“The Foundations of Geometry”

CALEB GATTEGNO

A film with this title was produced by Educational Solutions Inc in the Fall of 1979. The Editor watched a single run-through of the film in December and then talked to its scenarist Dr. Gattegno. The following is an edited transcript:

D.W Having looked at your film on the foundations of geometry, I’m very much aware that this is a “modern” geometry, this is a modern mathematics approach to geometry. But many examples that you have worked out in detail, in the past, of teaching have been concerned with traditional, non-modern mathematics. Yet I feel that in yourself you are aware that modern mathematics has brought insights to you that have been valuable for your teaching, and I wonder if you can say something about that.

C.G This film is an attempt to do a certain number of jobs which I have carried in my mind for many years, but I had the opportunity to do only on the occasion of the French Government adopting, last September, a reform of the teaching of mathematics which replaced the Lichnerowicz programme [1]. The latter was attacked very severely by many people in France as too verbal, so biased towards a working mathematician’s viewpoint that children ended up having the words but not having the notions, and not understanding how to tackle problems. And teachers had to work a great deal before they could make sense of these words, although the people who made the original proposal were of goodwill and were certainly dedicated mathematics teachers who wanted to serve the cause of mathematics in France.

The reform came from the influence of some mathematicians who are not very much in the camp of the Bourbaki (although Henri Cartan, who is one of them, is also one of the Bourbaki group.) Henri Cartan and Gustave Choquet were sympathetic to the Lichnerowicz programme, but both of them wanted to give youngsters experience, and they wanted to reintroduce geometry which had been banned by those, like Dieudonné, who were rabid against the geometrical approach that they had suffered under, sixty or seventy years ago.

There is a basis for rejecting the ancient geometry which I would call, from my point of view, the stuffifying approach: having units of knowledge given, one after the other, and the student memorising a definition, a theorem, a proof, without really having a grasp of what the problem is. Now what mathematicians do with geometry — and in the 19th century in France, any mathematician was called a geometer, as being the best description of how his mind worked — what makes the difference between them and their students, is that they use a dynamic approach to the many notions, and they put things together in wholes that enlighten, that light up, the various bits. Therefore they can, when they speak of one particular thing, put in all their experience which contains a great deal of material which doesn’t appear at all to the outsider. The possibility of making the students level with their teacher in this respect has been my preoccupation since I started making films, which goes back 25 years now. And we can see that what a film can do that no language can compete with is to offer a multitude of notions, intermingled and separated by the artifice of film-making, and to do it very quickly and several times over. So, without words, we can introduce our students to the wealth of geometrical experience, employing an intuitive approach, and then try to do in the classroom all the other things that films can’t do — that is, to verbalise every second of the film and to express it in the proper language; to feel that the scene on its own is not sufficient to convince, therefore it is necessary to work differently with well-tabulated notions, producing the axiomatics; to distinguish the definitions from the axioms; and then to try to put things together so that it becomes clear that what one had in mind is true within the frame of reference of the notions that have been singled out in the axioms and definitions. So the presentation of the foundations of mathematics on film, in the film form, is to provide students in a very short time with all the many impacts that they need to have in their imagery.

For instance, it is clear that if I draw a straight line on the chalkboard, I can put my hand on one side and say, “This is a half-plane”, and my hand on the other, “It’s a half-plane”, and I can be satisfied that I have given the students a definition that the straight line divides the plane in two halves. But in this foundations of geometry I have taken 12 seconds of film to show that by lighting only one half of the plane I can bring their attention to that half, and then, by lighting, the other half; and I show that one can go from one to the other, and put them together to produce a whole plane, indicating that the line is the thing that divides the plane into two halves. And then I show that when the line disappears in the plane it can be replaced by any other line, so that to divide a plane into two half-planes with a line, which is a definition, is also a definition in the optical sense, that one can see it. There are two half-planes associated with this line, and when this line sweeps the plane by moving parallel to itself (without telling you that it stays parallel) it indicates that there are an infinite number of such partitions of the plane into two half-planes. And when I make it swing about one point, you get another definition of an infinite number of half-planes. In reserve for the future you have other questions to put about the shift from sweeping parallel to rotating the line that still produces an infinite number of pairs of half-planes defining the same plane.

D.W And the definitions at which you want to arrive, are these the articulations of these awarenesses; is that essentially what a definition is, that it’s a way of putting into
words a particular awareness? I feel that possibly if a
teacher and some students were discussing this episode
from the film, they might in fact talk about other things as
well as the existence of the half-planes.

