Editorial

Assembling articles to make up an issue of a journal, as I have just done, reminds me that an editor is always tempted to hope that the whole he has put together is somehow greater than the sum of its parts I'm not sure the journal's readers, or its authors, see the situation in quite the same way. For them, the "whole", if it exists in any sense at all, is probably the set of related articles from whatever sources that they have read recently, or the set of articles they have written and will go on to write. The journal, in their eyes, may be like a package containing — if they are lucky — a couple of items of value wrapped in old newsprint. Keeping this in mind stops an editor from indulging in splendid fantasies about the impact of his efforts

An editor may want to ease into existence a journal with certain concerns, a certain style or tone, a certain level of discourse. (The inside front and back covers of this issue tell you something about what this editor has in mind The contents of the issue tell you some more.) But he has to attempt to do this within constraints, since a journal, whoever initiates it, is a social enterprise Most obviously it needs people to write for it and a larger number of people to buy it. At this level the continued production of a journal is an exercise in management

A journal is a product: a product — or perhaps a by-product — of people talking to people Like face-to-face talk, printed talk takes many forms and serves many functions It can entertain, inform, reassure, tell lies, beat about the bush, play on feelings, inflame, and do any of the other things that people use words for. Like face-to-face talk, too, printed talk is received in context, and the context may determine that the words tell much more or much less than they say.

Words, spoken words and printed words, are so much a part of our social furniture, so pervasive, so everyday, that we don't take them seriously most of the time A professional writer knows that words are hard to master: they insist on being opaque when he wants to be clear, and blunt when he wants to be subtle; they are — worst of all — glued to the page, stuck at a particular point of time. He can rarely expect from his readers a comparable effort of re-animation, of re-creation.

Words can be offered seriously and can also be taken seriously. They can be worked at by a reader, re-read and thought about, until they yield up meanings that may have escaped a first scrutiny At this level the production of a journal may also be an educational enterprise

My hope that this journal may grow into one that learns along with its writers and its readers is my justification for introducing a new journal into a crowded field. "Print, print, and still more print. Who needs it?", as several people have said to me, using other words. The dangers are worth risking, I think. A new journal makes no demands by itself: only people do that, on themselves or on others. And although life, including the life of classrooms, will no doubt go on much the same with or without *For the Learning of Mathematics*, well-chosen words *can* trigger awarenesses and stimulate reflections and give experience to those sensitive to them. If any who are reading this sigh at the prospect of yet more to read, I'd say they have missed the point. I want to do something to serve the interests of those who have to learn mathematics. I hope some who share that desire may find *For the Learning of Mathematics* a journal which it is in their own interest to read.

In this issue, Jack Easley picks up Joseph Agassi's use of the word "agenda", but that is hardly a coincidence as both delivered their papers at the same meeting. Other convergences are more coincidental; the words "magic" and "metaphor" seem to resonate gently across articles Geometry is given a good airing and will be returned to subsequently. I hope some readers will be moved to enter into dialogues about these and other matters, either by writing directly to the authors, or by sending comments for publication to me so that the discussion can continue in the open

About Geometry

DICK TAHTA

In the nineteenth century, biologists tried to account for the diversity of species by supposing that the gaps between species contained the departed ghosts of the "unfit". For some, it was as if the surviving species were defined and brought into being by the death of other species. The metaphor was clearly linear, relying on some sort of geometric image of a line, not unlike that invoked by mathematicians whose rational numbers are interspersed by a "denser" collection of irrationals. Such a picture was inevitably very crude and has had to be considerably refined. But however sophisticated they have become, biologists, in common with scientists in general, continue to rely on some kind of geometric imagery

The original scientific enterprise of quantifying the world was undoubtedly highly successful but it has become increasingly clear that a qualitative grasp is more fundamental and more urgently required. René Thom, the mathematician who has in recent years offered biologists a powerful new tool for the study of changes in form, has emphasised that some kind of intuitive geometric imagery is of primary importance. [1]

I am certain that the human mind would not be fully

For the Learning of Mathematics **1**, 1 (July 1980)
FLM Publishing Co. Ltd.. Montreal. Quebec Canada

satisfied with a universe in which all phenomena were governed by a mathematical process that was coherent but totally abstract... In a situation where man is deprived of all possibility of intellectualisation, that is, of interpreting geometrically a given process, either he will seek to create, despite everything, through suitable interpretations, an intuitive justification of the process, or he will sink into resigned incomprehension, which habit will change into indifference

Thom goes on to add that even the most successful quantitative thinking leaves us still very ignorant. In the case of gravitation, he points out that we have not today less reason to be astonished at the fall of the apple than had Newton. It is moreover important to realise that our grasp of number and magnitude itself relies crucially on geometric thought. Even the most abstract algebraic notions are described in geometric metaphors which are often idiosyncratic and personal. For Dedekind, a set was like a bag, whereas for Cantor, it was an abyss. At a more important elementary level, the failure of so many to handle numbers confidently may be due to the fact that they do not have any mental picture corresponding to the numerals on which they are required to operate.

