EXTENSIONS OF THE SEMANTIC/ SYNTACTIC REASONING FRAMEWORK

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Research on students' proof production is commonly based on a dichotomy between two broad forms of proof-oriented activity. Some students' proof constructions rely heavily upon analytical or sequential reasoning. Such reasoning often focuses on symbolic expressions or logical frames. Another group constructs proofs via more imagistic and intuitive reasoning (*e.g.,* Alcock & Simpson, 2004, 2005). The former approach adheres more closely to formal mathematical structure while the latter may suspend formal exactness to gain insight. However, students' mathematical reasoning shows much more diversity than symbols and images or varying degrees of "formality." What exactly is different about these types of reasoning? How can they be clearly distinguished?

Weber and Alcock (2004, 2009) have developed a very useful way of distinguishing two broad forms of proof-oriented reasoning using Goldin's (1998) theory of representation systems. Weber and Alcock (2009) describe the rules governing the *representation system of mathematical proof* (RSMP) and define any student's proof activity that remains within it to be *syntactic*, corresponding to more sequential, analytical, symbolic, logical, or formal reasoning. They define any proof activity that makes use of alternative representation systems, such as informal verbal or imagistic systems, to be *semantic*, corresponding to more visual, intuitive, or less formal reasoning. Research using this framework (Weber & Alcock 2004, 2009; Alcock & Inglis, 2008) suggests that the dichotomy is analytically fruitful, insomuch as many research participants appear to display a consistent preference for one form of reasoning or the other (Alcock & Simpson, 2004, 2005; Weber, 2009). Also, according to Weber and Alcock (2004, 2009) practitioners of each reasoning style exhibit similar struggles with regard to their ability to produce rigorous proofs (better among syntactic reasoners) and their sense of insight gained from proving (stronger among semantic reasoners).

This approach raises a host of new questions. Researchers disagree about whether syntactic proof production must stay completely within the RSMP (Alcock & Inglis, 2009) or only mostly in that representation system (Weber & Mejia-Ramos, 2009), or whether an entire proof production should be categorized as a whole, or be parsed into semantic and syntactic proof actions or chunks. Weber and Mejia-Ramos (2009) point out that the framework may differ according to whether its purpose is to describe particular students' reasoning or to distinguish proof attempts across a broad data set. In this article, I attempt to clarify criteria for distinguishing which representation system best models students' reasoning, especially as it pertains to language use. In so doing, I question Alcock and Inglis's (2008) assumption that every proof production "begins" in the RSMP because the statement to be proven is in that representation system. In contrast, I define a third category called *linguistic proof reasoning* for proof attempts that never operate within the RSMP. Secondly, I extend the use of Goldin's (1998) theory by characterizing some proof-oriented *competencies* that can foster successful proof production in a semantic style. I illustrate these ideas using two students' proof attempts in a paired, task-based interview related to an undergraduate, axiomatic geometry course.

The theory of representational systems and semantic and syntactic reasoning

Goldin (1998) defined representation systems to serve as models for coherent bodies of mathematical reasoning and inscription. A representation system consists of three primary elements: *characters, permitted configurations,* and *structure*. Characters comprise the un-interpreted elements of a system such as words, symbols, drawn curves, *etc*. The system also includes rules by which combinations of these characters are deemed to be permitted configurations. Finally, representation systems include operations by which permitted configurations are related to or transformed into other permitted configurations. For instance, in the representation system of formal logic, characters such as p, q, $\neg$, $\wedge$, and $\rightarrow$ can be combined to form permitted configurations such as $p \rightarrow q$ but not $\rightarrow pq$. An example of structure in this representation system would be the rule that the configuration $\neg(p \rightarrow q)$ is equivalent in truth-value to $p \wedge \neg q$.

Goldin (1998) formulated his theory to address the complexities of student problem-solving behavior (such as proof production). He called students' ability and propensity to perform certain productive problem-solving behaviors *competencies*. Goldin's construct is highly inclusive covering skills and knowledge such as solving quadratic equations and labeling the solutions "roots". It also includes more abstract or metacognitive abilities such as proving by contradiction or engaging in control processes. Goldin's notion of competency differs from terms like understanding, though, because it is linked to problem-solving contexts. He emphasizes that the representation system and attributes of a particular problem can strongly influence students' exhibition of a competency. I attribute a competency to a student whenever I identify multiple distinct and intentional exhibitions of a proof-oriented behavior.

