MATHEMATICAL OBSERVATIONS: THE GENESIS OF MATHEMATICAL DISCOVERY IN THE CLASSROOM

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At the undergraduate level, instructors generally employ the "Euclidean method." They list sets of definitions, axioms and lemmas, and introduce new notations. Then theorems are proved, often followed by exercises to reinforce them. These theorems were originally mathematical conjectures before being settled in the classroom. This deductivist style completes the process of mathematical discovery in the classroom. Lakatos (1987), however, argued that mathematical activities should be conducted in the classroom just as mathematics was first heuristically discovered, following a pattern of discovery with four stages:

a) a primitive conjecture,

b) proof,

c) global counterexamples (counterexamples to the primitive conjecture),

d) proof-re-examined.

Lakatos proposes that a proof of a theorem is a dialectic process starting with a conjecture that might be reformulated after counter examples are found.

In this article, I show how an undergraduate mathematics class may be guided to a "primitive conjecture" and its proof via the methodologies of Realistic Mathematics Education. Realistic Mathematics Education (RME) is a set of mathematics curriculum principles developed in the Netherlands mainly for mathematics education at the elementary school level. It has also been used at the undergraduate level of mathematics education in various topics, including Euclidean and non-Euclidean geometry (Rasmussen *et al.,* 2005), differential equations (Rasmussen & Blumenfeld, 2007), and group theory (Larsen, 2010; Larsen & Zandieh, 2008). In this article I share RME-inspired classroom activities in abstract algebra, designed for the guided reinvention of factor groups and identify some important aids for students to reach and prove a conjecture.

Mathematical observations and the scientific method

A mathematical observation is the detection or assertion of a truth in the context of a mathematical activity. Making observations is necessary in solving a problem, whether the problem falls in the realm of mathematics or the physical sciences. In fact, "from the recognition of a problem, to finding its solution, observation is essential" (Mak *et al.,* 2009, p. 17). A *sine qua non* for a mathematical observation is that it has to be followed by a deduction. Such deduction may come in the form of a conjecture. A conjecture is a statement that can be proven true or false, but has not yet been proven. A conjecture, in many instances, is a generalization of an observation. Once the conjecture is proposed, one may follow Lakatos's approach. Although a conjecture is always preceded by a mathematical observation, a mathematical observation is not necessarily followed by a conjecture, since it may be followed by a truth, a theorem, or a formula.

Observations in mathematics and in science are alike, in that they both initiate discovery. The difference between mathematical observations and observations in the scientific method is the steps that follow. Mathematical observations are usually followed by conjectures, which only admit *a priori* reasoning with no empirical considerations, while scientific observations are followed by hypotheses, which require inductive reasoning, and which are resolved with experiments. All in all, hypotheses are the scientific counterparts of conjectures in mathematics. Observations precede both hypotheses and conjectures.

There are both similarities and differences between mathematical observations and conjectures. One similarity is that they both emanate from mathematical activity. One difference is that observations in mathematics may be vague and informal in statements, since there is no "primordial language of observation" (Smirnov, 1970, p. 44). As such, observations do not necessarily use formal language of mathematics, while conjectures use formal language and mathematical notation. This formalism is to facilitate the eventual proof of the conjecture. Also, the observation-to-conjecture process is not one-way; more observations may be made after the proof of the conjecture, and more conjectures might ensue after such observations. In other words, observation and conjecture form a cycle. In the middle of a proof, for example, the author sometimes writes, "observe that ...." or, "we note that ..." in order to evoke a certain observation.

Although mathematicians do not perform experiments, for didactical purposes, students may be encouraged to appropriate some scientific techniques for the sake of discovery. As Cuoco, Goldenberg and Mark (1996) lament, this approach is rare in school mathematics: "simple ideas like recording results, keeping all but one variable fixed, trying very small or very large numbers, and varying parameters in regular ways are missing from the backgrounds of many high school students" (p. 378). This approach is also missing in many undergraduate mathematics classrooms. Cuoco *et al.* continue, by warning that "mathematics is more than data-driven results, and students need to realize the limitations of the experimental method" (p. 379).