There is no doubt then that geometry is important. But what it actually is and what part it should play in mathematical education are notorious and perennially difficult questions. Geometric thinking pervades all mathematics, yet curiously enough it is less and less pursued as an independent study in schools and universities. It provides a way of construing reality and yet is at the same time based on observation of the external world. It is not easily characterised and often it seems as if the only thing that can really be said about it is that it is what geometers do. Nevertheless, the epistemological and pedagogical status of geometry continues to be felt to be an important matter for discussion. What is the point of learning geometry? Is it to train logical thought by studying deductive systems? Is it to develop spatial awareness and ability by empirical study of the environment? Is it to learn a language rich in metaphor?

The questions pile up and a wide range of familiar answers are available. Yet the fact that the questions continue to be asked, that the status of geometry continues to be uneasily problematic, suggests that current solutions are inadequate. There are so many pitfalls... For instance, the classical arguments in favour of a training in Euclidean geometry have not been able to survive the experience of universal secondary education. The arguments in favour of the development of spatial awareness seem less convincing when it is recalled that babies learn to stand on their feet and walk without reflective self-consciousness or analysis of this ability. The argument that geometry is an empirical study becomes complicated, if not circular, when it is realised that space itself is a human construct. Nor is it very convincing that geometry might be a language worth independent formal study when it is observable that its metaphors are naturally acquired or that mathematicians themselves seem increasingly to embed the use of the language in their other studies.

In these circumstances general words like *geometry*, or for that matter apparently more specific ones like *space* or *linearity*, become devalued place-holders, merely indicating to cognoscenti that a certain kind of discussion is taking place. The situation is not unlike that of the status of words like *freedom*, *justice* or *democracy* in the appalling political aftermaths of the second world war. In that case it became particularly fruitful — notably among German philosophers — to take a so-called hermeneutic approach in which devalued abstract words in particular were given idiosyncratic and often frankly speculative interpretations. This enterprise can easily become highly metaphysical and runs the risk of being tortuously private and complex. But it has been successful in many areas and has, for example, been recommended in a recent useful discussion of research in the psychology of learning mathematics. [2]

A hermeneutic approach to geometry is almost inevitable and this preliminary study is deliberately intended to be highly speculative and divorced from familiar terms of reference.

One particularly useful way of speculating about the nature of geometry, or indeed mathematics in general, is to try to distinguish the activity from the technology used in the recording of its results. Cuneiform stylus, reed and parchment, typeprint and computer tape, all draw attention to that part of mathematical activity that is concerned with the creation and manipulation of symbols. It seems almost impossible to conceive of mathematics without drawn diagrams or written numerals and other signs. The social aspect of mathematics, the sharing of consensus and the attempt to agree universally valid truths, emphasise a need to externalise in some more permanent form the mental activity that mathematicians engage in. But whatever the technology used, the recorded form cannot fully convey the meaning attached to it by the recorder. Following St. Augustine, we say that the thing signified is not the sign. In their own context, mathematicians refer to the "arm-waving" that must necessarily accompany, say, work on a blackboard. Moreover it is notoriously difficult fully to understand how earlier mathematicians thought from the inherited records of their work — the history of mathematics is a complex interpretative enterprise.

The standard accounts of the beginnings of mathematics necessarily rely on what was recorded. The early tablets and papyri indicate the considerable achievements of the Babylonians and the Egyptians. These records give innumerable particular solutions to particular problems and the extent to which the scribes generalised has to be inferred. Clearly, however, the Greeks did generalise. They invented and recorded formal general *proofs* of much that was known in particular cases and it is this achievement that is often said to characterise mathematics as we now know it. But proof is a social matter depending to some extent on what is agreed by the people concerned to constitute a proof. The Greeks seem to have been especially interested in "knowing how we know"; but the nature of their achieved consensus was peculiarly dependent on the way they chose to record it. It is worth considering how some of their achievements were established and recorded more than a thousand years before Euclid.

