On Equivalent and Non-Equivalent Definitions: Part 2

ROZA LEIKIN, GREISY WINICKI-LANDMAN

Introduction

One of the central components of mathematical knowledge is defining and one of the main problems of mathematical education is learning to define (e g Mariotti and Fischbein, 1997; Vinner, 1991). A constructivist approach to mathematics teaching and learning asks teachers to provide students with various opportunities for active learning in general, and to emphasize the development of the students' abilities to define mathematical concepts in particular (de Villiers, 1998) One of the elements of de Villiers's experimental approach is encouraging students to realize that alternative definitions for the same concept are possible.

Mathematics teachers' competence in managing these kind of activities - providing their students with challenging mathematics (Jaworski, 1992) - demands their deep awareness of the significance of the defining process in mathematics and mathematics learning. Linchevsky, Vinner and Karsenty (1992) found that student teachers did not understand the nature of mathematical definitions and did not realize that several alternative definitions for a mathematical concept may co-exist Thus, we found it important and interesting to confront secondary school mathematics teachers with this problematic issue.

This second part of our article deals with the definitions of several mathematical concepts and the relationships between them based on the theoretical framework presented in the first part. [1] Mathematical activities associated with equivalent and non-equivalent definitions were constructed in the context of professional development workshops for mathematics teachers, in order to increase their sensitivity and awareness to the role of definitions in mathematics learning We developed various co-operative learning activities focusing on the following mathematical concepts: absolute value, circle, parabola, special quadrilaterals, distance from a point to a straight line, distance as a function, point of inflection, the number e, the Fibonacci series and the tangent line (Winicki and Leikin, 1998).

In this part, two examples of mathematical activities for mathematics teachers are presented. They are similar in their design but different in the mathematical content We discuss mathematical and didactic characteristics of the definitions used, as well as the teachers' mathematical performance when examining the mutual relationships between the definitions and their preferences regarding the use of a particular definition when teaching. We conclude with several remarks regarding the issue of arbitrariness of the definitions, its nature and its expression in our investigation.

The activities

Design

In order to involve the teachers in authentic discussions on the mutual relationships between different mathematical definitions of a concept, small groups of the teachers were presented with work cards consisting of three different definitions. Each teacher within a particular group got the same card. They were asked to examine mutual relationships between the definitions on a card and to justify their conclusions. Teachers in each small group received a work card which was different from the cards of the teachers within the other small groups The results of the small group discussions were presented to the whole group After the presentation of the results, participants discussed which definition they preferred to teach to students and explained their preferences.

Our analysis of the teachers' mathematical performance and preferences is based on written notes taken at the time of the discussion, as well as on the teachers' notes made during the work in small groups

Content

The first activity dealt with one particular mathematical concept - *absolute value*. Inspired by Vinner (1991), who refers to several aspects of mathematical definitions using the example of the absolute value concept, we chose this concept for our activity with the teachers Vinner presented several ways of defining absolute value when, for example, discussing the notion of elegance of mathematical definitions or discussing the stages of learning when a particular definition should be used.

Following Vinner's claim that:

> when coming to decide about the pedagogy of teaching mathematics one has to take into account not only the question how students are expected to acquire the mathematical concepts but also, and perhaps mainly, how students *really* acquire these concepts. (p. 67)

we found it important to explore teachers' preferences regarding different definitions of a mathematical concept via the concept of absolute value.

Overall, we presented five different definitions (Brumfiel, 1980) to the teachers, all of which are *equivalent definitions* of the concept (see Figure 1) Triplets of these five definitions were organized on the work cards as follows: (B, C, E),

For the Learning of Mathematics **20**, 2 (July, 2000)
FLM Publishing Association, Kingston, Ontario, Canada

(A, C, D), (B, D, E) (labelled according to Figure 1). In this way, each of the five definitions (except A) was included on two cards and logical relations among all five definitions were potentially able to be established during the whole-group discussion.

DEFINITION A:

Let x be any real number. On a coordinate line let X be the point whose coordinate is x. Then |x| is the distance between X and the origin.

