Mathematics and Being in the World: Toward an Interpretive Framework*

STEVEN R. WILLIAMS

The purpose of this paper is to outline a theoretical foundation for a hermeneutical analysis of discourse in middle school mathematics classes. This framework is being developed as part of a research program which attempts to study not only the social and cultural settings of classroom discourse but also the ways in which beliefs about mathematics are communicated, the ways in which mathematical structure is communicated, and the ways in which an appreciation of justification in mathematical discussion develops.

This paper grows in part out of my dissatisfaction as a mathematics educator with much of the world view which currently dominates the research and practice of our discipline in North America. Specifically, I feel that education, and particularly educational research, suffers by adopting the philosophical assumptions of the rationalistic tradition apparent in, for example, information processing psychology. At the same time, I recognize much that is intuitively appealing in constructivism and in the recent emphasis on societal and cultural influences as important aspects of the educational process. Much of what I find appealing in this work resonates with my knowledge of hermeneutics, and while it is certainly not my wish to use hermeneutics as an umbrella for diverse traditions and methodologies, I think that many educators will likewise find a great deal in the hermeneutic tradition which resonates with their beliefs. Thus one of my goals is to introduce this long-standing philosophical tradition and to explore its implications for certain paths we are currently pursuing, and others we are neglecting. Ultimately, I hope that this will lead to a framework for analysis of classroom behaviors, specifically, discourse among students and teachers on mathematical topics.

An example

The following except is taken from a seventh grade pre-algebra class, in which the teacher was introducing a method for solving equations. The teacher, Ms D, was using the problem $a + 8 = 17$ as an example.

Ms D: ...That's my goal, to get "a" on one side by itself, and all my numbers on the other side. Now, I have to get rid of adding 8. It's in the way and it's keeping me from having my magic equation. What can I do to get rid of adding 8? What will undo adding 8? Millie?

Millie: Subtracting 8?

Ms. D: Subtracting 8. Good answer. Bea, are you paying attention here? I said power listening here. Thanks. OK What will undo adding 8, again? Chris?

Chris: Subtracting 8?

Ms. D: Does everyone agree with that? Why will that work? Kurt?

Kurt: Because subtracting is the opposite of addition.

Ms. D: Exactly. Now there's just one rule I have to remember in algebra. Anything I do to one side of the equation I have to do to the other side. I call it Even Steven. If you do it to one side, you have to do it to the other side, otherwise it's not fair. When you look at an equation what you really have is like a balance. And what if I have plus 8 here and I subtract 8 here, what's going to happen to this side, if I don't do anything?

After a few minutes of this typical exchange, another typical exchange occurred:

Bobby: Wouldn't it be easier just to ..

Ms D: Figure it out in your head?

Bobby: Yes, . .You know, you can tell by yourself without having to write it down..

Ms. D: Good. I must have said this already this year, did I?

Anne: No.

Ms D: OK, he had a very good point. Why go to all that trouble, that is 6 steps, why go to all that trouble when probably I could look at $a + 8 = 17$ and know right off the bat the answer is 9? Why go to all that trouble? P?

Kelly: Because that won't work for all problems.

Ms D: We will have, if not this year next year, problems that can only be solved algebraically, and if you don't know the process you're going to be really out of luck. Would it make sense to show you the process for something pretty simple? And that is pretty simple, that's about as simple as you get. It's the process that I want you to know. Do I want you to follow the process? Yes. I'm pretty picky about it. I will decide as we go along that you can drop certain steps once I know that you understand the process. But I have to know that you understand what you're doing, and I'm going to insist that you follow MY steps. And the first step is.. what, Chris? Can you read it up there in the brown?

The above example serves to highlight several aspects of classroom life to which we will give some attention below. First, it is easily recognizable as an example of classroom talk; it follows familiar and specialized patterns usually found only in classrooms. Second, the discourse occurs on different levels, focusing at different times on formal mathematics, analogies to the physical world, the social norms of the classroom, and the discourse itself. In addition, it highlights students' and teachers' beliefs and *taken*

*Research reported in this paper was supported in part by grants from Washington State University and the National Science Foundation. Opinions expressed are those of the author and do not reflect those of the National Science Foundation or of Washington State University.

as shared [Cobb, Yackel, & Wood, in press] assumptions about both mathematics and life in schools. Third, there is a sense in which mathematical processes and understandings are described by actions and possibilities for actions. These strands are daily woven together into a complex tapestry that is the learning of school mathematics. It is the way in which this process is made meaningful to the participants that is the subject of this paper.