It is sometimes said that one of the underlying themes of Euclid's *Elements* is the construction of the five regular solids In early Greek cosmology the first four were related to the fundamental elements which were taken to be fire, earth, air and water. The discovery of the dodecahedron was at first an embarrassment but it was soon related to a mystical fifth element — or quintessence — so that the pentagram, for example, with its irrational proportions became an occult symbol. The culminating thirteenth book of the *Elements* ignored these connotations and presented a proof that "no other figure besides the said five figures can be constructed so as to be contained by equilateral and equiangular figures, equal to one another".

In a recent book, *Time stands still*, Keith Critchlow has considered some of the slowly gathering evidence of the geometrical achievements of Neolithic men. In particular, he discusses in one chapter some remarkably shaped stones found at numerous sites in Scotland For some time these had been classified by museums as miscellaneous — possibly ritual — objects relegated to closed drawers. Critchlow was intrigued by the regular indentations of these stones with their symmetrically placed sections. He stuck pieces of tape along the indentations and found to his surprise that they formed edges of regular polygons Moreover he found examples of all five regular solids. What were these stones for? How were they made? Critchlow delineates carefully what we can interpret from them [3]

> *Leaving the social function of these stones aside, because interpretations become subjective and finally land in that wastepaper-basket of a word, "cultic", into which all difficulties are thrown, what we have are objects clearly indicative of a degree of mathematical ability so far denied to Neolithic men by any archaeologist or mathematical historian We only "know" that the Neolithic producer of these objects had such a knowledge of three-dimensional mathematics because of the objects themselves When one inspects the great variety of these intricate and intelligent objects, one marvels at the time, patience and skill employed in their making. Here we have the hardest stone found in Scotland being chosen to create beautiful mathematical symmetries for no apparent utilitarian use.*

The issue is not here of course to show that our culture is not entirely due to Mesopotamian man, nor to confirm that the Greeks were not the first on the mathematical scene. What is important is the forceful reminder that mathematics is not dependent on its written record. ". . what is unavoidable in these small objects is their total clarity of 'language' They are as clear and concise a statement of their own terms as any that can be made in either verbal or written form " [4] (Figure 1)

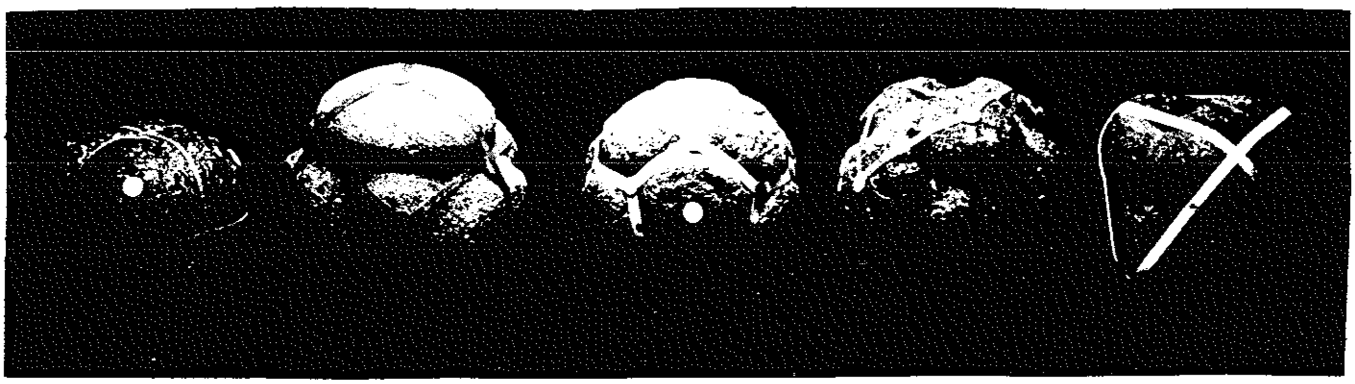
The same might be said of many ancient artifacts that have been studied in recent years — at first by various fringe writers but now also by a growing school of academic archaeologists. The artificial lines incised on the Peruvian plateau, the petroglyphs and medicine-wheels of the American Indians, the megalithic stone circles of Western Europe, all exhibit an extensive and confident geometrical knowledge without further evidence of interpretative written record [5] Whatever Stonehenge was built for, it was clearly a detailed and carefully planned construct. It does seem surprising to us that not even a scratched diagram or plan for this engineering feat has been found But the henge is its own diagram. Oral memory did not need to record It was not necessarily weaker and our talent for it has obviously atrophied. As Plato recounts, when the great god Toth introduced Rameses to the act of writing the king sadly wondered what would happen to memory now that it was to be kept on paper. One result certainly is that it is now very difficult to understand how such early geometrical knowledge was achieved let alone what it was all for.