DEFINITION B:

Let x and y be two real numbers. Let X and Y be the points on a coordinate line whose coordinates are x and y. Then |x-y| is the distance between the points X and Y.

DEFINITION C:

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

DEFINITION D:

$|x| = max\{x, -x\}$ $\quad$ $(max\{a, a\} = a)$

DEFINITION E:

$|x| = \sqrt{x^2}$

Figure 1 Definitions of absolute value

The second activity dealt with a family of geometric concepts, namely *special quadrilaterals*. First, we considered this topic as a central one in the geometry curriculum. Second, different definitions for most of the quadrilaterals can be provided, by defining equivalently a particular concept based on the variety of the necessary and sufficient conditions for different quadrilaterals. For example, a parallelogram may be defined as a quadrilateral having two pairs of parallel sides or as one having bisecting diagonals.

In other cases, definitions can be made non-equivalent by relying on small changes in a particular property of a concept, as for example a trapezoid may be defined as a quadrilateral having either at least or exactly one pair of parallel sides (see, for example, Vinner, 1991). Additionally, a special quadrilateral can be defined by referring to the properties of its angles and/or its sides or in light of its symmetry properties (see, for example, Ellis-Davies, 1986). Finally, the question of hierarchy in mathematical education is a valuable issue for pedagogical discourse (Hart, 1981) and this topic allowed us to refer to the hierarchical approach to the definitions of mathematical concepts (de Villiers, 1994).

Overall, we presented nine different definitions of quadrilaterals to the teachers (see Figure 2). The quadrilaterals were named by letters in order to try to neutralize the teachers' familiarity with their standard names. All but one of the quadrilaterals were defined in two ways:

(1) in a manner traditional for the Israeli curriculum – referring to sides and/or angles of the defined quadrilaterals (see definitions T_1, R_1, S_1 and K_1 in Figure 2);

(2) by referring to symmetry properties of the quadrilaterals (see definitions T_2, R_2, S_2, K_2 and P in Figure 2).

Triplets of these nine definitions were organized on the work cards as follows: (T_1, T_2, P), (R_1, R_2, K_1), (S_1, S_2, K_2). These combinations of the quadrilaterals on different work cards ensured the teachers had to deal with pairs of definitions which exemplified different mutual relationships among them: *equivalent* definitions, *consequent* definitions and *competing* [2] definitions (see for example Figure 3).

T_1 is a quadrilateral having a pair of parallel sides while sides in the other pair are equal

R_1 is a quadrilateral having four equal angles

S_1 is a quadrilateral having four equal sides

K_1 is a quadrilateral having two pairs of equal consecutive sides

T_2 is a quadrilateral having a symmetry axis, which is not a diagonal

R_2 is a quadrilateral having two symmetry axes, which are not diagonals

S_2 is a quadrilateral having two symmetry axes, which are diagonals

K_2 is a quadrilateral having a symmetry axis, which is a diagonal

P is a quadrilateral symmetrical with respect to a point

Figure 2 Definitions of special quadrilaterals

Teachers' performance and views

Our exploration was conducted in a number of in-service workshops with different groups of mathematics teachers. These workshops always facilitated a productive and rich learning environment based mainly on the uncertainty the teachers felt regarding the posed questions. The issue of equivalent and non-equivalent definitions was always new to these experienced mathematics teachers. The most frequent questions we were asked when the task was presented were: "What do you mean by mutual relationships between definitions?" and "What does it mean to prove them?" We did not answer this kind of question, so as to allow the teachers freedom to make sense of the questions and to decide by themselves how to work on their responses.

Teachers strategies when examining the relationships between definitions

Two main strategies were identified when the teachers examined the relationships between two given definitions. One of these strategies was based on exploring logical relationships between the defining properties of the concept (*the properties strategy*). The other relied on the comparison between the sets of objects that exemplify each definition (*the sets strategy*). An additional strategy, which was rarely observed, was based on the teachers' reference to a representation of the defined concept mentioned in its definition (*the representation strategy*).

The properties strategy

Although this strategy was mainly manifested when proving that definitions were equivalent, the equivalence of the definitions was not sufficient for its use. Thus, teachers mainly used this approach when dealing with the definitions of absolute value, but in some cases in this workshop the teachers also used the 'sets' strategy. The properties strategy was rarely used when the teachers dealt with the definitions of quadrilaterals.