C G. If they want to. But since it is an axiomatics that we
want to put in the film, since we want to give the founda-
tions of geometry, if I am a teacher our conversation is
essentially on: a line divides the plane into two half-planes
which together make the whole plane.

So. It can be done in an infinite number of ways, and we
want to know, to have an experience of that infinity of
ways. I can stop there, although there are lots of other
things that I would like to do. But I wouldn’t want to
jeopardise the future of the sequence that is in the film, that
has been selected so that it gives an alternative to Euclid’s
or Hilbert’s presentation of geometry. This short sequence
indicates that we have an entity called a plane that we can
perceive, and which we perceive as different from another
entity we call a straight line, which we can also perceive;
and the connection between these two has been given as the
entry into the foundations. So we don’t say, like Hilbert,
that there are three categories of things — although we shall
end up also able to say it.

We start with the screen, and the screen is an object, it’s
in the physical universe and will remain at the level of
knowing which is connected with the perception of the
world. And we’re going to do something to this perception
of the world so that what it generates is mathematics. And
we shall talk about this as we go along. So the viewers who
see these 12 seconds of film will have to tell what they saw,
what they perceived, what they think it’s all about. When I
hear their words I will know it is connected with imagery,
and it is this imagery that I want to establish in them. The
words will be triggered by the images, and later the images
will be triggered by the words.

D W. I think you’ve probably already answered a question I
had in mind earlier while you were talking, and this is that
the fact that children come with a great deal of geometrical
experience to the classroom, they don’t come as white
sheets without any geometrical awarenesses at all, and I’m
wondering how the link is made. But perhaps this is covered
by your remarks about the fact that the perception of the
film is part of their mode of perceiving the world, indeed.

C G. You’ve raised a question which is a little more
difficult to handle within this frame of reference. What we,
you and I, call the geometry they bring with them may not
be called geometry by the mathematicians. They may say,
“It is a preparation to enter geometry,” and I am playing
their game in this film.

D W. The mathematical game.

C G. That means I am addressing myself to those who have
not given their time as I did, and you did, to the relation-
ships between teaching and learning. I know that these
mathematicians want to succeed. I know that they want to
give students an understanding of what they say, and that
they accept the film as a medium to bridge the gap between
their words and the meanings that the children have attached
to the words. For me, it is another process. We know that
the students have seen walls, that they’ve seen distances,
that they have walked on lines on the pavement, that they
have played with string, that they have tightened it, and that
they know that a taut string is a substitute for a straight line
since they can chalk it and put it on a chalkboard and get a
straight line as a drawing from it. Now, I don’t want here to
goto into these subtle philosophical matters — which are
exciting, and should be gotten into — but I want to say to
people that we have made some progress in teaching mod-
erm mathematics through the medium of the film by incor-
porating the assimilation, the acceptance, of very rigorous and
demanding mathematics to our viewpoint since what stu-
dents see when they speak about the film resembles what
mathematicians speak without having seen it. And that’s
really the job I did in the film. I started with Choquet’s
words and said that if I wanted to convey them to youngsters
I needed to do it differently, needed to give them the
imagery, to give them the dynamics, and then make them
talk; and those will be my lessons. So 12 seconds of film
may take 2 hours of class time, but I will refine their lan-
guage; and when you come to my class after 2 hours you
will think that they know how to speak of geometry. You
won’t know how it has been achieved because we have only
worked on the awareness of these few seconds of film and
have recognised that we want to talk only about this, we
don’t want to say anything else. And this is the discipline
of the verbalisation of what is seen, so that the words are
triggered by the images, and not by invention, or by
fantasy, not by anything that I could add if I wanted.

D W. but which would not be part of geometry as the
mathematician sees it and understands it.

C G. At another stage more will be required, but not at the
stage of introducing students to the beginning. What is the
beginning? We go back to Hilbert, who has been more
concise than Euclid — who also had the same beginning,
but in his intuition and in his classification of what
thoughts we need to have as separate thoughts before we
begin: that is, that there exist planes, straight lines, points.
Then if we link them, we say we are concerned with axioms
of connection. And so, having learned from what Hilbert
told us, and having learned our lesson from the fact that
children come to this with their perception highly de-
veloped, but not necessarily the logical machinery, I give
them an experience of this. So the beginning of the film
contains planes through half-planes because I want to in-
trouduce the straight line with the plane.