A confident nineteenth-century account used to claim that mathematics found its origins in the needs of kings to collect taxes, make inventories and allocate resources According to this view, geometry was, as its name implies, based on mensuration — of land and of the rich alluvial mud of the Nile This, or some similar version, still remains attractive to those who would emphasise the "practical" value of mathematics, be this in science, in technology or in the marketplace. But it cannot explain, say, the "pure" sophistication of Babylonian number theory or the intricacies of Vedic mathematics. It does seem more likely — as has been suggested by A. Seidenberg — that these were originally developed in response to ritual needs. [6] The ownership of property demands the development of a certain sort of mathematics, but long before the need for this, men were naturally preoccupied with the harmony of the heavens and — by a universal and naturally-related shift — with the harmony of scales of music It has been suggested, notably by Giorgio di Santillana and Hertha von Dechend, that most ancient myths enshrine elaborate descriptions of the movements of the constellations. [7] And a convincing recent interpretation — by Ernest McClain — of the Indian epic, the *Rig Veda*, suggests that this also contains a hidden account of sophisticated early solutions to the problem of tuning an octave. [8]

Myths — and mathematics is also a myth — have their own clarity but like other human artifacts they disguise how they were interpreted by their makers In any case what is always missing for us is any certain sense of the imaginative reconstruction in the mind of another listener or viewer. We cannot be sure we share the same images with our contemporaries, let alone men of the past. Yet we continue trying to do so and are often magically successful. It could be fruitful to seek the nature of geometry within this magic. Now some geometry is clearly magical in the sense that its formal idealised language can adequately describe the real world Thom has also proposed the converse: *all magic, to the extent that it is successful, is geometry*. [9] Such hermeneutics usefully shift attention from a search for the stuff of geometry in land measurement, stars or scales, to the mental processes involved in the creation and exchange of images. And it does also suggest a way of dignifying the teaching of geometry with a purpose that could be granted serious attention

Figure 1

This illustration is reproduced from plates 114, 115 and 116, by Graham Challifour, in K. Critchlow, *Time stands still* (London: Gordon Fraser, 1979) by kind permission of the publisher



At this stage it should be emphasised that an image, at least in this present usage, need not be solely a visual construct. Clearly it is very powerfully so for some people but equally clearly it is not for others. The Russian psychologist, A R Luria, reports the case of a man with an amazing ability to memorise, due to a highly developed power of imagination. His imagery was not all visual [10]

> *I recognise a word not only by the images it evokes but by a whole complex of feelings that image arouses. It's hard to express ... it's not a matter of vision or hearing but some overall sense I get. Usually I experience a word's taste and weight, and I don't have to make an effort to remember it — the word seems to recall itself. But it's difficult to describe. What I sense is something oily slipping through my hand ... or I'm aware of a slight tickling in my left hand caused by a mass of tiny, lightweight points.*

To think mathematically is to work in some such way with images. But are there distinctions between types of imagery that might be described as algebraic or geometric? One might suppose for example that combinatorial thinking depends on some tactile or kinaesthetic sense. It is certainly tempting to characterise geometric thinking as based on the visual sense until one recalls the famous and deeply moving example of Helen Keller, who was deaf and blind. She described a straight line in terms of having something to do that could not be set aside. "I feel as if I were going forward in a straight line, bound to arrive somewhere, or go on for ever without swerving to the right or to the left." [11] Even if one begins to appreciate the strength of the tactile and other images that compensate in some way for lack of sight, it still seems astonishing that blind people can work geometrically using the usual conventional words. A classic of mathematical education by Benchara Branford contains a very moving description of some geometry lessons with blind children taken by their blind teacher. [12]

Clearly the distinction between algebra and geometry cannot be a matter of the type of imagery involved. In any case such distinctions — often made in discussions or indeed syllabuses — are unfortunate because the activities are really complementary. Physicists are accustomed to working with apparently exclusive complements; in certain situations they invoke a wave-theory of light, in others they need the apparently contradictory corpuscular theory. The real distinction between algebra and geometry may be a matter of stressing certain awarenesses according to choice at any particular time. In this case geometry is certainly not some specialised study that can be widely ignored. Nor can it be interpreted solely as being a study of shape or space, quite different from, say, the study of number and magnitude. In fact, shape and number are not particularly helpful distinctions either when considering the basic material that mathematics works on.