The sets strategy

The teachers used this strategy more frequently. It appeared either when proving or when refuting the equivalence of the definitions. It must be noted that the teachers applied this strategy in a rather implicit way, namely the relationships between different quadrilaterals were proved by naming a defined quadrilateral by a known word and then relying on known relationships between the known special quadrilaterals.

For example, R_1 and R_2 were both named 'rectangle' and by this naming the equivalence was proved. T_1 was identified as an isosceles trapezoid [3] or a parallelogram; T_2 was identified as an isosceles trapezoid or a rectangle. Thus, it was established that the definition of T_1 follows from the definition of T_2. P was named 'parallelogram' and its definition was considered as *competing* with the definition of T_2 and as *consequent* to the definition of T_1 (see Figure 3)

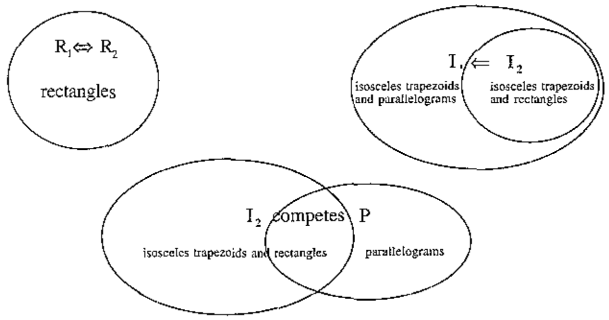

Figure 3 Sets strategy for demonstration of different relationships between special quadrilaterals

The following excerpt exemplifies the use of the sets strategy, as can be seen when the teachers named one of the quadrilaterals (K_2) by a known name - *kite* - and found an object which exemplified K_1 but was not a kite

Bob: We saw that every K_2 is a kite. Then we examined whether every K_1 is a kite. We thought so. But we found that every kite is a K_1 but the converse is not true. [see Figure 4]

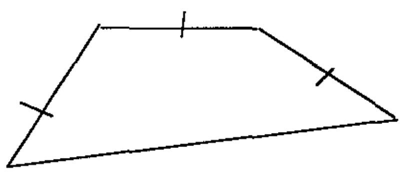

Figure 4 Not every K_1 is a kite

The next excerpt demonstrates a case when the teachers made use of both of these strategies. At the beginning, a teacher (Alexis) explained how her group proved the implication $S_1 \Rightarrow S_2$ using the properties strategy when this implication was suspected to be true. Then, she explained that when conjecturing the lack of the implication ($S_2 \not\Rightarrow S_1$) they used the sets strategy. The group came to the conclusion that the definitions are equivalent, but failed to prove this statement.

Alexis: S_2 is a property of S_1. We proved ... We took a quadrilateral S_1 and proved that its diagonals are axes of symmetry. So every quadrilateral S_1 is a quadrilateral S_2

It seems that every quadrilateral S_2 is a quadrilateral S_1. We did not find any example of S_2 which is not an example of S_1

We concluded that S_1 and S_2 are equivalent definitions because we could not find a counter-example.

It must be also mentioned that sometimes the teachers incorrectly used the sets strategy in order to 'demonstrate' that some of the presented definitions of absolute value were not equivalent. This happened when they decided that definition II follows from definition I (see Figure 1), "because definition I is a special case of definition II for y = 0".

Definition V and definition III were considered by some teachers as "appropriate only for real numbers", while definition IV could be used for the complex numbers too. Hence, definition V and definition III were considered as non-equivalent to definition IV. Note that the meaning of set in this case was different. The teachers referred to the domain of the definitions instead of the sets of their exemplifying objects and this reference brought them to a mistaken conclusion. In the whole-group discussion, all the definitions of absolute value (Figure 1) were accepted by the teachers as equivalent on the set of real numbers

Probably the differences in the use of these two strategies may be explained by the following considerations. First, the definitions of the special quadrilaterals presented triggered teachers to name the concept defined. This naming was referred to as establishing the set of the exemplifying objects for the definition. On the other hand, when dealing with the concept of absolute value, the naming process was avoided by the naming of the concept in the presented definitions.

