30	40	50	60	70	80	90	100

Figure 2

class, although other teachers may well use alternatives to the preferences used here 3×2 is called a "number name". So is 6, but 6 is the "standard equivalent" for 3×2. Conversely, 3×2 is one of the non-standards for 6 and a "product". $2 + 4$ is a non-standard "sum", $7 - 1$ a "difference", $12 \div 2$ a "division". There is, of course, an infinity of non-standards corresponding to every number name. Some are mixtures of products, divisions, sums and differences, like $(8 \times 2 \div 4) + 3 - 1$.

Two large display charts are suggested as important materials to aid the coming study, with a plentiful supply of dittoed copies for individual use. The first is a "product chart" (Figure 1). On the first occasion any student constructs his own the only guide needed is that teacher draws the headings on the chalkboard with a few entries already written in. Any student can complete the table, no further information being necessary.

The second table can be developed by accumulating those entries known confidently by at least one member of a class or, if the teacher thinks it appropriate, by work there and then which enables everyone to produce the standard names necessary for completion as far as 10×10 (Figure 2). This, of course, is the traditional multiplication or "times" table commonly reproduced on the backs of exercise notebooks, the publicly perceived objective of what is to be learnt.

It is not suggested that the study of the two tables be concentrated every day for a set duration. On the contrary it can stretch over years, depending on many circumstances such as students' ages, their knowledge, their previous experience, the teachers' wishes and the other current mathematics being covered. There are many variations too, in the details of what is actually studied. What follows here is by no means necessarily preferable to other equivalent studies in respect to order, language, form, emphasis, inclusion or depth. It would be absurd, for instance, to think that thorough command of the rational number system, in the intuitive fraction form of the primary grades, or in the more algebraic sophistication of the secondary school, should or could come solely from a study of the multiplication table. The point, however, is being made that it can certainly begin thus, or be seen to have close connections.

Speaking personally, I find it interesting that the multiplication table, with its common pressure "to be learnt", has peppered all my years as a student and mathematics teacher, yet it was not until recently that I became aware of the treasures I had missed, grouped together in one such cache! It brought no new mathematics to me, but the surprise that there *was* so much there within the confines of the old, obvious, culturally accepted dull tradition — that *did* teach me a lesson.