Hermeneutics

Briefly put, hermeneutics is the art of interpretation. What separates so called "modern" from "traditional" hermeneutics, and separates various factions within the community of hermeneutical scholars, is the view of what is interpreted. Traditionally, hermeneutics dealt with the interpretation of sacred or mythical writings, most particularly scripture. Through a long process of evolution in which Schleiermacher, Dilthey, Heidegger, and Gadamer all took part [see Palmer, 1969 for a brief historical overview], hermeneutics has moved from concern with the epistemological problem of coming to know the meanings of a text and the intents of its author, toward concern with the human being's lived experience of interpreting, and the interpretation of being human. Thus modern hermeneutics, that which begins with Heidegger, can be seen not as a method, but rather as a view of what it means to be a human being in the world. The problem of modern hermeneutics then becomes not so much one of methods of interpretation; interpretation is seen as our mode of being. Rather, modern hermeneutics is the study of the significance and implications of this interpretive viewpoint. It is this view of hermeneutics from which I draw for this paper.

Hermeneutics views human beings not as organisms in an environment, but as agents in a world. The "being" of humans cannot be divorced from "being-in-a-world", or in a context. We always have a *pre-understanding* or *horizon*; we find ourselves *already* in a world. This pre-understanding comes from our personal and cultural tradition, our history; it is not an "object out there," but is rather that which makes our knowing possible. As such, it cannot be made totally explicit. Humans' interpretation of experience is their fundamental way of being in the world. In this view, knowing becomes a process of interpreting our experience by way of our horizon. Understanding becomes attuned particularly to the possibilities we perceive within our world. We have a view of our personal possibilities by way of which the natural joints of the world become visible; we see the world as containing numbers as they are part of the possibilities we perceive.

Closely connected with pre-understanding is the concept of the *hermeneutic circle*. The hermeneutic circle refers to the inescapable fact that we understand new phenomena in terms of what we already know, but that what we know is altered and enriched by the new phenomena. We approach new phenomena with a sort of preliminary knowledge, or *fore-knowledge*, of the phenomena and what they might mean to us. Thus understanding is a dialectical process of relating phenomena to prior context, of relating part to whole. It is a spiral process of current understanding being enriched by experience and serving as the basis for subsequent interpretation. Here again, the possibilities we move toward help us to interpret the world as meaningful; our experience enriches our world and our view of those possibilities.

That all human understanding is understanding *in a context* is often described by saying that there is no absolute frame of reference from which other frames can be judged; there is no way to step back from our own pre-understandings and see the world (in particular, ourselves and our viewpoints) totally objectively. As we perceive and make sense of our world, we do so always already in a certain context and with a certain viewpoint. We discover ourselves already acting, already possessing an interpretive framework and a set of goals. As an example of this, Winograd and Flores [1987] consider chairing a meeting of 15 people; we could as well consider a teacher in a classroom. We observe that as the teacher steps into the classroom, she cannot avoid taking action, nor can she step back and reflect upon her actions without missing part of the action of the classroom. If she chooses to sit and let the class run its own course, she is implicitly acting. At the same time, she cannot control the effects of her actions, nor know with certainty how others will interpret or react to her actions. During the classroom experience, what she chooses to do is largely based on unconscious, non-reflective coping within a cultural context. After class, as she reflects upon the events of the day, it is possible to explore emergent patterns and develop an account of what (from her particular viewpoint at this particular time) went on. As she is engaged in the classroom, however, these representations and patterns are not stable, but instead ebb and flow, changing direction and importance as the actions of those present unfold. Moreover, even the later reflection and representation is based upon pre-understandings and is grounded in a nonreflective coping with the situation which calls for reflection. In summary, Heidegger suggests that we participate in our life thrown [1], as it were, upon our instincts and intuitions, not as detached and rational observers. Such detached and rational thinking do exist and are valuable, but are derivative upon much more basic, non-reflective coping within a context.

Aside from offering an alternative to the rationalist tradition of research which modern psychology, mathematics, and as a result, mathematics education has embraced, hermeneutics has a number of specific implications for the study of classroom interactions. I will address a few in the space remaining.