Second, encountering the sets of exemplifying objects is much easier for a quadrilateral than for the concept of absolute value. This probably follows from the differences between the nature of the defined concepts. Quadrilaterals have a strong visual nature while the concept of absolute value is commonly perceived as an algebraic one and rarely exemplified by a geometric (visual) object (as, for example, by the graph of the absolute value function).

The representations strategy

As we mentioned earlier, in a small number of cases the teachers referred to the nature of the representation of the elements used in the definition. For example, definition I of absolute value referred to a geometric representation of the concept - length of the segment, while definition III referred to an algebraic representation of the concept - the absolute value of a number, considered as max $\{x, -x\}$. Thus, the teachers decided that these two definitions of the concept were not equivalent. In the whole-group discussion, the incorrectness of this conclusion was demonstrated using the sets strategy by comparing the sets of objects exemplifying these definitions.

Comparing triplets of equivalent definitions
As described earlier, the teachers in both tasks considered were presented with triplets of definitions. The triplets of the definitions of different quadrilaterals made it possible for the teachers to consider different types of mutual relationship between the definitions while all the definitions of absolute value were equivalent.

The absolute value activity illuminated ways in which these teachers went about proving the equivalence of three definitions, ways naturally connected to the strategies they used when proving the equivalence of two definitions. Thus, the *properties strategy* was implemented in the two following ways of showing implications of the statements:

1) cyclic proof of equivalence: I ⇒ II, II ⇒ III, III ⇒ I;

2) transitivity proof of equivalence: I ⇔ II, II ⇔ III

In the other cases, following the sets strategy, the teachers tended to compare sets of exemplifying objects. Teachers performed this comparison in a transitive way, by proving that the sets are equal:

$A = B, \qquad B = C$

On these occasions, the possible cyclic way of proving ($A \subset B$, $B \subset C$, $C \subset A$) was not used by the teachers. In the subsequent discussion, teachers claimed that use of the statements' implications was easier and more clear than set comparison.

Teachers' preferences regarding different definitions of the same concept

Main attributes of the mathematical definition influencing the teachers' preferences
The concluding whole-group discussion at all the workshops we conducted focused on the teachers' preferences regarding different definitions. The teachers usually discussed both pedagogical and mathematical characteristics of the definitions, and often used didactical considerations in order to explain their mathematical preferences. When talking about mathematical aspects of the definitions, the teachers claimed that they preferred the definitions that are more accurate.

The teachers pointed out that more precise definitions help teachers to avoid student confusion when teaching mathematical concepts. We found that with respect to *didactical aspects*, the teachers preferred definitions that they considered as fitting the following attributes more:

(a) intuitiveness;
(b) matching students' knowledge and needs;
(c) clarity to the students or ease of understanding;
(d) convenience for applying to problem solving;
(e) enabling mathematical generalizations

For example, at the 'absolute value' workshops, most of the teachers reported that they preferred definition A (see Figure 1), since they considered it the most intuitive and the easiest to use when solving simple equations and inequalities. A similar level of 'popularity' was addressed to definition C. Although this definition was found to be less intuitive, its functional approach and the possibility of explaining it strongly verbally influenced the teachers' preferences regarding it.

These findings naturally arise from the dominant presence of these definitions in the school textbooks. Hence, the teachers were very familiar with them. Interestingly, Vinner (1991) considers this definition of absolute value as a formal definition of the concept and points out the need to use this definition at the later stages when solving algebraic equations and inequalities with absolute value. Many teachers during the discussion agreed that definition E is "not less preferable" when used in the upper grades of secondary school. Some stated that this definition might help the learner avoid a well-known mistake: $x = \sqrt{x^2}$ Definition D was not considered intuitive at all, thus judged difficult for the students.