Goldin organizes representation systems into five categories roughly corresponding to different forms of cognitive processing:

1. verbal/syntactic systems,
2. imagistic systems,
3. formal notational systems of mathematics,
4. a system of planning, monitoring, executive control,
5. a system of affective representation. (Goldin, 1998, p. 148)

It is beyond the scope of this article to fully explain all five categories, but Weber and Alcock's (2009) account of basic rules for permissible configurations and structure in the RSMP place it in the third category. Even though Goldin (1998) uses the term "syntactic" in conjunction with linguistic representation systems, his explanation of the categories indicates that he intends "verbal/syntactic systems" to refer to less technical language and "formal notational systems" to refer to technical language. For instance, Weber and Alcock's (2009) descriptive criterion for reasoning within the RSMP include:

> Words used in proofs, especially nouns, have a precise, unambiguous mathematical meaning [...]
>
> Valid proofs must employ acceptable proof frameworks [...]
>
> Arguments cannot be based on impermissible informal representations of [concepts]. (p. 326)

Such syntactic reasoning may include any representation system in Goldin's third category, such as algebraic manipulations. Because they assume that statements to be proven are always in the RSMP, Alcock and Inglis (2008, 2009) suggest that successful *semantic proof production* (involving reasoning outside of the RSMP) generally involves at least two translations: one from the RSMP to an alternative representation system and another to translate the reasoning back into the RSMP (except when disproving by counterexample). These classifications imply, contrary to definitions of proof in research at lower levels of instruction (*e.g.,* Balacheff, 1987; Stylianides, 2007), that students in advanced mathematics courses must move beyond the production of convincing arguments, which may remain outside the RSMP, to producing proofs exhibiting well-formed configurations and appropriate structure within the RSMP. However, the constructs of semantic and syntactic reasoning are not limited to those students who produce such proofs successfully (Weber & Alcock, 2004, 2009).

Classifying the representation system of students' reasoning

Identifying the representation system of any particular reasoning can be challenging (Alcock & Inglis, 2009; Weber & Mejia-Ramos, 2009). Furthermore, researchers must carefully maintain that representation systems are models of student reasoning rather than the other way round, so "structure" must reflect patterns or relationships in students' thinking rather than in mathematical theory. Whenever multiple problem solvers interpret the same inscribed characters and configurations (sentences, equations, diagrams), the same words or grammatical frames may admit different structures in different representation systems (Epp, 2003, provides good examples, such as the understood meaning of multiply quantified statements such as "There is a time to every purpose under heaven," or limit definitions). Similarly, formal characters are often attributed less technical meanings (*e.g.,* "=", ">", "∞").

How can one distinguish whether a student's interpretation of a written statement should be modeled by a representation system in Goldin's verbal/syntactic category or his formal notational system category? I argue that structure is the most determinant element of a representation system, and structure is rarely represented directly in the permitted configurations of these systems. Structure, like grammar or logic, mostly resides in the activity of the prover in relation to a given configuration. Epp (2003), for example, argues that many students' difficulties with logical structure stem from interpreting mathematical expressions using everyday linguistic conventions.

Thus, to model students' reasoning, one must identify the structure their reasoning exhibits. With respect to language use, I identify two pertinent forms of structure that distinguish formal and less formal language use: (1) logic and (2) the distinction between the meaning and sense of a statement. The *meaning* of a statement includes its direct linguistic content, relative to the "precise, unambiguous mathematical meaning" (Weber & Alcock, 2009) of terms. The *sense* of a statement includes the larger body of meaning that may be associated with statements or utterances. For example, the *meaning* of the theorem "If *c* is any point on $\overrightarrow{AB}$ with $0 < AC < \omega$, then $\overrightarrow{AC} = \overrightarrow{AB}$" includes the equivalence of two sets (two rays) under given conditions. Any proofs of this statement must adequately establish set equivalence using acceptable warrants (definitions, axioms, theorems). However, the book from which this theorem was taken (Blau, 2008) describes it by saying: "[This] theorem shows that there is nothing unique about the point B in the name for a ray $\overrightarrow{AB}$" (p. 85). This explanation about uniqueness of names is part of the sense of the statement, part of how the statement is understood. While informal language use usually does not strongly distinguish between meaning and sense, formal language use must restrict itself to direct mathematical meaning (Weber & Alcock, 2009). If students (a) consistently fail to interpret a mathematical statement according to normative logical and mathematical structure and (b) apply alternative structure incompatible with standard interpretations, then I assert that the student is not reading the statement in the RSMP.

