moreover, in the current era of increasingly sophisticated data analytics, we must be vigilant about the ways different persons might be differently targeted in various media.

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Notes

[1] Mathletics, a product of 3P Learning, https://www.mathletics.com/nz/
[2] Emojipedia, https://emojipedia.org/

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Congruence theorems in the past, present and future

ANTONELLA PERUCCA

Triangle congruence theorems are a central part of geometry teaching, but in spite of their familiarity, there is much to be learned about them, both for students and researchers. They form a collection of results that fit nicely together, so they constitute a small *mathematical theory;* they have what Freudenthal (1971) called 'local organization'. Moreover, they provide a playground for pupils to *distinguish truth from falsehood:* some sets of assumptions give a true theorem, others are not sufficient and lead to counterexamples. As it often happens for mathematical statements, in the classical congruence theorems none of the assumptions can be removed without invalidating the result.

Congruence theorems around the world

Everyone can list the 'triangle congruence theorems' (or 'properties' or 'postulates' or 'criteria'). But the list might differ according to who you ask. It is likely to include some, and perhaps all, of these:

> (SSS) Two triangles are congruent if three pairs of corresponding sides are equal in length.
>
> (ASA) Two triangles are congruent if two pairs of corresponding angles are equal in measurement, and the included sides are equal in length.
>
> (AAS) Two triangles are congruent if two pairs of corresponding angles are equal in measurement, and one pair of corresponding sides, different from the included ones, are equal in length.
>
> (AAcorrS) Two triangles are congruent if two pairs of corresponding angles are equal in measurement, and one pair of corresponding sides are equal in length.
>
> (SAS) Two triangles are congruent if two pairs of corresponding sides are equal in length, and the included angles are equal in measurement.
>
> (SsA) Two triangles are congruent if two pairs of corresponding sides are equal in length, and the pair of angles opposite to the bigger of these sides are equal in measurement (if the two sides are equal, then we can use any of the two sides).
>
> (HL) Two right-angled triangles are congruent if their hypotenuses and one pair of corresponding legs are equal in length.

All these ensure that two triangles are congruent, making use only of conditions involving the length of some of the sides and the measures of some of the interior angles. Within this description there are at least two other, less familiar, congruence theorems:

> (acuteSSA) Two acute triangles are congruent if two pairs of corresponding sides are equal in length, and one pair of corresponding angles are equal in measurement.
>
> (perimeterAA) Two triangles are congruent if they have the same perimeter and the same angles.

It is interesting to examine the list of triangle congruence theorems that one finds on Wikipedia (under 'Congruence (geometry)' in 2022) and how that list changes on the corresponding pages in various languages. Table 1 shows which theorems are listed. We can see that, although in mathematics what is true can be proved, what is important is a matter of choice.

Language	SSS	ASA	AAS	AAcorrS	SAS	SsA	HL
English	✓	✓	✓	(✓)	✓		✓
Chinese	✓	✓	✓		✓		✓
Italian	✓	✓		✓	✓		(✓)
French	✓	✓	✓		✓	✓	
Spanish	✓	✓	✓		✓	✓	
Japanese	✓	✓		✓	✓		✓
Portuguese	✓	✓	✓		✓		✓
German	✓	✓			✓	✓	
Russian	✓	✓	✓		✓	(✓)	
Arabic	✓	✓			✓		
Hindi	✓	✓	✓		✓	(✓)	

Table 1. Occurrence of congruence theorems across languages.

A checkmark in parenthesis stands for a result which is mentioned but not with the status of a congruence theorem. Note that AAS and AAcorrS are variants of ASA because knowing two angles amounts to knowing three angles. The low rate of inclusion of SsA may be because knowing two pairs of corresponding sides and one pair of corresponding angles is not sufficient to ensure congruence. HL is a special case of SsA, and is often included as a substitute. The abbreviations are standard, aside from translations (for example, Spanish uses LLL because 'lado' is the word for side). Languages without an alphabet (or with the unfortunate coincidence that the words for angle and side have the same initial) can still use full names like Side-Angle-Side. In the Chinese, Japanese, and Hindi Wikipedia the English abbreviations are used. What implications this choice has didactically could be a topic for research.

A mathematical subtlety

In the triangle congruence theorems there are six important quantities, namely the measures of the three interior angles and the lengths of the three sides. Given two triangles, we must ensure that some of these quantities match, and for this reason one speaks of 'corresponding' sides or angles. This notion of correspondence is crucial.

Consider for example the triangles that are similar to the right triangle with side lengths 3, 4, 5. If we want such a triangle with one side length equal to 1, then it is not clear whether this side is the short leg, the long leg, or the hypothenuse, so we cannot determine the triangle up to congruence even if we know three angles and one side.

Even worse, there are triangles that are similar and share two side lengths (of non-corresponding sides), known as *5-con triangles.* Notice that there are 5-con triangles with very nice side lengths, for example (8, 12, 18) and (12, 18, 27). If we are not careful with matching, then there are non-congruent triangles which share three angles and two sides. To see for yourself how neglected the notion of correspondence is, simply ask (as I have) some pupils or teachers if knowing the measures of three angles and the lengths of two sides of a triangle is sufficient to determine the third side. I believe that all prospective teachers should reflect on this correspondence issue and should know about 5-con triangles, and schoolbooks should speak about both.

Part of the problem is that the word 'corresponding' is not mathematically precise. What is usually implicitly understood is that, once we have explicitly assigned corresponding vertices, then 'corresponding angles' are simply angles at corresponding vertices, while 'corresponding sides' are simply sides between corresponding vertices. But this is rarely made explicit, though it could and should be. Once teachers and pupils have understood this hidden convention, then there is no problem in using it.