A distinction that is extremely useful in practice was offered by Caleb Gattegno in a brief, but seminal, article published in 1965. He approached the problem from the experience of systematically using a certain method of teaching that he has described in various writings. [13]

> *In a number of experiments I have asked my classes to consider with their eyes shut some situation in their minds which I generate by instructing them to produce some images and act mentally upon them. By doing so it became possible to perceive in the mental situation a number of distinctive features which when talked about sounded strangely like the geometrical statements we read in books as theorems or problems.*

For Gattegno then, *geometry is an awareness of imagery*. Such awareness arises from a dynamic process of the mind and any formalisation of such awareness is an algebraic activity. This means that "algebra differs from geometry in that the first describes mental dynamics while the other uses mental content, imagery." [14] Thus *algebra is an awareness of dynamics*, an awareness of "the mind at work on whatever content".

These interpretations have the merit of showing precisely how the two activities are complementary and intertwined. They also indicate why it is that there will always be a tendency to algebraicise. That this can become dangerously premature has been pointed out by various educators from St. Augustine to Mary Boole. Curiously enough, failure to master algebra may be due to neglect of the underlying geometry. There cannot be an adequate awareness of dynamics if there is nothing to act dynamically on.

A historically interesting discussion of another aspect of this issue occurred in a seminar of the newly-founded Ecole Normale at Paris in 1795. At the time many mathematicians were seeking to resolve the vexed problems posed by the Euclidean theory of parallels by trying to analyse more rigorously the definition of a straight line. In the seminar, Joseph Fourier, then a young student, criticised the definition of a straight line as the shortest distance between two points, attributed to Archimedes, and then being used in an influential text-book written by Lagrange. He suggested instead a definition of a straight line as a set of points equidistant from three fixed points in space and compared this with a definition of a plane as a set of points equidistant from two fixed points and a definition of the surface of a sphere as a set of points equidistant from one fixed point. The leader of the seminar was Gaspard Monge, whose course in descriptive geometry had been enthusiastically followed by many students and which clearly had inspired Fourier's thinking. Monge complimented the young student and then gently criticised the proposed definition in words that are particularly interesting in this present context. [15]

> *Citizen ... the notions you invoke in your definition are more complicated than the line you want to define. They assume a familiarity with geometry that could not have been acquired without the notion of a straight line. Certainly in order to define some type of object in geometry a property must be found that applies to the very object of that type and only to such objects. But this is not enough. The chosen property should be the simplest and easiest to conceive.*
>
> *... It is not even sufficient that the defining property be simple and easy to conceive. If possible it must — above all in geometry —* be able to offer an image. *For*

instance, suppose the straight line were able to be defined as follows. Consider an object turning about two of its points like a block of wood turning on a lathe Most of the points of the object will describe circles of various sizes But some points will not change position during the motion. The set of such points forms a straight line Such a definition would not be sufficiently simple because of the ideas of rotation that are used and because of the difficulty of imagining where the invariant points might be The definition does not show *a straight line It fails* — despite the concrete example — *because it does not offer an image of the line at all*

There is of course a longstanding tradition of effective practical realisation of this particular emphasis on the image in geometry teaching The work of Monge himself, the method of making mind-pictures described by Mary Boole, the rhythmic curve-stitching of her friend Edith Somervell, the play materials of various educators from Johann Pestalozzi to the present day, the imaginative drawing and dance exercises of Rudolf Steiner, the animated films of Jean Nicolet, the use of geoboards, films and other materials by Caleb Gattegno — these [16,17,18,19] are indeed some strands in a tradition reaching back to the Pythagoreans, a tradition that firmly refuses to atomise geometry into a series of written exercises but seems to offer a seamless web of thought, action and feeling

Why is it that this powerful tradition has not been able to withstand the various pressures from within and without schools that have, it sometimes seems, reduced geometry either to an outmoded and sterile exercise in logic, or to a vague, arbitrary and often mindless, empirical craft? A hermeneutic shift is again required to find some possible answers One possibility may be that geometry is indeed magic in the sense that it conveys but cannot state its own power To paraphrase the Chinese philosopher, the geometry that can be told is not geometry. Another possibility is that for various reasons people have mistakenly struggled to relate the magic to a different tradition, that of recorded proofs and universally agreed results, which can so usefully provide the material for the competitive examinations necessary to bureaucratic societies In any case it would seem that geometry might be more successfully pursued if it were to be explored in its own terms A possible way of doing this in the future lies in the field of computer graphics and automated control Seymour Papert in reporting children's work with computers has emphasised that their activity cannot always be interpreted in traditional terms [20]

Meanwhile in the ordinary classroom without sophisticated and expensive hardware there remains the powerful, albeit possibly atrophied, power of mental imagery and oral memory. The neolithic tradition can still be invoked and it seems appropriate here to offer some accounts of tentative work of this sort.