Minimality of the definitions
At all of the workshops, the teachers did not consider the minimality of the definitions as one of the critical characteristics of the definitions when discussing their preferences. We always raised this issue for discussion, especially because of the inaccuracy in the teachers' explanations that was often connected to the minimality of the justifications. For example, several times the teachers named quadrilateral R_1 (see figure 2) 'a rectangle or a square', while it is sufficient to call it 'a rectangle'. The following excerpt demonstrates one of the discussions focused on the minimality of the definitions

Vera: Why cannot we refer to a rhombus and to a square simultaneously? If we say a set of sufficient conditions, this set will include different conditions. Thus, it may include different conditions which sometimes are not independent. Why is it confusing if there are spare conditions in the definition? Is it confusing from a mathematical point of view? Is it unaesthetic?

Beth: It [minimality] is mostly a matter of aesthetics. It is also connected to the issue of understanding

Jody: Minimal conditions are those that you do not need to prove. If you can prove something, then it may be by definition.

Beth: If a definition is not minimal, we can make a compromise.

As we see from the above segment of the discussion, the teachers were not convinced of the mathematical and didactical importance of the minimality of a mathematical definition, as well as not understanding the reasons for this requirement. When Vera was asking "Why is it confusing?" having a definition which is not minimal, Beth found it "mostly a matter of aesthetics". There was neither a mathematical nor a didactical justification offered for this requirement in the discussion and moreover the teachers were convinced that they can "make a compromise" regarding minimality of the definition.

Following Vinner's (1991) assumption that a mathematical definition should be minimal, we accept this requirement as essential not only from an aesthetic point of view but also from a mathematical perspective. This point always served a very fruitful background for the workshops' summary discussions and the teachers were amazed by the presence of mathematical criteria in many places where they did not expect to find them.

Concluding remarks

This article has examined the issue of the equivalence and non-equivalence of mathematical statements which can serve as definitions of a mathematical concept. It focused on the mathematical and didactical aspects of this question According to Noss (1998):

> The essence of professional practice is difference and diversity: the efficacy of routine lies in the specificity of its language and conventions But the essence of mathematics lies in sameness: in the search for common structures, invariants and conventions (p 7)

Thus, activities in which secondary school mathematics teachers were involved when examining and discussing different definitions of a mathematical concept were offered as a springboard for their professional development. Analysis of the learning process which took place during these activities illuminated the close relationship between the issues raised by the teachers and those discussed in the educational and mathematical literature that we considered in the first part of this article

Attributes such as intuitiveness, matching students' knowledge and needs, clarity to the students or ease of understanding, convenience in applying to problem solving and enabling mathematical generalizations were raised by the teachers, in order to justify their preferences regarding a particular definition of a mathematical concept. Even though the teachers did not refer to elegance and minimality of the definitions as critical characteristics of the definitions by themselves, they attached them to the list of essential attributes of the definitions when they were involved in the discussion on these topics.

Two issues identified in the workshops were found especially worthwhile for the concluding discussion, namely:

(1) defining as naming;

(2) defining through properties or defining through sets of objects

Both of these are connected to the arbitrariness of mathematical definitions. We would like to reflect here on two different aspects of the arbitrariness of mathematical definitions.

Defining is not just giving a name, as can be sometimes misunderstood. As well (or, probably, mostly), it is establishing properties and indications of the concept defined or establishing a set of exemplifying objects The historical development of some mathematical concepts shows this face of the defining process.

> Important concepts have long histories before a definition is formulated. But, whatever trouble mathematicians may have in finding a definition that would suit everybody's needs and the existing examples, once they have agreed upon a definition, it is binding, and one has to accept all its logical consequences (Sierpinska, 1994, p 50)

Consequently, mathematical statements including different necessary and sufficient conditions, which are not always equivalent, can be called by the same name, as for example, in the case of the concept of a function

> Before Dirichlet and Bolzano, functions were those well behaved relationships that could be represented by almost everywhere smooth curves. After the general definition was introduced allowing absolutely any well-defined relationship between two variables to be a function, mathematicians started to come up with examples of functions that were real monsters to most of their colleagues (p 50)

As described earlier, in our workshops we presented the teachers with several definitions of the concept of absolute value. In this case, arbitrariness is expressed by the ways the teachers use the concept in their teaching practice and correspondingly by the teachers' preferences regarding different definitions