It may be helpful to provide a couple of examples of mathematical observations. The first example is based on a question that might be posed to students in elementary and middle school:

> An employee is placing symbols on 500 booklets. She affixes one symbol on each booklet in the following order ♦, ∇, ♥, O, ⊥, ♦, ∇, ♥, O, ⊥, ♦, ∇, ♥, O, ⊥, … What is the symbol on the 387th booklet?

There are several possible ways that primary school students might try to arrive at a solution to this problem. The observation that the sequence is a repetition of the 5 symbols ♦, ∇, ♥, O, ⊥ and the use of modular arithmetic will lead to the answer: ∇.

The second example is based on the following exercise:

> John has 6 orange marbles, 7 blue marbles, 8 yellow marbles, 9 white marbles, 10 green marbles, and 11 red marbles. He wants to separate each set of marbles into two containers, with no empty containers. How many ways can he do that for each set?

A student might proceed as follows:

Orange marbles: $6 = 5 + 1 = 4 + 2 = 3 + 3$

Blue marbles: $7 = 6 + 1 = 5 + 2 = 4 + 3$

Yellow marbles: $8 = 7 + 1 = 6 + 2 = 5 + 3 = 4 + 4$

White marbles: $9 = 8 + 1 = 7 + 2 = 6 + 3 = 5 + 4$ (4 ways)

Green marbles: $10 = 9 + 1 = 8 + 2 = 7 + 3 = 6 + 4 = 5 + 5$

Red marbles: $11 = 10 + 1 = 9 + 2 = 8 + 3 = 7 + 4 = 6 + 5$

A student familiar with prime numbers might observe that $6 = 3 + 3$, $8 = 5 + 3$, $10 = 5 + 5 = 7 + 3$, and see that the even numbers are sums of two primes numbers. By noticing that this does not hold for odd numbers (11, for example) and after having found more examples, one can make the conjecture, "all even numbers greater than 2 can be represented as the sum of two primes," and try to prove it. This is known as Goldbach's Conjecture (circa 1742).

In what follows, I show how undergraduate students use mathematical observations as the genesis of mathematical discovery in the classroom. Using the perspective of Realistic Mathematics Education, I also illustrate a connection between horizontal and vertical mathematizing and mathematical observations.

Realistic Mathematics Education

Three decades ago, a reform movement in mathematics education was started in the Netherlands, as a result of concerns with the quality of children's learning. Educators were disillusioned with the mechanistic, empiricist, and structuralist philosophies of mathematics education of the 1960s and 1970s (Freudenthal, 1991, p. 135). The result was the development and implementation of a new theory of mathematics education based on Freudenthal's view of mathematics as "human activity". This theory was later called Realistic Mathematics Education. The Netherlands, with its use of RME, has consistently performed well in both the Trends in International Mathematics and Science Study (TIMSS), and the Programme for International Student Assessment (PISA). This success is a testament to the efficacy of RME methods.

Learners' guided re-invention of mathematics is one of RME's practices. Guided re-invention of mathematical concepts is based on the precept that learners should be given the environment to invent mathematics on their own. Freudenthal (1991) argues that students should not be presented with ready-made problems, but should grasp something as a problem (p. 46). He also argues that "inventions, as understood here, are steps in the learning processes, which is accounted for by the 're' in reinvention, while the instructional environment of the learning process is pointed to by the adjective 'guided'" (p. 46). Freudenthal proposes that for the best learning outcomes, learners should be guided to create their own definitions and notations. He offers the following reasoning for proposing guided reinvention:

> First, knowledge and ability, when acquired by one's own activity, stick better and are more readily available than when imposed by others. Second, discovery can be enjoyable and so learning by reinvention may be motivating. Third, it fosters the attitude of experiencing mathematics as a human activity. (p. 47)

Instruction plays a key role in the re-invention process. As the instructor is designing the material, he or she will imagine ways that students will produce mathematical results on their own. Gravemeijer and Terwel (2000) assert that "the curriculum developer starts with a thought experiment, imagining a route by which he or she could have arrived at a personal solution. Knowledge of the history of mathematics can be used as a heuristic device in this process" (p. 786). As students are presented with well-conceived, realistic problem situations, they will use various thought processes to construct their own solutions.