Although the plane is not visible, the screen is visible and
the line is visible. Through the artifice of photography I can
insist that what we are looking at are planes: half-planes
and planes. And then I make dots appear. Now these dots are
blobs, they are big things, they are not mathematical dots,
but I don’t care about this. All I want to show is that there
are dots on the plane. And lots of them. The dots are in the
plane because they appear on the screen. As soon as dots
appear I can give another axiom of connection, that two
dots determine a straight line. So straight lines, planes and
points have been ensured to exist in the minds of our stu-
dents at the level of intuition, of imagery.
And now we want to develop geometry. That few moments that they see, that they look at, when they see that *any* two points generate a straight line, well, they can't deny that this happens on the film; and we also take it as a statement that will be true from now on between us. Every time we have two points, we can say there is a straight line, and only one. Now, since the lines are in all directions, it is understood that the decision to separate these things makes it possible to ask the students, “If there are more dots, like those you see there, but not on the screen, but I tell you that they are there in the plane, what can you say? Can you see anything of the line?” Maybe If it’s on the screen. On the screen one may see the line and not the dots. So the viewer can shift from actual perception to virtual perception and therefore establish that there are as many straight lines as he wants, and that every pair of points on one straight line can be replaced by another pair of points on the straight line — which is not necessarily explicit in any geometry book. You can take the lesson to make them come to the conclusion, “Whether I see them or not, I can place as many points as I want on a straight line I just need to think of them. And if I look at the plane, I can see it by itself, or with lines, or with dots, or with dots and lines; and I have now the levels of discourse that I want to entertain as a geometer. Sometimes I will want to talk of the straight lines only, sometimes I will want to talk of the plane, sometimes only dots, or I will want to have all three together.” And because of this, the foundations that are on the film are incomparably larger than any sequence that the film actually presents. You can stop it sometimes and say, “There we have a fork; we could have followed this line or this line, but we chose to follow on only that line.” So you get an arborescence of possibilities from the moment you let the film guide you in what you want to do. And by using the lesson essentially for such awareness, rather than for assimilating the content of the film, you find you will get that too at the same time.

**D W** Some of us think of, perhaps a lot of us think of, geometry as not only something to do with the experience of space, but also as deductive; and you say yourself that you’re doing an axiomatic approach here, so that there are some axioms. I can see how you are giving the students material out of which some axioms can be formulated, and some definitions.

**C G** Yes, but axioms and the axiomatic method are two different things. I assume that the students can deduce, can produce a deductive sequence.

**D W** You assume that they come with this?

**C G** I can tell them that if they are out and I am in, they know they could say that they are not in and I am not out: two ways of stating the same fact, by negating or affirming. They know that if I am out and you are out, we can shift from this to: we are out. Which is another statement than the ones already said. So I will take out of their experiences all these pairs of statements that become a third which is different from them but implies the two.

**D W** In a way this is some kind of a restriction on the way that we ought to talk about some of the things that we see in the film. Now, I may again be asking a question that is not particularly important, but you referred earlier to the “logical machinery” that is required. And it seems to me possible that this also is something that the students come with, but I don’t know how the film can build on that.

**C G** The film can’t. The film has only one purpose. And that is to make the viewer have the images for the sequences of words that the teacher may say. Without the teacher this film is totally not usable, whereas this is not true of the “Animated Geometry” films which take a topic and handle it through a certain dynamics. Here there is a guide, there was a guide to begin with.

**D W** A distinct sequence, a distinct structure.

**C G** It is one axiomatics of geometry that I am concerned with, one which is intuitively easier than Hilbert’s, and one that is acceptable by mathematicians as one that will one day be refined and replaced by a more exact one in which the intuitive content will be assumed without being talked about. In this case I wanted the intuition to be present.

**D W** Yes, because that’s where you start from.

**C G** But it doesn’t mean that anybody sees today what it will be possible to do with youngsters who have developed in their early years the proper imagery through being exposed to mathematical films, and who one day will be talking with each other of these things as they talk about a butterfly on a flower that they’ve seen on coming to school. Now we don’t know, because we are under the influence of the social authority of the mathematicians, of the logicians, of the administrators, and so on; and we who operate between them as teachers are trying to have peace with the mathematician who says we make blunders, the administrator who says we don’t deliver, and the students who show that they can handle the language of mathematics as well as the best in the past — or as the good ones in the past. So it is a confused situation today within which we are working, but I can say we’ve put an instrument in the hands of educators, we have made it possible for them to get results without knowing why. If one day we do know why, we may come back with suggestions for teachers and for everybody, that because of this-and-this that we have achieved through the medium, we can start presenting questions differently, we can recast the curriculum in such a manner that we ask of seven year olds what we dare ask only of fourteen year olds today.