Second, different names may be given to a concept. For example, right-bisector and mid-perpendicular are two names given to the objects having the same set of defining conditions. As presented earlier, in our workshops two different names were given to a concept of rectangle, which was defined equivalently in two ways (i e. R_1 and R_2), as if we had presented two new 'unknown' concepts

On the one hand, in this case we used the arbitrariness of a definition in order to allow the teachers to discover the real connections between the sets of objects defined and to try to neutralize the effect of naming on the exploration process On the other, arbitrariness is expressed by the way in which the teachers make connections between 'unknown' concepts and the concept having a known name. This was done either by searching for logical relationships between the defining statements (using the property strategy) or by comparison of the sets of the exemplifying objects (the sets strategy).

The freedom to use these two strategies may be considered as another demonstration of the arbitrariness involved in the defining process. A concept having a particular name can be created by arrangement of its necessary and sufficient conditions (properties and indications) or by establishing the set of exemplifying objects of the concept

Zaslavsky (1995) found that:

> the notion that they [teachers], as teachers, will always have more to learn, even with respect to what they think they already know, gave new meaning to what 'knowing' entails. (p. 19)

Similarly, our teachers' intuitions led them to discover these two ways of exploring defining in mathematics, which provided us with an excellent opportunity to reflect on this issue and we realized how much we have learned. At the same time, we were glad to know that the teachers were surprised and became aware of the opportunity to learn a variety of new things through dealing with different definitions of very familiar mathematical concepts.

Notes

[1] The first part of this article appeared in *For the Learning of Mathematics* **20**(1), 17-21

[2] For the description of different relationships between definitions, see part 1 of this article

[3] In the traditional approach, a trapezoid is defined as a quadrilateral having *exactly* one pair of parallel sides

Acknowledgements

We would like to thank Marjorie Henningsen for her helpful comments on the first draft of this article, which was prepared while Roza Leikin was a post-doctoral fellow with Ed Silver at the Learning Research and Development Center, University of Pittsburgh

References

Brumfiel, C (1980) 'Teaching the absolute value function , *The Mathematics Teacher* **73**(1), 24-30

Ellis-Davies, A. (1986) 'Symmetry in the mathematics curriculum , *Mathematics in School* **15**(3), 27-30

de Villiers, M (1994) 'The role and function of a hierarchical classification of quadrilaterals', *For the Learning of Mathematics* **14**(1), 11-18

de Villiers, M. (1998) 'To teach definitions in geometry or teach to define', in Olivier, A. and Newstead, K (eds), *Proceedings of the 22nd International Conference for the Psychology of Mathematics Education*, Stellenbosch, South Africa, vol 2, pp 248-255

Hart, K. (1981) 'Hierarchies in mathematics education', *Educational Studies in Mathematics* **12**(2), 205-218.

Jaworski, B (1992) 'Mathematics teaching: what is it?', *For the Learning of Mathematics* **12**(1), 8-14.

Linchevsky, L., Vinner, S and Karsenty, R (1992) 'To be or not to be minimal? Student teachers' views about definitions in geometry', in *Proceedings of the 16th International Conference for the Psychology of Mathematics Education*, Durham, NH, vol. 2, pp 48-55

Mariotti, M A and Fischbein, E (1997) 'Defining in classroom activities', *Educational Studies in Mathematics* **34**(3), 219-248.

Noss, R (1998) 'New numeracies for a technological culture', *For the Learning of Mathematics* **18**(2), 2-12.

Sierpinska, A (1994) *Understanding in Mathematics*, London, Falmer Press.

Vinner, S (1991) 'The role of definitions in the teaching and learning of mathematics', in Tall, D. O (ed.), *Advanced Mathematical Thinking*, Dordrecht, Kluwer, pp. 65-81.

Winicki, G and Leikin, R (1998) *Equivalent Definitions – Mathematical Activities for Teachers and Students*, Haifa, Department of Education in Technology and Science, Technion (in Hebrew)

Zaslavsky, O. (1995) 'Open-ended tasks as a trigger for mathematics teachers' professional development', *For the Learning of Mathematics* **15**(3), 15-20