Guided re-invention in the classroom is achieved through the process of mathematizing. As defined by Treffers (1987), mathematizing is the "organizing and structuring activity in which acquired knowledge and abilities are called upon in order to discover still unknown regularities, connections, structures" (p. 247). Treffers identifies horizontal mathematization and vertical mathematization as two of the main tenets of RME.

As explained by van den Heuvel-Panhuizen (2003), in horizontal mathematization, students take problems or realistic situations from daily life and link them to the world of mathematics. Similarly, Treffers (1987) describes horizontal mathematization as "transforming a problem field into a mathematical problem" (p. 247). For example, transforming a pizza-sharing situation into the addition or multiplication of fractions; using group-theoretic concepts to tackle a graph theory problem; and the use of Venn diagrams to count elements in intersecting sets, are all examples of horizontal mathematization. These examples involve the use of symbols, diagrams, or tables to approach a problem, or moving from one branch of mathematics to another branch of mathematics to find a solution.

In contrast, in vertical mathematization, one operates "within the mathematical system" (Treffers, 1987, p. 247). That is, in vertical mathematization, one performs computations and proves theorems within the mathematical construction resulting from horizontal mathematization. For example, in a pizza-sharing situation, the performance of addition or multiplication of fractions, or finding efficient algorithms for such operations, is a form of vertical mathematization. Performing a permutation or proving a theorem in group theory, and deriving and using a formula like $|A \cup B| = |A| + |B| - |A \cap B|$ for counting the elements in two intersecting sets are part of the domain of vertical mathematization. Hence, in vertical mathematization, formulas are found and theorems are proved.

Optimal teaching and learning is achieved through both horizontal and vertical mathematizing and the use of models. Models play a central role in the implementation of RME. They are used as starting points for problem representations, and so not only facilitate mathematization, but also help to create a dynamic and social teaching and-learning environment. As asserted by van den Heuvel-Panhuizen:

> with RME, models are seen as representations of problem situations, which necessarily reflect essential aspects of mathematical concepts and structures that are relevant for the problem situations, but that can have different manifestations [...] materials, visual sketches, paradigmatic situations, schemes, diagrams, and even symbols can serve as models. (van den Heuvel-Panhuizen, 2003, p. 13)

Guided re-invention of quotient group concepts with logic puzzles

My main goal in this project was to examine the application and effectiveness of RME in an undergraduate setting, demanding advanced abstract thinking from students. Since groups are important in mathematics as they measure symmetry (Armstrong, 1988, p. vii), I chose a course in abstract algebra. As a starting point, I used the solution of a Sudoku puzzle. The students also used Cayley tables [1] as models for representing and visualizing groups. I selected the topic of quotient groups, since students often have a difficult time understanding them. To evaluate the project, I collected artifacts consisting of tasks completed by students and excerpts of student responses and arguments in the classroom.

My mathematical goals in the instructional activity were as follows:

- to reinforce normal groups and introduce quotient groups;
- to solve the problem: If N is a subgroup of index 2 in a group G, then N is normal in G.

Subsequent to the completion of the activities, I found that mathematical observations were precursors to conjectures. In this article, I illustrate the importance of making observations as an activity in both horizontal and vertical mathematizing.

The concept of quotient groups (or factor groups) can be difficult for students to understand. To introduce the topic, I solved a Sudoku puzzle and used some observations on the structure of the solution for a guided reinvention of quotient group concepts. I presented the class with the following tasks in two consecutive sessions (see Figure 1, overleaf, for an example of a Sudoku).