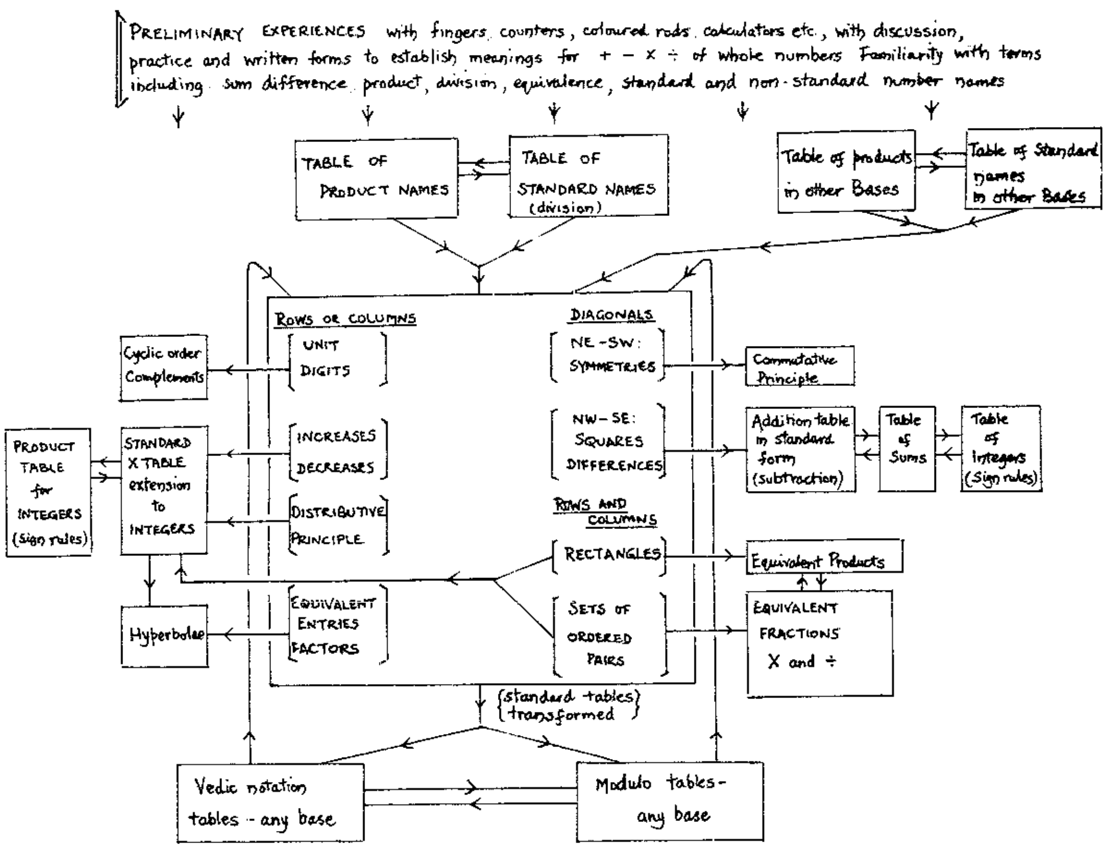


Figure 3

Let's get on then and open the treasure chest! One may do it by simply issuing an invitation to the students to "look for patterns" in the two tables, the product version and the standard form.

Here I resort to a listing, not altogether an attractive form for easy reading, but one which has the advantage of quick reference to what an individual wants. Figure 3 shows an even more concentrated lay-out of possibilities although there are many other equivalent diagrams, none of which is likely to show all the pattern possibilities nor the sum total of the inter-relationships which underlie those patterns

1 Students, even young ones, can spot patterns in various rows (or columns) of the standard table:

a (i) The right hand digits of all outside entries are zero

(ii) The right hand digits in column 2 are 0,2,4,6,8,0,2 ... and in column 8 they are the same but in the reverse cycle

(iii) The set of right hand digits in columns 4 and 6 are the same; as are those of columns 1 and 9; and those of 3 and 7 Those of column (or row) 5 are 0,5,0,5

(*Note* 2 and 8, 4 and 6, 1 and 9, 3 and 7 are complements of 10)

b A constant increase from one entry to the next is seen in every row (except the 0 row) Entries in row 2 "go up by 2's", in row 5 "by 5's", etc

Therefore an entry next after the last written, even if its standard is not known, can always be stated in non-standard form; e.g. row 7: after 70 the next entry must be 70 + 7, then 70 + 7 + 7. and so on, *ad infinitum*.

c Reverse the order of consideration of the entries in a row (or column) and the successive numbers decrease constantly. "What would be the entry after 0, in row 3?" "Zero minus 3" Then comes "Zero minus three minus three" or "zero minus two threes" or "zero minus six." These can be written 0 − 3, 0 − 6 ..., etc. and the standard name given, "negative 3" written $^{-}3$

From such a straightforward extension of every row and every column, an extended multiplication table is formed, two quarters showing negative entries, with a repetition of the original in the "Northwest"

d When students have met the integers a corresponding product table can be written. From it extracts can be made and practiced for familiarity; for example, $^{-}2 \times {}^{-}3 = 6$, $4 \times {}^{-}2 = {}^{-}8$, $^{-}5 \times 2 = {}^{-}10$ The adjective "positive" can then be used to modify and clearly identify those numbers not negative, the usual rule of signs emerging not solely because that's what mathematicians have decided, but from one's own conclusion from the extended table

2 The multiplication table is also a division table. By first selecting an entry, noting its column heading as the divisor, the standard equivalent is the corresponding row heading. This may motivate practice on important transformations: $^{-}a \times {}^{-}b = {}^{+}c \rightarrow {}^{+}c \div {}^{-}b = {}^{-}a$.

(*Note* division by 0 has to be resolved in its massive ambiguity).

3 Diagonals can be examined in two directions, Northeast to Southwest and Northwest to Southeast

a The entries in any NE-SW diagonal are symmetrically arranged This is a consequence of the commutative principle.

b Every adjacent pair in a NW-SE diagonal has a difference, which differences can be extracted and written separately to give another table, discovered to be the addition table (of course!) Since it has entries in standard form this may prompt students to produce the corresponding table of sums.

4 a In its turn the addition table can be extended to negative integers, as previously done with the multiplication table

b Each addition table, either in standard or in sum form, lends itself to subtraction interpretation. It can also be looked at from the point of view of every pattern that was revealed in the multiplication table, as well as for new patterns appropriate only to itself. In particular the sums and differences of integers can be investigated.

5 If any rectangle with sides parallel to the rows and columns is imposed on the multiplication table (the regular or the extended version), products of the entries in opposite corners are found to be equivalent.