Another, arguably mathematically more satisfying, solution to the correspondence issue would be speaking of just one triangle. The question shifts from comparing two triangles to asking whether a single triangle can be constructed when some of its parts are given. More precisely, we are given some angles and some sides (and possibly some specifications as to how they are related to each other) and asked whether we can construct only one triangle which has these properties. Instead of drawing two triangles, we just draw one. This is done in some countries.

Research on triangle congruence theorems

Although triangle congruence theorems have been known at least since the time of Euclid, c. 300 BCE, there are still open questions on this topic. An untapped advantage of congruence theorems is that they provide open research questions whose statements can be understood by pupils. Moreover, since there are many research directions, pupils can understand that a mathematical result can be generalized in many ways, as is common in mathematical research.

The first avenue for exploration is to seek new proofs. One congruence property can be taken as a postulate, and then the others can be proven from it. For example, Hilbert (1899/1902) takes SAS as an axiom (IV-6). Varying which we chose to begin with allows the exploration of the local organization of the theorems. It can also be investigated whether there is another postulate that can be used to prove all the congruence theorems.

Pupils can also explore other combinations of known sides and angles that determine congruence. For example, although there is no 'SSA Congruence Theorem', an *acute* triangle is determined up to congruence if we know two sides and one angle. Similarly, as we have already mentioned, knowing the three angles and the perimeter is also sufficient. As soon as one involves other objects than sides and angles, there are many possible theorems. Inscribed and circumscribed circles, and special points in the triangle can be used as the basis for congruence theorems. As a simple example, a triangle is determined up to congruence if we know the angles and the radius of the circumscribed circle.

We could also move beyond triangles. By applying the congruence theorems for triangles, it is easy to prove some congruence theorems for convex quadrilaterals, for example the SASAS Congruence Theorem where we know three consecutive sides and the two angles that they form. For further results on quadrilaterals, see Vance (1982), Laudano and Vicenzi (2017) and Perucca and Torti (2023). Vance states that exploring these results is an enriching activity for pupils: "The theory of congruence for quadrilaterals, can provide some rich applications of congruence of triangles and other concepts taught in high school geometry, and opportunities for valuable practice in constructing counterexamples" (p. 403).

Perucca and Torti (2023) prove congruence theorems for convex n-gons for arbitrary n. For example, that for $n > 6$ it suffices to know the lengths of enough diagonals to show congruence. The assumption of convexity is important. Knowing the exact position of $n - 1$ vertices and all sides and all but two angles is not sufficient to determine an n-gon up to congruence because the two sides at the missing vertex could point outwards or inwards. There remain open research questions in this area.

It is an instructive mathematical exercise to prove congruence theorems for circles (it suffices to know the radius or the diameter or the circumference or the area). One could also consider, for example, rhombi, trapezoids, T-shaped figures and so on. The class can, in principle, take any agreed-upon figure and analyze which conditions guarantee that the shape and size cannot be changed.

Congruence theorems can also be investigated for more complicated shapes in higher dimensions. One could investigate similarity theorems in the same spirit. Let us rejoice in the fact that there are plenty of open questions that are accessible research exercises.

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Teachers' subjectification as a dialogue with standardized assessment

FEDERICA FERRETTI, GEORGE SANTI

Di Martino and Baccaglini-Frank (2017) point out the need for a critical approach to standardized assessment that fosters "*Informational Potential,* seen as the information that can actually be obtained by interpreting and analysing students' performance results on standardized tests (this includes understanding the limitations of such potential) [and] *Developmental Potential*, seen as the educational opportunities offered by a critical approach to standardized tests and by a re-elaboration of the informational potential." (p. 39).

In the present communication, we focus on an *operative* use of standardized assessment for pre-service teachers' professional development based on data collected by the Italian National Evaluation Institute for the School System (INVALSI) and framed in mathematics education. In particular, we create a link between the Italian National Evaluation System and pre-service teacher professional development programs in order to *improve* mathematics teaching school practices.

Our model is made up of the following elements:

- an interactive database containing structured information regarding Italian standardized assessment that contains 1793 test items, spanning 10 years of the Italian National Evaluation System activity;
- a theoretical framework for pre-service teacher training that conceives teachers' learning as a process of *subjectification* (Radford, 2020).

The first element encompasses Informational Potential and the second one Developmental Potential.

Theoretical background

Our understanding of teachers' professional development ascribes a key role to the socio-cultural environment in defining individual agency and the production of mathematics teachers as new subjectivities, *i.e.,* a process of *subjectification.*

The Theory of Objectification (TO) outlines a dialectical co-production between individuals and their cultural and historical reality. Radford conceives the individual as:

> an entity in flux, in perpetual becoming—an entity who, through practical activity (like play) is continuously inscribing herself in the social world and, in doing so, she is continuously produced and co-producing herself within the limits and possibilities of her culture. (p. 43)

The cultural-historical context defines who we are, how we conceive of ourselves and how others conceive of us, the "fabric of our subjectivity" (p. 46), the space, the constraints, and the forms of our agentic maneuvering. It is also true that we hold a reflexive relationship with our cultural context, which means that "we *react agentically* to such a context" (p. 4).

The cultural-historical context unfolds as a *symbolic superstructure*—termed by TO as 'Semiotic Systems of Cultural Signification': a network of distinctive traits that makes up the fabric of a culture and its society.

We have selected and adapted from Radford the systems that are specific to the production of prospective teachers' subjectivities and have informed our professional development activities:

- Epistemology and ontology of mathematics
- Systems of truths
- Forms of rationality
- Accepted mathematical teaching practices, problems, and situations