Day 1 activity

1. Solve a Sudoku Puzzle
2. Subdivide the solution into three blocks B_1, B_2 and B_3, where B_1 forms the first three rows, B_2 the next three rows, and B_3 the last three rows.
3. Switch any two rows within a block, say B_1. Does the solution remain a Sudoku solution? (You may use scissors to cut out the blocks)
4. Take one of the blocks, say B_1, cut out the rows and label the first row *a* the second row *b*, and the third row *c*.
5. Rearrange these rows (in Part 4) to make all the possible Sudoku blocks. Make a list of these blocks.
6. Make a conjecture and prove it.
7. Construct a Cayley table by using an appropriate operation to determine whether this list of blocks forms a group.
8. What are the possible orders of its subgroups?
9. Find four distinct subgroups (not including the trivial one).
10. Find the index of each subgroup in the group.
11. Find the distinct left cosets and the distinct right cosets of each subgroup in the group.
12. What observations can you make on the cosets?

I brought to class a Sudoku puzzle to be solved in class, and gave a quick lesson on its solution, to ensure that everyone was familiar with the puzzle. To maximize class participation, the above questions were not handed to the class, but asked orally by the instructor, one by one. The assignment was handed out at the end of the session for completion. To the instructor's satisfaction, the class participated and was able to give the expected answers.

To Question 5, one student, Anton, offered, "there will be 3! blocks." We then proceeded to make a list, $\{abc, bac, \ldots\}$. Another student, Renzo, was able to recognize that this set is the same as the permutation group S_3, as an answer to Question 6. We then rewrote the list as $G = \{(a), (ab), (bc), (ac), (abc), (acb)\}$. As for Question 7, we agreed that the Cayley table is the same as that of S_3. The students recognized that Question 8 was a direct application of Lagrange's

Theorem. As there was not enough time, the class was assigned Questions 9 to 12 as homework.

The following activity was assigned in class in the next session to motivate the concept of quotient groups.

Day 2 activity

1. Let $G = \{(a), (ab), (bc), (ac), (abc), (acb)\}$ be the group formed by the rows of B_1. Find the subgroups $<(ab)>$, $<(ac)>$, $<(bc)>$, $<(abc)>$, $<(acb)>$.
2. Let $H = <(abc)>$.
    - a. Describe the permutations in H.
    - b. Find $|G:H|$, the index of H in G. Are the left cosets and the right cosets of H in G necessarily equal? In other words, does $|G:H| = 2$ imply that $gH = Hg$ for all $g \in G$?

We did Question 1 and Question 2a as a group and the class was given about five minutes to prove 2b: $|G:H| = 2$ implies that $gH = Hg$ for all $g \in G$. Again, to my satisfaction, at least one student, Ryan, wrote a good proof and was invited to share it with the group by presenting it on the board (see Figure 2). As the opportunity to introduce quotient groups presented itself, we attempted to construct a Cayley table with the cosets of $H = <(abc)>$ in G, namely, $\{H, (ab)H\}$. Not having an operation to work with, a few difficulties arose, so the students built a Cayley table. Later, one student, Kevin, conjectured and proved that "If H is a normal subgroup of G, then $(aH)(bH) = abH$ for all elements a and b in G". This became our operation. The class then realized that $\{H, (ab)H\}$ forms a group with identity H, under the operation $(xH)(yH) = xyH$. Following standard group theory notation, we called the group G/H. The class then made some conjectures about some properties of quotient groups. In particular, a student, Joe, conjectured that "if a group is cyclic, then all its subgroups must be normal." He then briefly explained why this holds.

Primitive conjectures, RME, and quotient groups

Lakatos's notions of primitive conjectures and RME's principles of horizontal and vertical mathematizing contributed to the didactics of difficult group algebraic concepts. I used them extensively in the classroom design experiment to support the learning of normal subgroups and quotient groups.

One of my main goals was to facilitate the discovery or reinvention of these algebraic structures. I accomplished this by guiding the class to make observations leading to conjectures and proofs. Students employed a Cayley table to "primitively" conjecture that "If H is a normal subgroup of G, then $(aH)(bH)=abH$." As indicated by Lakatos, this conjecture will lead to a proof, examples and counter-examples and re-examination of the proof. In our example, students used information about x, y, and H to find that under the operation $(xH)(yH) = xyH$, the set $\{H, (ab)H\}$ forms a group with identity H. The proof will make it easier to further explore properties of quotient groups.