Referring to the product table this is seen to follow from the way the table is set up. The rectangle suggested by the four entries determined by columns a and b, and rows c and d, gives the products $a \times c$ and $b \times d$ from one pair of its corners, with $b \times c$ and $a \times d$ from the other The products of *these* products are $(a \times c) \times (b \times d)$ and $(b \times c) \times (a \times d)$, equivalent to each other

6 a In some simple but clear way, sets of ordered pairs of entries can be selected from the table. One way is to identify those number pairs which constitute a rational number, by tapping entries in a regular table with a pointer. Beginning, say, in row 1, the pointer can tap 3, then 7, pause and then tap 6 and 14 in row 2, pause again before moving to row 3 to tap 9 and 21. Students can be invited to find others. There are plenty: (12,28), (15,35), 18,42)... (30,70).

The notion of a set of such ordered pairs can thus be engendered, to include the integers if the extended table has been met, but excluding pairs like $(n, 0)$.

b When the corresponding product table is used for the "tapping game" the form of each pair becomes more evident: (1×3, 1×7), (2×3, 2×7), (3×3, 3×7) ... and so on. Such pairs are seen as members of an infinite "class of equivalence of ordered pairs" — in this case the rational number whose standard name is "three sevenths", often written in fraction form as $\frac{3}{7}$.

c The multiplication of fractions, representing ordered pairs, can be intuitively defined in a similar way — by identification of selected entries using a pointer.

Identifying some of the entries in row 1, say, a teacher may tap the 4 and then the 9, pause slightly, tap 9 again, followed by a tap on 10, pause once more, finishing with another tap on 4 and another on 10.

Perhaps the students perceive the pattern of taps after one such demonstration. If not, a second or third can be shown until everyone sees that for any entries a, b, c (except 0) chosen in any row, the sequence is,

$a, b \quad b, c \quad a, c$

The corresponding conventional writing for this is

$4/9$ of $9/10 = 4/10$ or, generally, a/c of $b/c = a/c$

(or $a/b \times b/c = a/c$)

This, therefore, may be another surprising find for it certainly is not the traditional evolution of the "multiplication of fractions". It is not implied, let it be stressed, that one replaces the other

d Combining the pattern of 6a with the generalised product form of the equivalent ordered pairs, as in 6b, the multiplication of *any* two fractions, in either algebraic sense or form, can be reached. For in tapping six times in one row the pointer can substitute another pair for any of those originally chosen.

Instead of $a, b \quad b, c \quad c, a$ in row 1, for example, the new choices can be $m \times a$, $m \times b$ instead of a, b; (nb, nc) for (b, c); and (pa, pc) for (a, c)

The reader should practise with the table until s/he sees how the pattern of taps yields:

$$ma/mb \times nb/nc = pa/pc$$

and generally:

$$h/m \times p/q = hp/mp \times pm/qm$$
$$= hp/qm$$

because $mp = pm$.

e The division of fractions can also, of course, be attained by a "tapping game" One version is to use the opposite pattern to that for multiplication, namely $c, a \quad c, b \quad b, a$ This leads to $c/a \div c/b = b/a$ with generalisations as previously.

(*Note*. All the patterns alluded to from 6b through 6e may also be applied to an extended integers table, with appropriate restrictions in regard to the second term of any ordered pair not being zero.)

7 "Where in a multiplication table are all the entries 20?" is a question which can be applied to every other whole number, though for many choices the table headings and entries can well be extended "eastward" from 10, 11, 12, 13 . . up to 40 or even 100 Many more locations can then be found for some entries

If interpolations are considered corresponding to *fractional* headings, there are even more entries for any given number (for 20, for instance: $\frac{1}{2} \times 40$, $1\frac{1}{2} \times 13\frac{1}{3}$, $2\frac{1}{2} \times 8$ . .).

In this way a student may appreciate that there is an infinity of positions in the table for every given entry. Some of the sets of positions can be suggested by the drawing of a curved line through the approximate point position of each entry. (Figure 4) This curve, a hyperbola, has an image in the NW quarter of the extended table Many pairs of curves may be drawn, asymptotes discussed and the equation of a hyperbola: xy = constant, becomes compellingly sensible

8 Every other base of enumeration can lead to its own multiplication tables, as illustrated so far here in the common base 10

Each table developed in another base can consequently be examined for patterns, just as we have done for rows and columns, unit digits, integers, in the corresponding addition table The same algebraic structures for rational numbers and their multiplication and division are seen to hold good for every such table, whatever its base

Most of the other patterns also will be relevant to the tables of alternative bases, the main exceptions being those which depend upon patterns of the names and not the numbers Each base table has a different set of permissible digits in the compilation of its number names, some tables excluding digits seen in other tables.

