On the Foundations of Mathematics Education

WILLIAM HIGGINSON

In the fall of 1726 a book was published in London with the title *'Travels into Several Remote Nations of the World'* The author, described as, "first a Surgeon, and then a Captain of several Ships", was reputed to be one Lemuel Gulliver. Behind Lemuel and his fictitious journeys there was, of course, the brilliant mind and savage wit of Jonathan Swift. The fact that more than two hundred and fifty years later *Gulliver's Travels* can be read on several different levels with pleasure and enlightenment is a mark both of Swift's genius as a writer and the invariant foibles of human nature On his third voyage Gulliver pays a visit to the "Academy of Projectors at Lagado" where in their projects, "the professors contrive new rules and methods" with the intention of improving the lot of the citizens of Lagado. Unfortunately, however, "the only inconvenience is, that none of these projects are yet brought to perfection, and in the mean-time the whole country lies miserably waste, the houses in ruins, and the people without food or clothes". (p. 222)

It is clear that the target of Swift's satirical barbs in this section of the book is the Royal Society of London. Indeed, as later critics have pointed out, the apparently preposterous experiments described were, in many cases, ones which had been carried out by members of the Society. While Swift's contemporaries may have enjoyed a few chuckles at the time, at the "projectors" attempts to extract sunbeams from cucumbers, or to breed naked sheep, there is a sense in which the scientists have, in most ways, enjoyed a very long and hearty last laugh. The idea that, "all the fruits of the earth shall come to maturity at whatever season we think fit to choose", no longer seems ridiculous and there is more than a little foreshadowing of the computer in the description of a machine for artificial versifying.

From the section on one department of the Lagado Academy, however, it is not entirely clear just what progress has been made over the last two and one-half centuries.

> *I was at the mathematical school, where the master taught his pupils after a method scarce imaginable to us in Europe. The proposition and demonstration were fairly written on a thin wafer, with ink composed of a cephalic tincture This the student was to swallow upon a fasting stomach, and for three days eat nothing but bread and water As the wafer digested, the tincture mounted to his brain, bearing the proposition along with it But the success hath not hitherto been answerable, partly by some error in the* quantum *or composition, and partly by the perverseness of lads, to whom this bolus is so nauseous that they generally steal aside, and discharge it upwards before it can operate; neither have they been yet persuaded to use so long an abstinence as the prescription requires. (p. 231)*

On reading this one can imagine how delighted Swift would be in our age with programmed instruction, behavioral objectives and sleep-learning machines. But it also raises the critical question Why is it that, after more than a quarter of a millenium, mathematics educators are still at the stage of searching for the appropriate methodological "composition" and complaining of the "perverseness" of contemporary lads and lasses who seem to find their mathematical "boluses", or pills, quite hard to swallow too?

Before proceeding further with this question it is desirable to make explicit three assumptions about the nature, aim and efficacy of mathematics education which underlie the remarks which follow.

I: There are individuals who have, as a significant part of their professional responsibility, the consideration of and action on issues related to the acquisition of mathematical knowledge. These individuals, whose numbers include classroom teachers, curriculum writers, teacher educators and researchers, are "mathematics educators", and the discipline which embraces their professional concerns is "mathematics education"

II: The aim of a mathematics educator is to optimize, from both intellectual and emotional viewpoints, the mathematics learning experience of the student.

III: The mathematics learning experience for the majority of students has been neither intellectually nor emotionally satisfying; their exposure to mathematics has not been pleasurable, nor has it made them competent.

If these assumptions are valid, it would seem to follow that mathematics educators have an obligation to try to account for this state of affairs. Why should it be that so many children have so much difficulty in learning mathematics? The question is neither trivial nor easy to answer and there are many different responses which could be made. The position taken here is that we will not begin to make significant progress in dealing with this question until we more fully acknowledge the foundations of our discipline To return to Swift for an analogy, we have in this regard been like his, "most ingenious architect who had contrived a new method for building houses, by beginning at the roof and working downwards to the foundations, which he justified to me by the like practice of those two prudent insects, the bee and the spider". (p. 224)

Fundamental to what follows is the conviction that we have had an excessively narrow view of the factors which influence our discipline. We have failed to create any major, coherent theories or methodologies in mathematics education largely because we have ignored some essential aspects of its foundations. It is not possible in the space of a few pages to give a detailed account of this position. What follows is an outline of a framework for mathematics education, a staking-out of territory, the sketching of a tentative model. It is claimed that there are four dimensions to mathematics education and that by seeing the structural relationships among these dimensions in the form of a tetrahedral model we will be in a better position to understand what has gone on and what might go on in the future when learners encounter mathematics.