**D W** The teacher doesn’t necessarily understand how it works, or why it works; but the teacher has to use this tool, has to use this instrument. Have you any notions as to how a teacher readies himself to use an instrument which may be completely different from anything that he has tried to use before?

**C G** Yes. The job that I see for myself, as a teacher of teachers, is to show them that what I have obtained from them, with them, when we are working on the situation, is what they can obtain from their students. This doesn’t depend upon having taken courses in mathematics, doesn’t depend on having assimilated the vocabulary of modern
mathematics but only being prepared to be disciplined, to bow to the discipline of saying only what the situation permits one to say.

I take as an example the last bit of the film which is the Pythagorean theorem, as they call it in America. In Choquet's paper there was an algebra. He introduces the cosine, and out of the cosine he produces an algebra, then he obtains the Pythagorean theorem. And I know that algebra can't be put on film — at least, I can't do it. Therefore I had to make a device by which I would transfer the algebraic ideas into a geometrical situation. And I began by showing that if we start with a right triangle — which is possible according to the axiomatics that is in the film since we know what a train of segments is, and we know what a train of three segments is, when it is closed, and we know what perpendiculars are; so we know what a right-angled triangle is, and we know what an altitude is — if we have a right triangle with a perpendicular, with an altitude from the right angle, the altitude divides the hypotenuse into two segments, a small one and a big one. I take the right triangle which is on the right and I make it turn by $180^\circ$ around $BC$ as an axle, then I make it rotate clockwise around the vertex $B$ until its angle is on top of its original position. Now one of its sides is parallel to $AC$, which demonstrates the similarity. We can repeat this on the other side. And again, because of the rectangularity, we get parallelism, we get a parallel line there. Now, this is how I want to introduce the cosine that Choquet introduces algebraically in the formula.

(Figure I)

Now we can begin by turning the small segment $BD$ counterclockwise through a right angle, and sweep it along $BC$ producing a rectangle. Now this rectangle has a certain area, which is the product of $n$ by $c$, it's $nc$, if $c$ is the length of $AB$. Now if you take this rectangle, it is equal in area to the square on one side; and that's what I want to convey. I can't write a square until the notes, with the algebra, have been introduced, but we have one relationship: the square of one side $a$ is equal to $c$ times $n$, and the square on the other side is $c$ times $m$. So now if you add them you'll get $c$ times $c$, and therefore you obtain your algebra given all the imagery that's required there, and I only did it this way because I can't do algebra on the film.

It may be objected that this isn't a proof of Pythagoras; and it isn't a proof that I want to use here. I don't want to prove Pythagoras. I want to get a property through the medium of vision. And the fact that I use a square and its area is well, people, may say, "We don't know well enough what area is." But intuitively we're going to use it since we have defined the square on the hypotenuse in writing "the square on the hypotenuse equals the sum of the squares" as a numerical relation. So we could say, for the square on the side, "I'll draw a square" — and we know it's a square because we studied rectangles, we studied parallelograms, in earlier parts of the axiomatics. So squares will be special rectangles, or special parallelograms. And to each of these squares we associate a number which we call its area. And we haven't actually said that the area of a rectangle is such-and-such, we haven't passed to $\mathbb{R}$, the real numbers. We've done it without reference to the numerals, although we have established in earlier parts,
through Thales, all the work with the rationals which we require [3].

So there are gaps in the construction, but gaps which teachers of mathematics can fill. Through discussion they can come to the point of seeing that they can use this sequence in order to get the algebra on the blackboard. Once the algebra is there, we put aside all these things that we were saying and we construct the axiomatics that Choquet wants. But we will know that we’re talking of a square when we put a little 2; we’ll speak of a times a. Now if we have all the reals on the straight line this follows. If we don’t have all the reals, it only follows for that core of numbers that we have been able to place on the straight line through the measure which has been studied earlier. We use a great deal of time in part of the film to make sure that the viewers know that on a straight line one can put two segments one after the other and define the segment which is the sum of the two segments. And if we take a segment and repeat it instead of adding a different one, then we have a multiple of a segment. There we can define these things and, through Thales, define all the rationals on the straight line. So we have all this at our disposal. But we have other points than the rationals and now we want to introduce them, intuitively, by accepting the continuum as a given, and we’ll struggle with the details later.