B_1 a	3	4	2	1	7	6	9	5	8
B_1 b	7	8	6	9	5	4	1	3	2
B_1 c	9	5	1	3	8	2	6	7	4
	4	7	9	6	2	5	3	8	1
	2	6	3	4	1	8	5	9	7
	8	1	5	7	3	9	4	2	6
	1	2	8	5	6	3	7	4	9
	5	9	7	8	4	1	2	6	3
	6	3	4	2	9	7	8	1	5

B_1 a	3	4	2	1	7	6	9	5	8
B_1 c	9	5	1	3	8	2	6	7	4
B_1 b	7	8	6	9	5	4	1	3	2
	4	7	9	6	2	5	3	8	1
	2	6	3	4	1	8	5	9	7
	8	1	5	7	3	9	4	2	6
	1	2	8	5	6	3	7	4	9
	5	9	7	8	4	1	2	6	3
	6	3	4	2	9	7	8	1	5

Figure 1. Sudoku in which Block 1 (B_1) has rows a, b, and c (top); when rows b and c from Block 1 are switched (bottom), the result is still a Sudoku solution.

The steps leading to the example provided in Figure 2 and comments made by the students illustrate an observation-conjecture-proof paradigm for mathematical discovery in the classroom. Through discussions, the instructor guides the students in an observation-conjecture-proof process. This process may sometimes be cyclical. Learners achieve the observation-conjecture step through thought experiments. Students might make different observations leading to various conjectures. Lakatos theorized how to go from conjecture to proof. Throughout the process, RME principles of horizontal and vertical mathematization are applied, as illustrated in the following examples.

Kevin's case

Observation: From his Cayley table (shown in Figure 3), Kevin observed that the operation he constructed works for a group with 2 elements.

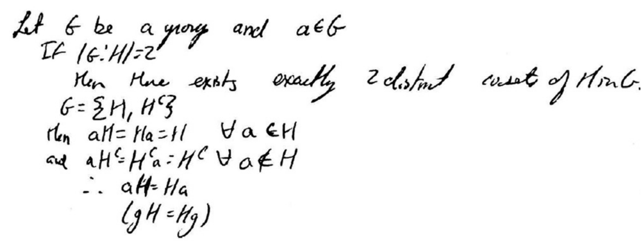

Figure 2. Ryan's proof.

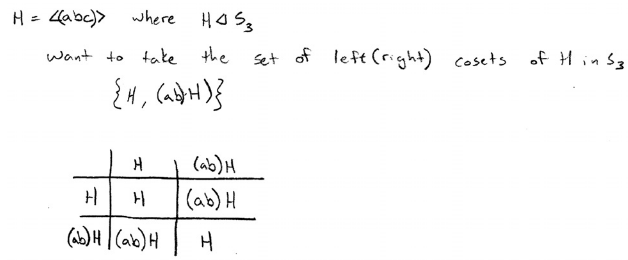

Figure 3. Kevin's Cayley table.

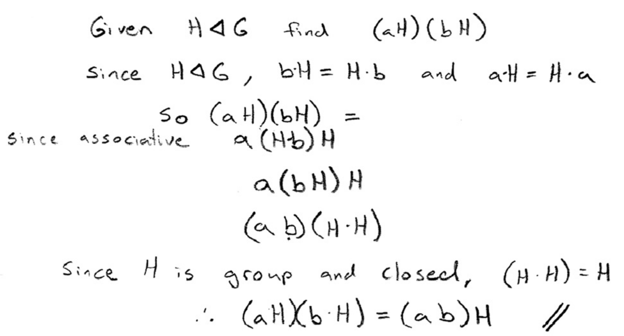

Figure 4. Kevin's proof (note: in the proof, Kevin used a triangle symbol to mean "is a normal subgroup of").