The dimensions of mathematics education

Any conception of mathematics education must be founded on the discipline of mathematics. The "What is Mathematics?" question which comes to the fore at this point is not one, as many books and papers will testify, which can be answered concisely. Nor is it an area without controversy. However, for our purposes here, we will take it that this foundation stone is an obvious and firm one. The issue at stake immediately is whether there is anything other than mathematics significantly involved in mathematics education.

This question is at the root of one of our most serious problems, the gap of incomprehension between mathematicians and mathematics educators. For it seems to be the feeling of some research mathematicians that nothing other than mathematics really counts in mathematics education. The classic statement of this view was made by G. H. Hardy in the context of his Presidential Address to the Mathematical Association in 1925. In the teaching of mathematics, Hardy claimed, "there is one thing only of primary importance, that a teacher should make an honest attempt to understand the subject he teaches as well as he can, and should expound the truth to his pupils to the limits of their patience and capacity". (p. 309) In reading this it is probably appropriate to recall that Hardy's educational experience consisted of student days at Winchester and Cambridge and teaching positions at Cambridge and Oxford. It is the sort of view which leads immediately to bad feelings between mathematicians and mathematics educators, since in the eyes of the former the latter are reduced to being non-productive and hence inferior colleagues. Hardy's long-time collaborator, Littlewood (1953), for instance, in his recollections of his first teaching post, observed that part of his duties was to lecture to pupil-teachers on "Principles of Mathematics", a task which he described as, "an 'Education' stunt, naturally a complete failure" (p. 80)

To anyone other than a few cloistered mathematicians, however, it seems obvious that there is a second fundamental dimension to mathematics education, the psychological one. Even from Hardy's conservative viewpoint, the importance of the mental abilities and interests of individuals is indirectly acknowledged. The teacher must "understand" as much of the subject as he can and it is recognized that pupils have limits to their "patience and capacity". Involved as he had been for so many years in the competitive examination system of the Cambridge tripos, Hardy would not have argued for the homogeneity of student ability.

The door is then open to the investigation of numerous questions about the psychological functioning of the individual in the context of mathematics learning. In the past mathematics educators have looked to learning theory and the literature on individual differences and motivation for many of their insights. In more recent times increasing awareness of the many mathematical features of general thought processes has meant that the rapidly growing sub-discipline of cognitive development has become the part of psychology of most interest to mathematics educators

One can contend that the battle for the recognition of a psychological dimension in mathematics education has been won, for almost all purposes, for some time now. The recognition of the role of social and cultural factors is, however, a process which is still ongoing. At one level this recognition is coming about because of increased sensitivity to the interpersonal dynamics of classrooms and the social role played by schools. Most of the teaching and learning of mathematics takes place within these complex institutions. At another level, the prevailing cultural values, economic conditions, social structure and range of available technology are seen to exert considerable influence.

The psychological dimension of mathematics education is mainly concerned with the way in which the individual attempts to learn mathematics. The social-cultural dimension deals with the influence of groups of individuals and their creations on this experience. It is here, for example, that the role of language comes to be seen as being very important

The argument to this point has been that mathematics education has its roots in the three relatively distinct areas of mathematics, psychology and sociology (with the last term being used in a somewhat unorthodox sense to stand for a number of "social sciences"). To some it might appear that the gates have been opened too far already, but even with this wide range of informing disciplines there is still one very large gap in the foundations and it is a gap which is particularly important because it is so infrequently recognized as such. All intellectual activity is based on some set of assumptions of a philosophical type. The particular assumptions will vary from discipline to discipline and between individuals and groups within a discipline. They may be explicitly acknowledged or only tacitly so, but they will always exist. Reduced to their essence these assumptions deal with concerns such as the nature of "knowledge", "being", "good", "beauty", "purpose" and "value". More formally we have, respectively, the fields of epistemology, ontology, ethics, aesthetics, teleology and axiology. More generally we have the issues of truth, certainty and logical consistency.

As in the case of the psychological dimension, there is in the philosophical dimension a particular field which is intimately interconnected with mathematical ideas. In this case it is the area of epistemology. From the time of the Greeks much of the discussion of the nature and limitations of human knowledge has taken place in the context of mathematical concepts.