D W This is presumably something the film itself can’t solve, can it?

C G If you use the film, you imply the continuum, and you need to make use of it. But if you want to, you can make it explicit; so you can say that we have through the film a foundation of fluxions, therefore Newton’s calculus is there (since we can say “the point describes a line, the line describes an area”, we can use these things as a matter of course.) To become aware of them will keep the students in a state of alertness. “In sweeping does it really take all the possible positions?” Well, intuitively, yes. Whether all this can be obtained in discussion with the youngsters is not yet known.

So what I’m talking about is no longer the film itself but how one can have a dialogue with teachers and make them examine the content of their own minds in a manner which is there for them to perceive. And if it is a prejudice of mine that algebra can’t be put on film, as such, someone one day will come up with a proposal for it. But if it is true, then we shall have to do the kind of thing that I did at the end, which is to invent the visual language for the algebra.

D W I’m still a little puzzled about one particular thing. You talk about the discipline of discussing the film, of talking about what is actually there. You also, I think, said that teachers didn’t necessarily need to know the language that the mathematicians use. I can’t quite understand how this could be so.

C G They know some language. They know the language that they come with, and I don’t know whether they have thought about what is the best system of axioms with which to start geometry.

D W Oh, I’m sure they have not, in general.

C G If we start with this, I could say they don’t have the language of the mathematician; but I can make them recognise that this sequence yields what they know in geometry. Had we chosen another sequence, it would still yield what they know in geometry.

D W I accept that. But I was thinking that the teacher is going to have to be in the situation of being with a group of students and being able to enter into a dialogue with them, and produce something with them that, as you say, the mathematician will recognise as sounding the same as the thing that he does, that he talks about. I’m still not quite clear how the teacher who may not have all of the mathematician’s knowledge, all of the mathematician’s vocabulary, and so on, is going to be able to know that he’s doing this in a reasonable sort of way.

C G Well, I can’t say that this is going to happen, that they will do it in a reasonable kind of way. What I know is that it is possible to sharpen their awareness of the problems so that they can sharpen the awareness of children. And since it’s the children who will be the proof that they’ve done it, not themselves, if they manage to give problems to children who, because of the equipment they have, can throw them up and bring them down broken, with the solution visible, and then write the solution, they will have their students working like mathematicians. And, indeed, I think that’s feasible even for a clumsy teacher who is disciplined enough not to want to make the other like him, but to give him a chance to learn what techniques there are within reach to use for sorting out this question or that question.

For instance, when Thales is run in the film, I also took the liberty of showing the mathematicians something they hadn’t seen, which is that it is easy to divide a segment into two parts through the axioms but it’s impossible to do it in three parts through the axioms. Unless you have the idea — which is the importance of Thales in the foundation of geometry — unless you know that you can put three segments one after the other, and that if you do that you can draw parallels from the endpoints, then you don’t know that you can provide a division into thirds. (If it’s in thirds, it’s in any number.) I knew that if I’m thinking of students I have to give them the truth, and the truth is that you can only divide a segment in two. You can have 2, 4, 8, 16 parts, as far as you want, but you can’t have 3. In order to have 3 you need another foundation, and it is this foundation that I give in the film. And so I take about two or three minutes of the film to do Thales in my way. Although it is in the notes that we have to introduce Thales at this point, Choquet and the others were not sufficiently alert to see that, look, that question has not been solved. Because for them it was the case that you can divide a segment in three parts; which you can’t. There are no means except by using Thales.

D W Was it thinking about making the film which made you aware of this?

C G No. You can see it in the things I did in England in 1945 or 47 or 48. But it is true that, you see, if you are only concerned with addition, subtraction and multiplication, you can do all of it on the straight line. You can do these
things; you can have a segment taken out of another segment, you can add two segments, you can take a multiple of one segment. But to divide one segment, you can’t. So this is really one of the walls that had to be surmounted. A genius had to come in, and Thales was the genius who did. I was never told at school that he did this. I was told, “Thales: Take two straight lines. On one you have a segment that you want to divide, on the other you put one segment after another, and you join, and then you have divided.” Of course, that’s Thales’s theorem. But what mathematicians do is to ask what are the foundations for this to be made obvious? So we find we have a theorem rather than a definition, which in fact it ought to be. It is the creative moment, to have thought of putting them one after the other.