A class of nine and ten year olds of various abilities have been working regularly at making mind-pictures At first they squeeze their eyes tight and report "seeing" stars and flashes of light After careful relaxation exercises and instruction in "palming" their eyes they still report vivid pictures, usually described in terms of simple elements — points, lines, circles — rarely in terms of motorcars or people Then people try to create a blank "screen" and impose upon it an image derived from someone else's prescription.

Think of a line segment rotating about one end For the children a line is often finite — indeed one specific exercise is concerned with growing the line in both directions without limit The term "line segment" is then used to describe the finite case

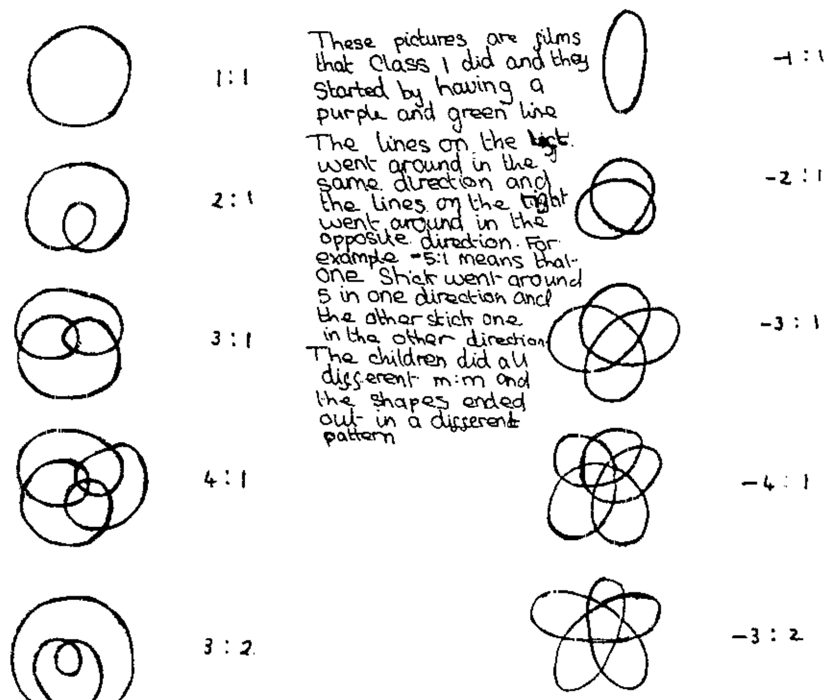
Think of another line segment rotating about the same point After a silent period of concentration people start talking without being prompted. There seems to have been various choices — between different relative lengths of the lines, different senses of rotation, different rates of rotation, different starting positions People try following other people's choices There seems to be a wealth of possibilities and some time is spent seeking control of some of them in the mind Some days later the teacher returns to the theme

Think of the moving endpoints of the lines as they turn. . Now think of a point halfway between these endpoints This is more difficult There are some particular cases that are easily coped with. If the lines rotate in the same sense and with the same speed the midpoint in question describes a circle But suppose they turn in opposite senses with one of them moving twice as fast as the other? Some particular positions can be invoked and there is a suggestion of a locus with loops. Clearly this could be investigated further on paper, but the mental imagery already has the dynamic of a film and it seems natural to attempt to make an animated film.

The children work in groups of three or four at a time spending an hour or so at a time with the animated camera. A polar grid is drawn on a large piece of paper and this is covered by a blank sheet with two strips of wood pinned at the centre The pieces are moved appropriately and at each frame the midpoint is marked. As the work slowly progresses all are involved in how the locus will turn out The teacher labels the first choice — same speed, opposite sense — as $-1:1$ without further explanation. This notation is naturally taken over and applied to other choices. The final film investigates the cases $\pm n : 1$ for $n = 1,2,3,4$ and $\pm 3 : 2$ The makers now have a confident knowledge of how the loops appear and can talk fluently about this The only writing they do is a brief statement on the actual pieces of paper each group used for filming and a summary of results on one extra page (Figure 2 overleaf) There still seems plenty to talk about and explore.

The reader is invited to attempt a classical analysis of these curves. Are any of them well-known? Can they be defined in other ways? Does an algebraic description offer any useful way of looking at the changes in form as $m : n$ varies? The work of the children seems to have a life of its own It has its own clarity and its own justifications It is not examinable in any traditional sense and it does not solve any "useful" problem. But it surely has a magic that serves some deeper educational purpose.