Summing the parts: The MAPS-tetrahedral model of mathematics education

In the previous section it was contended that mathematics education is informed by the four disciplines of mathematics, psychology, sociology and philosophy. Intellectual hybrids as such are not unusual in today's academic world, accustomed as we are to physical chemistry, economic geography, sociobiology and the like In most cases, however, these marriages are binary and just from the equation, "mathematics education equals psycho-philo-socio-mathematics", we begin to get some idea of why our problems of long standing have not been resolved easily. There are several ways in which one might visualize the contributions of the four foundational areas to mathematics education. Perhaps the most powerful image is that of mathematics education as a tetrahedron in which the four faces are the four contributing disciplines. This geometric image, which is referred to henceforth as the MAPS-Tetrahedral model of mathematics education [M-Mathematics, A-Philosophy (arbitrary?), P-Psychology, S-Sociology], gives a more vivid picture than other (say of four parallel planes) images of the dynamic and interactive aspects of the model. The fact that the tetrahedron is closed may be one way of quickly perceiving the claim that the four foundational areas are not only necessary, but also sufficient, to determine the nature of mathematics education

To provide some form of test for this last claim we can look briefly at other ways of approaching a definition One test might be called that of the journalistic criterion or the five "W's". To pass this test any proposed framework for mathematics education must show that it incorporates features which will make it possible for adequate responses to be generated to the five questions, "What, When, Who, Where and Why" (According to journalistic lore these are the five questions which an editor will expect a well-written news story to answer We can also add "How" to the list for our purposes.) Using the MAPS model we would contend that the question of "What" concerns mainly the mathematical dimension, "Why" the philosophical, "Who" and "Where" the social component and "When" and "How" the psychological one.

In what might be considered a second preliminary test of the model we can observe that by removing mathematics from both mathematics education and MAPS, we are left with education in the one case and philosophy, psychology and sociology in the other. Traditionally these three disciplines have constituted the foundations area of education faculties, so in this way we see that our usage is consistent with this approach. The fact that so many mathematics teachers have found it difficult to relate studies in these areas to their personal concerns may demonstrate the weakness of trying to consider some aspects of the model in isolation from the rest.

Returning to a consideration of the model itself we note that one can accentuate either its continuous or its discrete aspects From a continuous viewpoint one can postulate the existence of some point of optimality which varies with time. The idea in this is that over time there are significant changes in all four of the constituent dimensions; new apparatus is invented, more mathematics is created, better understanding is achieved of human psychology, societal values change Hence if at time t_1 the best mixture of M, A, P and S produced an optimal position of p_1 somewhere in the interior of the tetrahedron, at some later time t_2 the point of optimality will have shifted. One can apply this image at either the societal or the individual level An immediate and very important consequence of this approach is that there is no one ideal mathematical education for all places or for all individuals in one place.

If one accentuates not the continuous aspects of the model, but rather its discrete or structural aspects, some different features of mathematics education are highlighted. In particular, one can approach the model systematically from a logical, combinatorial point of view Any set of four elements has a power set with sixteen elements Excluding the empty set and the set itself we have fourteen combinations left With reference to our tetrahedron these are respectively the four faces, M, A, P and S; the six edges, MA, MP, MS, AP, AS and PS and finally the four vertices, MAP, MAS, MPS and APS. From this structural perspective one can begin to look consciously for the influence of one factor on others Some of the interactive features of the model are recognized areas of academic work. The PS edge, for example, which represents the area of joint interest to psychology and sociology (what is the influence of the group on the individual and vice versa) is the active field of social psychology. In so far as it represents the merging of psychological, philosophical and mathematical ideas, much of Piaget's work on genetic epistemology can be seen as falling near the MAP vertex of the model

Applying the model

To speak, as we have, of the MAPS *model* is in one way to say nothing more than that the construct postulated has a particular structure; it is a geometric object. When considering the construction of theories there is another meaning of the term. This is the idea of the model as analogue. Analogical models are simplified representations of complex entities. The purpose of studying the simplified model is to gain insight into the working of the complex entity The power of an analogical model is a direct function of its ability to facilitate understanding. A good model reveals relationships which previously were unclear; it suggests new questions which prove, upon investigation, to be fruitful; it helps to resolve old questions.

In this section of the paper we consider the MAPS construct as an analogical model. Given the ultimate objective of activity in mathematics education, that is, providing an intellectually rich and emotionally satisfying experience for the learner, of how much use might the MAPS model be? Does it provide novel insights, generate new questions and clarify old problems? Bearing in mind that it is an unsophisticated construction, using as it does, only four large 'beams', it seems that it may function quite well as an analogue for mathematics education. We now attempt to substantiate this claim with a series of briefly-sketched observations which touch upon several different aspects of the discipline.