An example: Kirk and Oren

The two students featured in this discussion, whom I shall call Kirk and Oren, participated together in a series of nine weekly task-based interviews in conjunction with their fifteen week, third year university geometry course. The course began with an informal introduction to non-Euclidean geometries by examining Euclid's Fifth Postulate (EFP), Playfair's Postulate (PP) and alternative planes such as the spherical, hyperbolic (Beltrami-Klein model), and Minkowski planes. The text and lectures also included units on:

- formal proof techniques such as direct proof and proof by contradiction,
- logical operations on statements such as negation and contraposition, and
- basic set notation and operations.

Over the rest of the semester, the professor introduced a series of 21 axioms for neutral, planar geometry by which any plane could be proven to be "Euclidean," "spherical," or "hyperbolic".

Kirk and Oren were both mathematics majors with emphases in applied and pure mathematics respectively. They both exhibited strong mathematical ability and intended to pursue graduate study in mathematics. The interview referred to in this article occurred after the introduction of the first few axioms. During much of the first interview, the students tried to prove that Euclid's Fifth Postulate implies Playfair's Postulate. This requires proving another claim that I shall refer to as Theorem *. The students expressed these three statements in the following way:

> EFP: "Given two lines cut by a transversal, if the two interior angles on one side of the transversal sum to less than 180°, then the lines will intersect on that side of the transversal."
>
> PP: "Given any line and a point not on that line, there exists only one line through the given point that does not intersect the given line."
>
> Theorem *: "Given two lines cut by a transversal, if the same-side interior angles sum to 180°, then the two lines do not intersect."

Prior research on this task indicates that students find it especially challenging because of the "conditional implies conditional proof frame" it requires (Zandieh, Knapp & Roh, 2008; Zandieh, Roh & Knapp, 2011). Contrary to the basic proof frame for conditional statements ($p \rightarrow q$) in which one begins with the antecedent (p) and, by a chain of deduction, concludes the consequent (q), statements of the form $((p \rightarrow q) \Rightarrow (r \rightarrow s))$ should be proved beginning with the second conditional's antecedent ($r \Rightarrow (p \rightarrow q) \Rightarrow s$).

In using this task, I adapted a paired interview protocol from Alcock and Simpson (2004, 2005). Similar to their experience whenever students with different reasoning preferences (visual and non-visual) were paired, Kirk and Oren often worked somewhat independently and struggled to communicate effectively. The two students also differed in their relative success at proof production. This motivated my inquiry into their *proof production competencies.* The competencies I identify in this article appeared repeatedly during later interviews, but the "EFP implies PP" task particularly highlighted Kirk and Oren's differences because: they began the proving process in relative agreement; they co-constructed an informal argument; but they disagreed about whether and how to translate that argument into an acceptable proof. This disagreement revealed incompatibilities between the ways they interpreted the task and assessed potential proofs.

I initiated the session by asking the students to recall and explain the two key postulates (EFP and PP) before asking them to produce a proof of their equivalence (first proving "EFP implies PP" and then the converse). Because Kirk and Oren struggled to communicate, much of their progress through the task proceeded somewhat independently, so I describe their proof production attempts sequentially.

Stating the conjecture to be proved

When asked to recall and explain EFP, Kirk began by drawing the transversal configuration (two lines cut by a transversal) in the air with his fingers. On paper, he produced a diagram that directly mirrored the diagram for EFP in the textbook (Blau, 2008) and the professor's lecture (Figure 1). Oren said that he would explain the statement in the following way: "any two lines that aren't parallel [extending his hands in parallel paths in front of him], will eventually meet." Further questioning revealed that by "parallel," Oren did not mean "non-intersecting" but the term referred to his perceptual image of Euclidean parallel lines (non-intersecting, equidistant, maintaining congruent corresponding angles with any transversal).

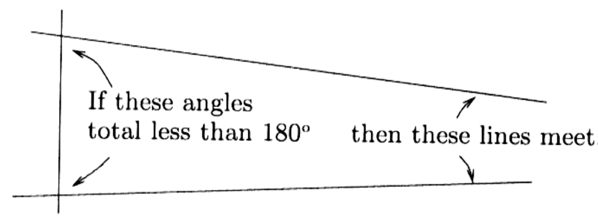

Figure 1. Textbook diagram for EFP.

When the interviewer asked whether his new statement was equivalent to EFP, Oren explained the following thought experiment:

> I just imagine two straight lines that are parallel [extending his arms in his parallel gesture] and if you were to shift a line in any [wiggling his arms], if there was like an axis point and sort of turn it any degree [acting as if he is turning one line between his fingers and thumb by twisting his wrists], even the slightest bit, then they will eventually meet and that's not parallel.