Engaged activity

From the hermeneutic viewpoint (especially as expressed by Heidegger), the existence of humans is typified by engaged [2] activity in the world. A human being "cares about its life and projects some plan for its life as a whole. As such a projection, it exists for the sake of certain goals" [Guignon, 1983, p. 96]. Thus *being-in-the-world* is an activity through which we know the world, and we know things in the world by virtue of what they do as part of that activity. Again, "... our goal-directedness and the practical contexts in which we are engaged are welded together into a unified totality. Our purposive agency is always directed and under way. This goal-directedness lays out conditions of relevance for the equipment we encounter." [Guignon, 1983, p. 97]

Thus human beings have primary engagement with the world as they go about their business in the world. Tools and objects are part of a context of everyday activity, in which human beings are engaged. Packer [1985] identifies three *modes of engagement* in the works of Heidegger. He makes the distinction between an object's being *ready-to-hand* (i.e. handy, able to be used in the ongoing activity), or *unready-to-hand* (i.e. not handy, but still viewed as part of ongoing activity), and its being *present-at-hand*, or merely present in an abstract sense. To use Heidegger's classic example, we have primary access to a carpenter's hammer through the act of hammering, not by detached consideration of the hammer. As we hammer, we are not aware of the hammer as such, we are aware of hammering; the hammer is ready-to-hand, which for Heidegger is the basic mode of engagement, the way in which we perceive most of the world most of the time. Only if the hammer breaks, or we encounter a bolt rather than a nail, does the hammer come to our attention. It then becomes unready-to-hand, but is still seen in the context of our overall goals and the activity in which we are engaged. It is still part of the *manifold* or ensemble of objects which are part of our overall activity. As we divorce the hammer from the context in which we are engaged, we can consider the hammer more scientifically, and may speak of the hammer's properties, its weight, shape, and so forth. But in this act, the hammer has been pulled from its context of action; it is no longer ready-to-hand, but becomes present-at-hand.

For one example of these distinctions, consider again an experienced classroom teacher. As she is engaged in teaching a lesson, she is thrown into the activity and acts in terms of everyday, culturally correct ways of understanding and responding. She discusses problems, maintains order, asks and answers questions, and effects transitions in the lesson without conscious effort being directed to her teaching. Her teaching is, for her, transparent. For the most part, she doesn't stop to consciously reflect upon her actions, and can act without conscious goals or intentions. Her fundamental mode of engagement is thus the ready-to-hand mode. At any time, a breakdown can occur, in the form of an extreme behavior problem, a student's question which transcends the norms of classroom dialogue ("Why don't you ever answer our questions?"), or making a mathematical mistake. The teacher then becomes aware of her actions; she can examine ways of resolving the breakdown, but always within the context of the overall activity of teaching the lesson. Here, the mode is that of the unready-to-hand; there is a need for conscious reflection upon actions, and for resolution of the breakdown in a way which is contextually meaningful. Finally, the teacher can later view videotapes of the teaching episodes. Here, she can theorize about her goals, plans, and strategies; she can make hypotheses about student understanding or the meaning of students' questions; she can analyze the breakdowns and make plans for dealing with them in the future. In this present-at-hand mode, the teacher moves farther from the context, and deals in more abstract ways with the happenings of the classroom.

Another example of the ready-to-hand mode might be the relative ease with which most adults deal with simple arithmetic problems involving, for example, the counting of change. Most adults become aware of arithmetic processes only when forced to by some breakdown: for example, when the merchant with which they are dealing disagrees with the results. The arithmetic then becomes apparent as both merchant and customer attempt to arrive at the same answer. Should the event attract sufficient attention, the participants might later give some thought to how they typically *do* arithmetic, and enter the present-at-hand mode, searching for general patterns which emerge across several instances of using arithmetic.

If we take the ready-to-hand as the primary mode of engagement for human beings (as Heidegger does), there are several implications for the study of discourse in mathematics classrooms. One important implication for the analysis of classroom discourse is that as researchers, our primary participation in classroom life is by way of it being ready-to-hand for us, and that our understandings change as our engagement changes and as the happenings of the classroom become present-at-hand. That is, we understand happenings in the classroom primarily through our participation in a common culture in which that classroom is a part. Our fore-understanding of the classroom, upon which all our other understanding is built, is precisely that which would allow us to operate in a ready-to-hand mode within that classroom. It is exactly that which would make us able to enter the classroom and perform within it sensibly. Thus our ability to understand classroom activity is based primarily upon our own informal, activity-based abilities to interpret, that is, our own horizon. From a hermeneutic viewpoint, this is an inescapable aspect of any attempt to study human activity, and should be borne in mind both as we attempt to make sense of what we see, and as others attempt to make sense of our interpretation.