(i) *Understanding traditional rationalizations*. It was suggested earlier that mathematics educators have an obligation to try to account for the fact that many students have considerable difficulty in learning mathematics. There are several fairly standard responses which are made to this question. The MAPS model allows us to see how the partial truths of each of these responses may fit into a larger whole

Let us exemplify this with four different rationalizations selected from numerous possibilities. Response one: the Discipline Difficulty response; children have problems learning mathematics because it is a particularly recondite discipline. Response two: the Limited Ability argument; only a minority of children have the intellectual capacity to cope with any formal academic discipline. Response three: the Social Instrument argument; large numbers of children fail at mathematics precisely because that is the role mathematics plays in contemporary society, it is a filter. Response four: the Evolution argument; we are slowly understanding more about the ways in which humans acquire mathematical knowledge; considering that mass education is a relatively recent and very ambitious endeavour, we are making slow progress.

Consideration of the validity of these four arguments will take one into the four different dimensions of the model. The Discipline Difficulty and Limited Ability positions are firmly based respectively in the mathematical and psychological dimensions. The Social Instrument and Evolution arguments are both rooted in the social and philosophical areas of the model.

(ii) *Understanding what has happened*: The MAPS model can be of assistance in understanding what has happened historically in mathematics education. To pursue briefly two examples at the national level. At the turn of the century in Great Britain there was a very strong movement, led by John Perry, for the introduction of practical mathematics into the school curricula. In the nineteen-sixties in the United States the largest of the "new math" projects was the massively funded School Mathematics Study Group. The conception of mathematics portrayed in the SMSG materials was a very formal one.

Neither of these two examples of curriculum change can be fully understood without an awareness of the powerful interaction between the social situation of the time and a commitment to a particular view of the nature of mathematics.

It would be an interesting and fruitful exercise to plot the movements of the "centre of gravity" for the American MAPS tetrahedron from 1965 to 1980. There would be, for instance, a very distinct shift in the mathematics plane toward an applied view of the discipline. Psychologically there would seem to have been a near-complete cycle from didacticism to discovery and back again. Paralleling the changed social conditions and the movement in the mathematical dimension one could observe a shift in philosophical rationale from intrinsic reward to a largely utilitarian view.

(iii) *Understanding what might happen*. The temporal focus can easily be shifted and rather than looking backwards, one can use the MAPS framework to try to understand what the future may bring to mathematics education. Indications are that the dominating influences for at least the next decade will come from the social-cultural dimension. Educational systems will be high on the list of institutions which will be forced to make major alterations because of social changes related to "world problematique" issues such as inflation, population shifts and resource depletion. A second, and perhaps in some cases a countervailing force will come from the impact of advances in microelectronic technology. If, as some forecasters seem to feel, we are on the edge of an Information Revolution on the scale of the Agricultural and Industrial Revolutions, there is no doubt, because of the centrality of mathematical concepts to the idea of information, that major changes are in store for teachers and learners of mathematics. Concomitant with social change there will be shifts in philosophical values which will rebound through educational systems.

The vision of mathematics inherent in and fundamental to industrial society as it has evolved over the past century and one half has been centred on quantity. Reaching, as we are, the limits of this particular worldview, it seems likely that a conception of mathematics which emphasizes the qualitative-aesthetic aspects of the discipline may be of considerable use in constructing an alternative. There is now no doubt that "bigger is better" is a dangerous ground-rule. It may well be that small is not necessarily beautiful either. What we need is a well developed sense of balance; optimal *is* beautiful.

(iv) *Changing conceptions of research in mathematics education*: The structure of the MAPS model is such that it invites a broadening of the domain of research in mathematics education. Traditional work in this area has often met with a hostile reaction from classroom practitioners who have criticized its relevance to their concerns. At the root of much of this difficulty was an epistemological problem, a commitment to an outmoded and inappropriate model of scientific method. By involving a higher percentage of the mathematics education community in research activity and by expanding the philosophical and psychological assumptions which underlie this activity it seems possible that this aspect of the discipline might undergo something of a renaissance.

As well as recasting some classic areas of research in a different light the MAPS model also suggests the legitimization of some new questions. Among these are the investigation of some of the tetrahedron's edges. Consider the MS edge. Historically many thinkers have denied the existence of any relation here. Perhaps with a vision of some Platonic ideal forms in mind they have argued that mathematics is, in some fundamental way independent of and above social concerns. In the philosophy of science the question of the nature of objectivity has been of intense interest in recent years and it seems that some light may be thrown on the social roots of mathematics from this debate. The essential distinction would seem to be between the end product of mathematical activity and the processes which lead to this end product. While the former may be seen as removed from the influence of social factors, the latter most certainly are not.