Oren's explanation of EFP differed from the meaning of the postulate in several ways because it did not require a transversal nor did it refer at all to angle sums. Considering Oren's explanation, Kirk pointed out, "Well that is sort of like Playfair's postulate because, you know, there's a line and a point." They recalled the statement of PP, but expressed confusion about properly distinguishing the two statements:

> *Oren:* I feel like we are going around in circles. It is sort of like circular reasoning. Euclid's implies that and Playfair's implies, proves Euclid's Fifth [tracing his finger in an ellipse alternating between pointing to the two statements imagined as locations].
>
> *Kirk:* They're like the same, that's why. Right?

Kirk and Oren interpreted their shared sense of statement equivalence differently: Oren articulated a relationship of mutual implication while Kirk simply identified them as "the same." Oren described a *structure* for statement equivalence that is acceptable within the RSMP. Kirk appeared to link the senses of the two statements such that they entailed the same thing. I then asked the students to prove the claim "EFP implies PP."

Kirk: linguistic proof reasoning

Kirk's initial argument classified all possible transversal configurations obtained by rotating line l at point P in the diagram they drew (see Figure 2), consistent with Oren's thought experiment. Unlike Oren's initial explanation, Kirk referred to angle sums ($\alpha+\beta$) and used language very similar to the formal statement of EFP. He verbally classified the cases in the following way:

- if <180, then the lines meet,
- if >180, then the angles on the other side are <180 and the lines will meet on that side, and
- if the angles sum to 180 on either side, "we know through class that the lines will not meet, lines l and m will not meet."

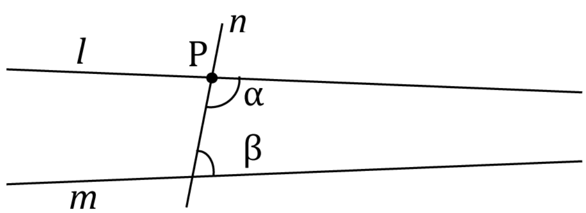

Figure 2. Kirk's diagram for proving "EFP implies PP".

Kirk's final statement refers to the proof of Theorem * developed in class. Kirk articulated his cases in an "if… then…" frame. At this point, Kirk appeared to use syntactic reasoning by applying EFP to the first two cases and Theorem * to the last case. However, he did not seek to connect his three cases to the meaning of PP. Instead, at least 7 times throughout the conversation he paraphrased his three cases with some form of the claim: "there's only one instance where the lines will not meet." He continually asserted that this proved PP because:

> If the angles add to 180° exactly, then the lines will not meet, and it's like only one instance where the lines will not meet, so basically you know there is only one time where the lines will not meet and Playfair's postulate basically states that there's only one instance or case where the lines will not meet. And so they kind of like, either way, there's only one case.

This exemplified Kirk's competency for paraphrasing mathematical statements, which he displayed on multiple occasions ostensibly to reduce cognitive load. Were Kirk reasoning in a manner compatible with conditional logic (appropriate to the RSMP), then the following invalid syllogism would model the logic of his reasoning: EFP ⇒ "only one instance," PP ⇒ "only one instance," thus EFP ⇒ PP.

Kirk's reasoning, though, was not compatible with conditional logic or the structure of the RSMP. It appeared that Kirk's argument asserted that both statements classify the same set of arrangements of lines in the plane, from which he concluded that they are equivalent or say the same thing. In other words, Kirk interpreted EFP and PP using an informal linguistic representation system in which he failed to appropriately distinguish the meaning and sense of EFP or PP. He interpreted these statements as classifications or descriptions of transversal configurations with identical results. For instance, he explained there was only one line l with angle sum 180 saying:

> Well you can't really develop that line in any other instance cause the instance, the cases that we are trying to develop these lines in, like we try to develop a line in the case that alpha plus beta is greater than 180°, so you have a bunch a different lines there, but each one will meet m […] But in the case where $\alpha+\beta=180°$, we know there's only one instance that can have it.

Kirk's use of words like "develop" or "instance" indicate that he imagined drawing different lines and then using EFP and PP to classify the occurrences in a quasi-empirical sense. He did not treat the conditionals EFP or Theorem * as inferential links between distinct properties of those transversal configurations (angle sums and line intersections), but rather as descriptions of property co-occurrence (when this happens, that also happens). He made no clear distinction between the meaning of PP and his paraphrase that parallels are unique. As further evidence, when later asked whether EFP and PP were true on the hyperbolic plane, Kirk tried to contradict PP by drawing three non-intersecting hyperbolic lines. Figure 3 recreates his diagram. Kirk's example did not negate the uniqueness of parallels *through a point* (which is possible in the hyperbolic plane), but rather negated the uniqueness of parallel lines in general (which is also possible in the Euclidean plane).