A second implication is that, for better or worse, all mathematics is viewed by students in the context of *their* concerns and engaged activity. Mathematics, when students encounter it, already has an involvement in their ongoing activities and understandings. This is one interpretation of what we mean by saying that mathematics can be seen as meaningful by the student, which is by no means a new idea. Other scholars [e.g. Romberg, 1992] have implied that to know mathematics is to do mathematics. Lave [1988] has addressed in detail the imbedding of cognition in practice and the "direct, persistent and deep experience of whole-persons-acting." A hermeneutic viewpoint, however, would give practice, or engaged activity, a position of primacy in our efforts to understand classroom life. The connection of mathematics with life activity is not a result of successful education, but rather a fundamental mode of being. In Heidegger's words, "In interpreting, we do not, so to speak, throw a "signification" over some naked thing which is present-at-hand, we do not stick a value on it; but when something within-the-world is encountered as such, the thing in question already has an involvement which is disclosed in our understanding of the world, and this involvement is one which gets laid out by the interpretation" [Heidegger, p. 191]. We encounter the hammer as a tool for hammering when our world is one where we need to hammer; we encounter mathematics as useful when we need to use it.

This is not to imply that mathematics has only a pragmatic element, or that instruction should somehow be tuned solely to the concerns of the student. Indeed, a major part of education seems to be the widening of students' horizons and concerns so that more mathematics is seen as meaningful. This will include an appreciation of the mathematical in its formal, present-at-hand sense. This view of mathematics is powerful and useful, but it is derivative from the everyday, ready-to-hand understandings and possibilities that emerge in going about our business in the world.

Closely related to this is the fact that as we observe communication about mathematics in the classroom, we must remember that it is the activity of *the classroom* which we are observing, and to which we have access. This is not to say that such activity has no legitimacy, or less legitimacy than mathematical discourses which might occur outside of class. It is simply to state that the communicative acts which we observe are part of the context of the classroom, and have meaning based on the activity of the classroom. A hermeneutical viewpoint insists that mathematics considered as a part of concernful school-based activity is fundamentally different than mathematics in non-school-based practice. Thus, even though we may agree to view mathematics as being grounded in the ready-to-hand, actions interpreted as mathematical in one context may not be seen that way from another context. Arithmetic computations viewed as mathematical in the classroom, may not be so viewed on a construction site. Everyday algorithms used by Lave's *just plain folks* are not always valued as being mathematical by the more sophisticated [1988]. By the same token, we must bear in mind the difference between mathematics as ready-to-hand and mathematics as present-at-hand. We sometimes forget that mathematics separated from specific concernful activity (i.e. present-at-hand), is very different from the mathematics of practice, just as studying the properties of a hammer is very different from hammering itself. Thus we come to the conclusion that what we study in school is almost inescapably school mathematics. It need not be considered less valuable for that, but we must remember that the mathematics cannot be divorced from the context in which it is used. We should also remember that to suggest that mathematics should be learned in general and should "transfer" to other situations is to fail to understand the intimate connections between understanding and practice.

In short, the perceived division between "school" mathematics and "real world" mathematics is not a delusion under which our students labor. Rather, it is an admission of the importance of activity and context to the understanding of mathematics.

Thrownness and communication

In the sense of being thrown which was discussed above, students, teachers, and researchers are thrown together in the classroom, and are able to make sense of what goes on there because we share, in some sense, a common pre-understanding of classroom life. This is due in part to our common cultural and historical heritage, and to our common experiences in school: we share a common language. What other researchers have called scripts or routines [Leinhardt, 1988], which are employed by the teacher to maintain the order of the classroom, are a large part of this common understanding. And, so long as the scripts are not violated, our engagement of the classroom is ready-to-hand: the classroom and its unstated rules are part of the largely unnoticed background. However, we are all aware of times when there is a *breakdown,* when we are suddenly forced to notice what was part of the background, when the ready-to-hand becomes unready-to-hand or present-at-hand. The established patterns of classroom discourse function in this way, as part of the common pre-understanding, until there is a breakdown. Such a breakdown may result in a question, often procedural, or in confusion and dissolution of established norms of behavior. When this happens, it is often a primary function of discourse to negotiate and establish new shared understandings. It is expected that similar breakdowns can occur in the context of mathematical activity, and that during these breakdowns, mathematics is seen as present-at-hand, an object of discussion, decontextualized. This is similar to the distinction which Cobb, Yackel and Wood make between "talking about and doing mathematics" and "talking about talking about doing mathematics" [Cobb, Wood, & Yackel, in press]. It is a primary goal of the current research project to determine how these breakdowns are related to the learning of mathematics, as well as determining what students learn as a result of talking about mathematics (whether as ready-to-hand or as present-at-hand) and what it is about that talking which gives rise to understanding.