Figure 2



A quite different sort of exploration — in this case of feelings — is provided by the following example of a drama session with older students. Preliminary activities include moving round the room with eyes closed. People distribute themselves far and near. The group is arbitrarily partitioned. There are now two groups and people move towards those others who are suddenly near. As a further preliminary each group is asked to arrange itself in a star with people lying on their backs, heads in a circle and feet pointing outwards. Close, but not touching. Everyone is now asked to say, whisper or shout, a succession of verbs, repeating words if they wish. Each group is charged with the task of seeking a common word, so that eventually every group will be mouthing the same verb. The task is presented as an exercise in group sensitivity.

Meanwhile two pieces of paper have been prepared with a single incomplete sentence on each. The sentences are: *none of them ... us* and *none of us ... them*. The verbs communally achieved in each group are entered, one in each sentence. Each group is then given the sentence containing the word it agreed. And left free for the rest of the session — a situation for exploration and improvisation.

And topological research. For to investigate *proximity* in general it is enough to have some connection between collectives defined. Think of groups of people and any relationship such groups may or may not have to each other. Whatever the nature of this relationship, refer to it as one of *nearness*. Does this offend preconception? Try it. Can an individual be "near" a group? According as the collective containing him and only him is or is not near... The individuals near a particular group are of obvious interest. Collectively they can be called anything you like. Say the "kin". Collectives that contain their own kin are well-knit, cohesive, bounded, well defined, supportive, buttressed, self-contained, complete, clear-cut, separated, compact, united, safe, defended, removed, out-of-reach, stable, invulnerable, confederated, tribal ... If these words are reasonably chosen then it would certainly seem useful to investigate such collectives in further detail.

In the drama session there is this experience: everything outside one is far away, none of them can touch us, none of them ... us, we include all who are near us, the collective contains all that is close. *But there is also what it feels like from outside*: every part of me is remote from what is not me, none of us can touch them, none of us ... them, we exclude all who are near them, all but the collective is as the other case. These experiences can be simultaneous or distinct. This and the nature of the experiences depends on the nature of the "space" in which they occur for it is the space that determines what we may understand to be near or far. What is space? Space (it might be said in this context) is a whole whose parts have possibility of relation. More particularly the part-to-part relation may be designated as nearness — however arbitrary this may seem.

It could now be a technical exercise for professionals to characterise these properties of the nearness relation that would be required to define, say, a proximity space. [21] But much has been learned already about the nature of non-metric spaces.

One other, final, example may serve to highlight the fact that geometrical "knowing" can be very different from that which is often expected. Inspection of the points of intersection of the diagonals of a regular polygon for the first few easily drawn cases yields a remarkable distinction between old and even-sided polygons. When drawing polygons children have sometimes observed something equivalent to the statement that *when n is odd the diagonals of a regular n-gon only intersect in pairs*, whereas when n is even there are points where three or more intersect. It is difficult to test this conjecture with larger polygons because of the inevitable inaccuracy of pencil drawings. Moreover there seems to be no elementary way of exploring what seems to be a very accessible and simply stated observation.

Computer-generated slides showing the complete diagonals of some regular n-gons provide fascinating and accurate evidence for further empirical observations. Even a casual and rapid viewing of the slides reveals very different textures. When n is even the multiple intersections stand out like twinkling stars, but when n is odd the texture is intricately uniform and, for larger values of n, as delicate as a piece of fine lace. The distinction is also perceived in posters displaying various cases. [22] The empirical evidence supports the conjecture, though on ordinary slides the interior of the polygon is so dense with lines when n is, say, 45 that it does seem remarkable that the diagonals can go on interweaving yet avoiding meeting internally in threes.

The conjecture was in fact part of mathematical folklore for some decades — possibly much longer. If it is true, then the diagonals of odd-sided regular polygons intersect at even-order nodes so that they can be traced unicursally. This is often used in the curve-stitching or nail-and-thread constructions of complete odd-sided polygons. After a decade of circulation in books of puzzles and problem pages of professional journals, the conjecture was proved for prime n in 1961, and a complete solution — for all odd n — was given a year later. [23,24]

Both the last two proofs used algebraic manipulations of complex number representations of the vertices and hammered these out in a verification of what was expected to be true. But such proofs do not reveal much further insight into the phenomenon. Despite traditional training many people find that it is the inspection of the slides that really commands assent. These are in some ways mind-blowing — they induce a contemplative trace; viewing them in a dynamic sequence yields a strong sense of the difference in texture between the odd and even cases. The algebraic proof does not, in Monge's phrase, "offer an image". But how is it that we can *know* the result without it?