D W You were addressing a particular problem, making this film, as you explained. This is a film which will fit into a particular situation in France. It has shown that it is possible to put an axiomatic approach on film. But there are many countries which have virtually abandoned entirely any deductive approach to geometry and where the only kind of geometry that is in the curriculum is informal geometry, sometimes described as intuitive geometry. I feel that teachers are very confused about intuitive geometry. Not that they don’t believe it exists, but because they don’t know quite what the content of this might be. Can you say anything about the film in connection with this more general question?

C G We know that geometry has been taken out of the curriculum; but it’s coming back. It’s coming back under different headings. And it’s not coming back because the French have loved it in the past. They were radical in their elimination of it. It’s coming back because the mathematicians who are not rabid algebraists have realised that they need some substance; and the substance is easiest to offer in the geometry course. There will still be a modern approach, and they’ll still be concerned with the axiomatic method that has been used with sets and the theories that have been developed in the last 50 or 60 years. It’s coming back because the basis of analysis in mathematics — which applies to physics, to the theory of potential, to nuclear models — needs some of these notions that are found to be difficult to give once you have polarised your mind so much towards algebra. What is obvious is that any teacher who brings in a diagram to indicate, say, the graph of a function, makes assumptions, geometrical assumptions. If you try to develop the branches of mathematics which are needed by the chemists, the physicists, biologists, and all the other people, you can’t do it simply on the basis of an abstract system. You have to give the imagery — which those who removed it from the curriculum had, but didn’t know they had. And they have made people poorer by not giving them that experience.

We don’t want to bring back Euclid. I certainly know that it is an approach which is too costly in terms of children’s time, and not good enough for the time they put in it. So I will recast all the geometry in small segments. But I would like the students to use the powerful instruments that have been generated over centuries and centuries and which are needed to do any mathematics. You can’t really get into topology without having reflected on geometry. You can’t do analysis without being a geometer, a good kind of geometer. So I am telling teachers that there are other ways of handling the matter. If you want to give examples of functions, you give examples that are perceptible. The book may say, “Take two sets and make pairs of them.” Well, if you take two axes and you take pairs of points you create a function. It’s the same thing. The content of the example is not what you will retain, but the way in which you handle it is what you will retain.

So it’s impossible to produce any system without an axiomatic Teachers who are not aware that they have implicitly made assumptions will be brought to this realisation in our courses. I shall tell them, “Now tell me what you do.” And as soon as I hear what they do, I will write down what they do, and I will say, “Has it been proved, or is it assumed? And how do you get from there to there?” Therefore I force them to the awareness that when you produce a model, for whatever it is, you are saying, “This is what I need to begin with; without this I can’t start.” And having started with this, and having these dynamics at my disposal, I can get more out of the beginning.

Every model is a reduced system which is sufficient to allow one to get new results. And so model-making, which is universal in all fields of thinking, can be illustrated in an easy way by taking examples out of geometry. And the Animated Geometry films are essentially that. They are the exploration of a notion, or a word, in terms of, “What can I say about the circles of the plane?” And that exploration then produces chapters of geometry.

D W But when you’re looking at an Animated Geometry film and responding to that, although it’s true what you say that one is making assumptions, and that in that sense there are implicit axioms, nevertheless that seems very different from having an axiomatic system for one’s geometry — because one could take one or two of the geometry films and just explore those. There isn’t any feeling that there is a sequence of necessary notions that one has to develop.

C G We have come a long way from the time, around 1900, when it was thought that we could put the whole of mathematics together as a system that would be deductive. It has been recognised that this is a futile activity. What we know is that mathematics evolves, changes with the injection of some definitions which are fruitful; and this is what we want to put into circulation with the teachers. An axiomatic system is only for the presentation of what I have found; I don’t begin with it; I end with it. So I’ll give experiences, I’ll give a lot of experiences to the people, and then the experience will build it in such a way that it is the least costly in terms of words and propositions and so on; so that if you know this, you know all the rest.” Why? Because the mind functions; and the principle of “if you know a little you know a lot” is pedagogically sound everywhere and we can use it in any foundation of mathematics. And whether there is a strict minimum for everything is of no interest any more, only to crazy people.