A second example

Much effort has been made recently to close the perceived gap between school mathematics and "real world" mathematics. This is often done by attempting to create a community of learners which in some way approximates the community of expert users of mathematics. Like their expert counterparts, the students in the class have the responsibility of making mathematics meaningful, communicating that meaning to each other, and developing the norms which surround its use. The work of Lampert at Michigan State University or Cobb, Wood, and Yackel at Purdue represent exemplary programs of this type. However, our work has suggested that the real world, school mathematics, and so-called real mathematics interact in unexpected ways.

In a recent class in our project, for example, students were given the following problem:

> *In a drawer there are four blue socks, ten black socks, and ten brown socks. What is the minimum number of socks a person must select in order to assure having two socks of the same color?*

In the following brief section of dialogue, aspects of reality and classroom mathematics are interacting with the work of the classroom:

Ms D: Now, Lance brought out an interesting point yesterday: why doesn't that person just open up his eyes, or her eyes, and look at the socks in the drawer? Well, if that happened, then the mathematicians wouldn't have any word problems, and also, people do get dressed in the dark. All right, how did you think about it?

Here a student went up to the overhead projector, and provided a mathematically accurate analysis of the problem, stating that "on the fourth draw you would get a pair." Following this, Lance raised his hand.

Lance: Usually when you have socks in a drawer, aren't they folded? So that you'd ...

Ms D: First off, you're pretty darn lucky. Some people just throw the socks in the drawer and let the person to whom the socks belong sort them themselves. And some mothers I know even just put all the clean clothes on the bed and let everybody sort out what they need. So if your mom sorts your socks you're lucky.

In one sense, this is a fairly typical scene of an irascible student "mixing it up" with a teacher. But what does it say about the life of the mathematics classroom, and what does this conversation between Lance and the teacher accomplish for each? Does the problem really represent a breakdown in the norm of the classroom for Lance, or is this question part of a ritual which allows both Lance and the teacher to fulfill expected roles? What is the concernful activity in which each is engaged during this exchange? Although answers to such questions are always open to debate, our conclusions in this case suggest a complex interplay between shared understandings and breakdowns. The fact that the teacher herself brings up the question of how "real" this problem is, and the ease with which she deals with this problem (independent of how effectively we might feel she deals with it) indicates that the background in which both she and the class operate includes a willful suspension of disbelief about reality and mathematics. Thus mathematicians referred to by the teacher (not students and not teachers) create mathematics problems, and they do not have to make sense in terms of everyday experience. At the same time, the teacher spends a considerable amount of time justifying certain aspects of the problems, showing that it is at least conceivable that people might get dressed in the dark, and that some people do not pair their socks. This would seem to suggest that problems are expected to make some sense in terms of a certain set of rules for the mathematics classroom. they must be within an unspoken realm of possibility for story problems.

Thus, the conversation played the role of establishing for the class the norms for reputable story problems. Lance perceived a breakdown, not of classroom norms, but of his view of mathematics as ready-to-hand. He sees the world as it relates to mathematics in a different way than the teacher. It was in all probability a breakdown which he and other students had experienced before. Yet the mathematical breakdown was subsumed in the business of the classroom as usual; the conversation was an expected and accepted way of establishing norms for classroom mathematics, as distinct from activity outside the classroom. It is clear that much of the success of programs designed to bridge the classroom-to-real-world gap depends on the interplay between mathematical breakdowns and pre-understandings of classroom life.

Later, during the same period, the teacher made what might be viewed as a well-intentioned effort to shift the responsibility of determining the sensibility of actions and answers to the students. Here again the unspoken assumptions about the interplay of mathematics and classroom life affect the conversations. Students were given the problem of evaluating $3x^2 + 5x - 1$ if $x = 7$. A student came to the overhead and eventually arrived at 181 as an answer. She asked if the answer was correct. The teacher, wanting at this time to keep the responsibility of determining correctness of answers with the students, responds with "Well, I don't know. Who can go to the overhead projector and check it?" Louis came to the overhead and worked the problem again, evaluating $3x^2$ as $(3x)^2$, and obtaining 475 as a final answer. The teacher then asked, "Who has a different answer? Mark? Go up there [to the overhead].