Even more mysterious is a development from this theme pursued by mathematics teachers working in Steiner schools. A complete regular polygon is defined by joining a set of points equally spaced round the circumference of a circle. Imagine these points to have been defined by a "star" of equally spaced lines on the centre of the circle. Now imagine displacing the star — in the mind or perhaps with a piece of acetate sheet or tracing paper. The star now defines another set of points on the circumference, not of course now equally spaced. What properties of the original configuration are preserved? How do the diagonals of the new polygon intersect? What sort of transformation has taken place? Is it a collineation — that is, does it preserve incidence properties? Questions like these are accessible to children. But certainly a very keen geometric imagination is required to deal with them. They are peculiarly unsusceptible to algebraic analysis as the reader may care to verify. Teachers and pupils sometimes create geometries which are difficult to relate to the usual familiar ones.

In this article I have attempted to describe some ways of thinking and acting geometrically that almost seem at times to be part of a lost tradition. Its neglect and the behaviourism that is still paradigmatic in Western education mean that any attempt to resurrect a way of describing the tradition seems speculative and mystical. The language of the remarkable books about projective geometry by Heidi Keller-von Asten and Olive Whicher [25, 26], written from the experience of the Steiner schools, are quite unusual for mathematics texts. But the contents of the books do offer some exhilarating insight into what geometry can be and what it can do for people. There is also some growing evidence from other people who work with children with an active trust in their powers of imagery. It seems appropriate in a new journal of mathematical education that there should be an attempt to tackle a perennial problem with new terms of reference. I hope that others will take the discussion further. The great god Toth need not have the last word.

References

[1] R. Thom, *Structural stability and morphogenesis*. Benjamin 1975 (p. 5)
[2] W. M. Brookes, "A hermeneutic approach to psychology and its applications to learning mathematics". In: *Papers presented to the first meeting of the International Study Group for the Psychology of Learning Mathematics*. Warwick University, 1977
[3] K. Critchlow, *Time stands still*. Gordon Fraser, 1979 (p. 148)
[4] K. Critchlow, *op cit* (p. 149)
[5] K. Brecher and H. Feirtag, *Astronomy of the ancients*. MIT, 1979
[6] A. Seidenberg, "The ritual origin of geometry". *Arch for Hist Exact Sci*, 1 (1960), 488; *also* "The ritual origin of counting". *op cit* 2 (1962) 1-40
[7] G. di Santillana and H. von Dechend, *Hamlet's mill*. Macmillan 1970
[8] E. G. McClain, *The myth of invariance*. Shambalah, 1978
[9] R. Thom, *op cit* [*1*], (p. 11)
[10] A. R. Luria, *The mind of a mnemonist*. Penguin Books, 1975 (p. 28)
[11] H. Keller, *The story of my life*. Hodder and Stoughton, 1969
[12] B. Branford, *A study of mathematical education*. Oxford, 1908 (ch 4)
[13] C. Gattegno, "Mathematics and imagery". *Mathematics Teaching* 33 (p. 22)
[14] C. Gattegno, *op cit* [*13*] (p. 22)
[15] G. Monge, "Une discussion sur la ligne droite". *Mathesis*, 9 (1883) 139-141 (author's translation)
[16] D. Tahta (ed), *A boolean anthology*. Association of Teachers of Mathematics, 1973 (p. 38)
[17] E. L. Somervell, *A rhythmic approach to curve-stitching*. National Council of Teachers of Mathematics, 1978
[18] J. Nicolet, *Animated geometry*. Film series, remade in seven parts, Educational Solutions, NY, 1980
[19] C. Gattegno, *For the teaching of mathematics*, 3 vols. Educational Explorers, 1964
[20] S. Papert, Teaching children thinking. *Mathematics Teaching*, 58 (1972) 2-7
[21] S. A. Naimpally and B. D. Warrack, *Proximity spaces*. Cambridge, 1970
[22] Leapfrogs, *Diagonals of n-gons* (various n), posters, 1976
[23] H. T. Croft and D. M. Fowler, "On a problem of Steinhaus about polygons". *Proc Camb Phil Soc*, 57 (1961) 686-8
[24] H. Heineken, "Regelmässige vielecke und ihre diagonalen". *Enseign Math*, 8 (1962) 275-8
[25] H. Keller-von Asten. *Encounters with the infinite*. Keller-Dornach, 1971
[26] O. Whicher. *Projective geometry*. Rudolf Steiner Press, 1971