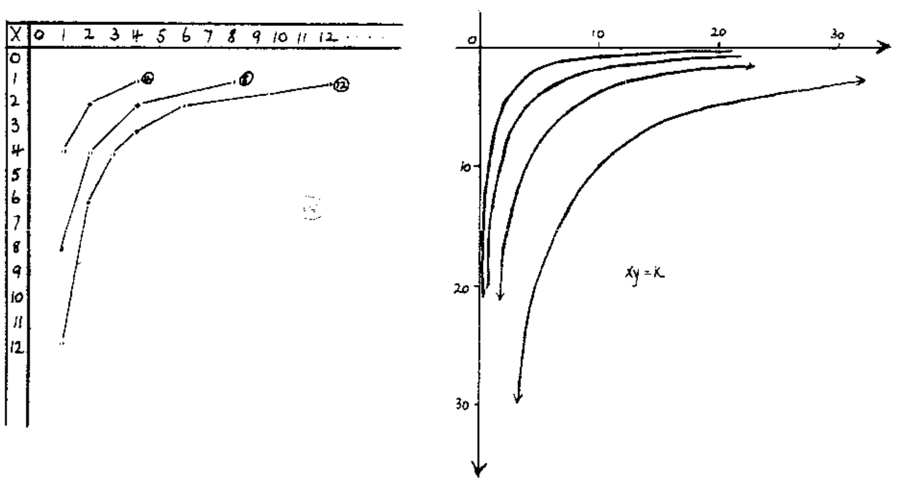
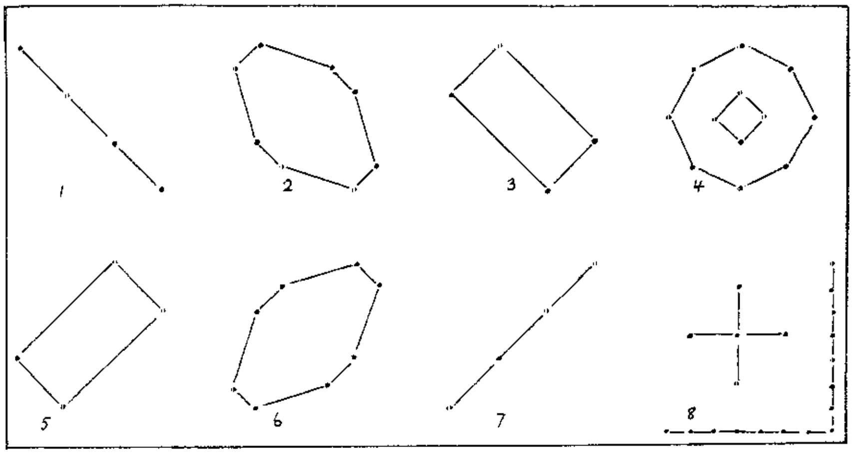


Figure 4



Locations of each digit in Vedic transformation of Multiplication Table Base IX

Figure 5

9 The so-called Vedic notation is obtained from the sum of the digits of a standard number name. For example, in the common base 10 the entry for 12×13, or 156 would produce $1+5+6$ or 12, with $1+2$ or 3 finally as its Vedic name.

Any multiplication table in standard form investigated so far, can therefore be transformed into a new Vedic table by writing every entry in the new notation. It can then be studied for patterns involving the location of specific digits (Figure 5) for symmetry, repetitions and extensions of the table in all four directions.

10 a Finally a modulo transformation can be used to change the entries of a given table. Each regular entry is divided by any chosen whole number, the "remainder" becoming the new entry. Thus to any regular table there are many transformations available, one for every divisor!

Once more such a newly evolved table may be examined in the light of every previous pattern: its rows, diagonals, symmetries... and so on. Perhaps some new patterns appear with this transformation, not applicable to the standard table.

b Having evolved Vedic transformation tables in any base, comparisons can be made with modulo tables.

For instance there are fascinating similarities between the Base X Vedic Table and the Base X Remainder Table with 9 as the divisor;

— between Base IX Vedic Table and Base X Remainder Table, divisor 8

— between any base, Vedic Table and Base X Remainder Table, when the divisor is $(10 - 1)$ in the base used for the Vedic.

The traditional multiplication table can provide, therefore, entry points into many viable studies using an initial situation known to every teacher and valuable for every student; challenges and patterns for all tastes, interests and grades, to be tackled lightly or penetrated deeply; connections to many curriculum topics traditionally considered isolated from each other; an algebraic emphasis for most patterns wherein generalities can be the stress though not necessarily couched in traditional language; strategies which spill over into other mathematical work — all this with the one extra inevitable consequence: the familiar use of the tables leaves everyone knowing the "basic facts"!

Note

For more information about Vedic notation, the reader may like to start with:

Albarn, Smith, Steele and Walker, *The language of pattern*. London: Thames and Hudson, 1974 (pages 10-15)

Buckminster Fuller, *Synergetics*. New York: Macmillan, 1975 (section on "Indigs")