Mark: [Explaining at the overhead] "OK, when you're doing algebra, see, you always do the exponent first, so you do x times x.. x squared, first, so that would be 49..

Ms D: Who says?

Mark: Me.

Ms D: OK, I guess it's Mark's rule that you do exponents first, can anybody add to that, or agree with that or disagree with that, and why? Nancy?

Nancy: It's not Mark's rule, 'cause everybody tells you you have to evaluate exponents first, and then do multiplication and division, and then ..

Ms. D: What do we call that?

Seth: Order of operations.

Ms D: What does it mean? When do we use it?

Mark: In algebra

Here, the teacher appears to make some effort to suggest that students are responsible for establishing mathematical meaning and standards for its use. In fact, it is not clear from the events in the preceding three weeks that this was her full intention. Nevertheless, the students perceive a slight breakdown in the pre-understandings about who does mathematics and who sets the mathematical rules. The students appeal to the ever-present "everybody", the doers of mathematics (consisting of parents, other teachers, perhaps textbook authors) to restore business as usual. Students learn rules, rather than set them; and efforts from teachers like the one in this scene, are perceived as pedagogical tricks which the students are supposed to "pick up on" rather than as legitimate efforts to establish a new order of mathematical activity in the classroom

Summary

I have touched very briefly on two examples which in general terms discuss beliefs held by teachers and students about school mathematics. I have suggested that such beliefs can profitably be seen as part of the pre-understandings which make comprehension possible, and display themselves to our view primarily when there is a breakdown in this pre-understanding Such a framework keeps us aware of the difference between the view of classroom life as it appears to participants and as it appears to detached observers. It holds promise for a useful interpretation of discourse in classrooms.

Notes

[1] This concept is most often translated as *thrownness*, and refers to the fact that humans find themselves always *already* in a certain mood, with a certain viewpoint, acting in certain ways, and believing in certain things.

[2] The adjective *concernful* is often used to describe human action, in order to emphasize the degree to which humans' activities are made meaningful by their concerns. Here, *engaged* activity is that which reflects this concernful coping with the world.

References

Cobb, P., Wood, T. & Yackel, E. [in press] Discourse, mathematical thinking, and classroom practice In N. Minick, E Forman, & A Stone (Eds) *Education and mind: institutional, social and developmental processes.* Oxford: Oxford University Press

Guignon, C.B. [1983] *Heidegger and the problem of knowledge.* Indianapolis, Indiana: Hackett Publishing Company

Heidegger, M. [1962] *Being and time.* New York: Harper and Row

Lave, J. [1988] *Cognition in practice: mind, mathematics and culture in everyday life* Cambridge: Cambridge University Press

Leinhardt, G. [1988] Expertise in instructional lessons: an example from fractions. In D.A. Grouws and D. Jones (Eds.), *Effective mathematics teaching* (pp. 47-66) Hilldale, NJ: Lawrence Erlbaum Associates

Packer, M.J. [1985] Hermeneutic inquiry in the study of human conduct. *American Psychologist*, 40(10), 1081-1093

Palmer, M.J. [1985] *Hermeneutics.* Evanston, IL: Northwestern University Press

Romberg, T A [1992] Problematic features of the school mathematics curriculum. In P.W. Jackson (Ed.), *Handbook of research on curriculum* (pp. 749-788) New York: Macmillan

Winograd, T., & Flores, F. [1987] *Understanding computers and cognition: a new foundation for design* Norwood, NJ: Ablex

---

(From Greenleaf's Intellectual Arithmetic, 1869)

13. If the sum of two numbers is 5, and their difference is 3, what are the numbers?

ANALYSIS — *As the greater number is* 3 *more than the less, their sum, or* 5, *must be* 3 *more than twice the less; hence twice the less must be* 5 – 3, *or* 2 *and the less number must be* 1/2 *of* 2, *or* 1 *The greater number is* 1 + 3 *or* 4

NOTE — *In the same way* 5 *is* 3 *less than twice the greater, and the greater must be* 1/2 *of* 5 + 3, *or* 8. *The sum of two numbers added to their difference always gives twice the greater and their difference taken from their sum gives twice the less*

Benjamin Greenleaf