Conjecture: If H is a normal subgroup of G, then $(aH)(bH) = abH$ for all elements a and b in G.
Proof: In class, by Kevin (see Figure 4).

Ryan's case

Observation: Ryan observed that the cosets of H in G partition G.
Conjecture: If H has index 2 in a group G, then its left cosets and right cosets in G are equal.
Proof: In class, by Ryan (see Figure 2).

In each case, a student made an observation, stated a conjecture and proved it. The discovery process incorporated RME tenets of horizontal mathematizing and vertical mathematizing. In Kevin's case, schematizing by constructing a table and the use of the table for exploring group concepts is a form of horizontal mathematizing. He effectively used a mathematical tool to re-invent an operation. Also, Kevin's moving from an example of a set of two elements to deduce an operation for the general case, and his use of axioms and operations in the proof, constitute vertical mathematizing. In the vertical mathematization, he has used symbols and formal mathematical language to solve the problem. Ryan's case is similar to Kevin's case with respect to vertical mathematizing. He uses formal mathematical language to articulate a proof within the algebraic system. As for Ryan's horizontal mathematizing, a good example is his visualization of the problem in terms of partitions. He discovered a relation between partitions and normal subgroups.

Mathematical observations as horizontal mathematizing and vertical mathematizing

In this instructional design experiment, I found that mathematical observations form the genesis for discovery. Although not explicitly asked of the class, the key observation in Day 1's activity is made in Question 3: when switching any two rows within a block, the solution remains a Sudoku solution. This observation leads to the further observation in Question 6, that the rows within a block of a Sudoku solution are similar to S_3. As the language of these observations is vague, they are typical of horizontal mathematizing.

Mathematical observations can also be a form of vertical mathematizing as they lead to conjecturing. In this case, the observation that "the set of rows within a block in a Sudoku solution is isomorphic to S_3" is an example of vertical mathematizing. Observations may be very useful when proving theorems. In the second part of my instructional design, Ryan was able to come up with a very good proof during class discussion. When interviewed briefly after class about his proof, Ryan revealed that his observation that cosets partition the groups was very helpful.

Conclusion

In this article, I have discussed the genesis of mathematical discovery in the classroom using RME. The starting point was Lakatos's method for mathematical discovery outside of the classroom. I observe that the primitive conjecture plays the same role as hypotheses do in the scientific method. I also conclude that conjectures are preceded by mathematical observations, just as hypotheses follow observations in science. Mathematical observations are important in the discovery or the re-invention of mathematical concepts; they are, in fact, commonly used in primary school classrooms that use the precepts of Realistic Mathematics Education. Learners can make mathematical observations through thought experiments, such as searching for patterns, for example. For optimal pedagogical outcomes, and also to encourage students to think mathematically and to discover mathematics in the way a mathematician would (which can be achieved through horizontal and vertical mathematization), I encourage the use of RME principles in undergraduate classrooms. As mathematical observations are useful for horizontal mathematization and vertical mathematization, they are effective tools for mathematical discovery. Also, since they tend to be informal in nature, making observations in the classroom encourages student participation, which facilitates the discovery process.

Notes

[1] A Cayley table, named after 19th century mathematician, Arthur Cayley, is a multiplication or addition-like table whose entries are the elements in a finite group.

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The outstanding characteristic of anthropological fieldwork as a form of conduct is that it does not permit any significant separation of the occupational and extra-occupational spheres of one's life. On the contrary, it forces this fusion. One must find one's friends among one's informants and one's informants among one's friends; one must regard ideas, attitudes, and values as so many cultural facts and continue to act in terms of those which define one's own commitment; one must see society as an object and experience it as a subject. Everything anyone says, everything anyone does, even the mere physical setting, has both to form the substance of one's personal existence and to be taken as grist for one's analytical mill. At home, the anthropologist goes comfortably off to the office to ply a trade like everyone else. In the field, the anthropologist has to learn to live and think at the same time.

From Geertz, C. (2000) *Available Light: Anthropological Reflections on Philosophical Topics,* p. 39. Princeton, NJ: Princeton University Press.