A second quite fascinating tetrahedron edge is the one where the philosophical dimension meets the mathematical one. As Thom (1973) has pointed out, "all mathematical pedagogy even if scarcely coherent, rests on a philosophy of mathematics". (p. 204) Despite this there has been little done to explore the educational implications of different philosophies of mathematics. This is partially due to the stalemate which seems to exist in the common, three-schools (Logicism, Formalism, Intuitionism) approach to the philosophy of mathematics. It appears, however, that recent developments generated by the ideas of Lakatos may provide a basis for productive work on these foundational questions

(v) *Changing conceptions of teacher education in mathematics* · If, as was stated earlier, one of the serious gaps which exists within the wider mathematics education community is between mathematics educators and research mathematicians, a second is the one which separates teacher educators from classroom practitioners. In what appears to be a near-universal complaint, the teachers, and very often the teachers in training as well, claim that education programmes devote too much time to "frills" and not enough to practical skills such as lesson preparation. Fundamental to this communication problem is an excessively constrainted conception of the role of the mathematics teacher held by many practitioners. It envisages the classroom teacher, à la Hardy, doing little more than flashing mathematical facts at his pupils. Teacher educators, who will themselves in almost all cases have been classroom practitioners for some time, attempt to make prospective teachers aware of some of the wider concerns of the truly professional teacher of mathematics. One of the main reasons why they have considerable trouble in doing this successfully is that they lack a coherent framework in which to anchor these wider concerns. The MAPS model goes a considerable way toward providing such a framework.

With the existence of an overarching model established, preservice education can legitimately focus on those aspects of the model which account for the survival skills which are of paramount concern for student teachers. Inservice education programmes, which at the moment in many places seem to be piecemeal efforts, could then, perhaps, be devoted to the consideration of some of the less immediate aspects of the MAPS model

Conclusion

The thesis expounded in the foregoing pages has been that by visualizing mathematics education in terms of the tetrahedral interactions of its four foundational disciplines, mathematics, philosophy, psychology and sociology, we might make some progress toward resolving some long standing problems. From even the preliminary examination of the model which has been possible here, it seems that some interesting insights emerge. One which bears restatement is the relation of our discipline to others. As in other areas in life, it is possible to be so sensitive to the inadequacies of one's discipline that one can lose a sense of overall perspective. To say that we have major disciplinary problems which have resisted solution for a very long time, is by no means to say that we are in a unique, or even unusual position. A sampling of the professional literature of other disciplines, particularly in the social sciences and the humanities, reveals much talk of crisis and badly needed reorientation.

The self-image which we can legitimately carry to discussions with our disciplinary colleagues, therefore, is one of a group of responsible and concerned scholars looking for assistance with important questions affecting the day to day lives of large numbers of people. In exchange for such assistance we are in a position to offer support to many of them. Problem solving research in cognitive development and areas of artificial intelligence in theoretical computer science are examples of fields which could benefit from the active collaboration of experienced mathematics educators.

It is quite possible that the MAPS model will be shown to have little long-term use. It may be incomplete, logically flawed or of very limited use in generating questions or insights. This, in itself, is not particularly important. What is important is that we as mathematics educators begin to talk about who we are professionally, what our problems are and how we can begin to tackle them. If the MAPS model can stimulate this sort of activity it will have been of considerable value. If we maintain the muddled silence which has existed until now, the percentage of children who find mathematics classrooms confusing and frustrating places will not decrease.

Acknowledgements

This paper was written while I was on sabbatical leave (1979-80) at Cambridge University. Earlier versions were presented to colloquia at Cambridge, Milton Keynes (Open University), Nottingham and Warwick. I am indebted to my colleagues at these institutions for their warm hospitality and constructive criticism.

References

Hardy, G. H. 'What is Geometry?' Presidential Address to the Mathematical Association, 1925. *Mathematical Gazette* XII, 175, March 1925, (309-316)

Littlewood, J. E. *A Mathematician's Miscellany*. London: Methuen, 1953

Swift, J. *Gulliver's Travels*. Harmondsworth: Penguin, 1967 (1726)

Thom, R. 'Modern Mathematics: Does It Exist?' (194-209) in A. G. Howson (ed.) *Developments in Mathematics Education. Proceedings of the Second International Congress on Mathematical Education*. Cambridge: Cambridge University Press, 1973

NEXT

The next issue will contain articles by Stephen Brown, Herbert Ginsburg, Brian Greer, David Pimm, etc., etc.