Kirk's reasoning about the proof was like semantic reasoning because he interpreted the meaning of the statements in terms of an imagistic representation system. However, his reasoning did not fit Alcock and Inglis's (2008) definition of semantic reasoning. This is because Kirk did not interpret

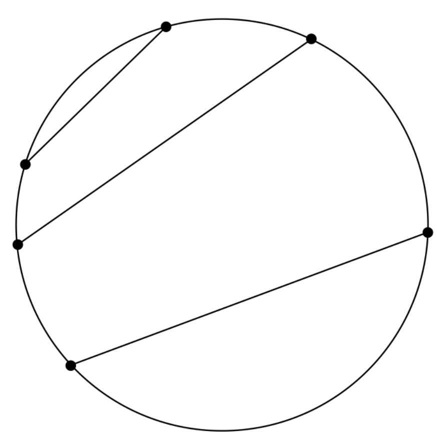

Figure 3. Kirk's intended counterexample to PP on the hyperbolic plane.

the statement using structure appropriate to the RSMP, meaning he did not "begin" there. One reason is that Kirk's "descriptive" or "empirical" interpretation of EFP and Theorem * did not maintain any "directionality." Later in the interview he said:

> Euclid's Fifth is that these angles are less than 180 deg and that would meet on that side.
>
> You know that through [Theorem *] that angle β plus α has to equal 180° if it's parallel.
>
> Cause if they don't meet then this angle would have to be supplementary to this angle here.

In the first quotation, Kirk appears to treat EFP as conjunctive ("and" statement) rather than conditional. His reasoning about Theorem * clearly reversed the "direction" of implication, using non-intersection to conclude the angle sum. Many previous studies (*e.g.*, Damarin, 1977; Hoyles & Küchemann, 2002; O'Brien, 1972) show that the logic of people's informal use of conditional statements may resemble the standardized logic of conjunctions or bi-conditionals (if and only if) as Kirk's reasoning does here. In Goldin's (1998) terms, Kirk connected permissible statements in the task via *structure* inappropriate to the RSMP.

Kirk's reasoning did not adhere to RSMP structure for two reasons: he did not distinguish meaning from sense and the logic of his reasoning was not compatible with formal, propositional logic. As such, the claim Kirk attempted to prove differed non-trivially from the claim understood by the mathematical community. Consequently, he did not prove the claim by advanced mathematical standards. Figure 4 contrasts my account of Kirk's (unsuccessful) *linguistic* proof reasoning with Alcock and Inglis's (2008) accounts of (successful) syntactic and semantic proof productions. The essential distinction is not that Kirk used an informal verbal representation system, which students often do in semantic proof production (as the "other" representation system). The essential difference is that throughout the proving episode, Kirk never understood the formal statement in the RSMP because he never applied appropriate structure in his reasoning about it. Theoretically, visual reasoning would not necessarily pose the same problem because students must translate statements from verbal form to an imagistic representation [1]. However, when Kirk interpreted the statement of the theorem using inappropriate structure, no translation took place because he interpreted the same set of characters and permissible configurations via a different representation system. As such, students' nonstandard logic and informal meanings for mathematical terms may often keep them from interpreting mathematical statements "within" the RSMP. Consequently, what the statement "says" to a mathematician or researcher differs from what it "says" to that student. Researchers accordingly must attend carefully to the structures students use to interpret and act upon mathematical claims.

Oren: competencies supporting semantic proof production

Oren initially agreed with Kirk's proof approach constructing three categories. However, they disagreed about whether it constituted proof:

> *Kirk:* So it's all there. It's just basically putting into words now that make sense.
>
> *Oren:* [But] that's proof. That's what a proof is.
>
> *Kirk:* It's all there though. You got it all there.
>
> *Oren:* There's something we're missing.

Oren's initial formulation of the equivalence of EFP and PP differed from Kirk's in that he explained "sameness" in terms of mutual implication via proof. Kirk paraphrased or reformulated the two claims by the same "only one instance" expression. Oren treated the two statements as distinct entities, as indicated by pointing to the two statements imagined as locations. I claim this reflected Oren's competency for *reifying statements or propositions*. One characteristic of reifying statements or propositions is the ability to suppress

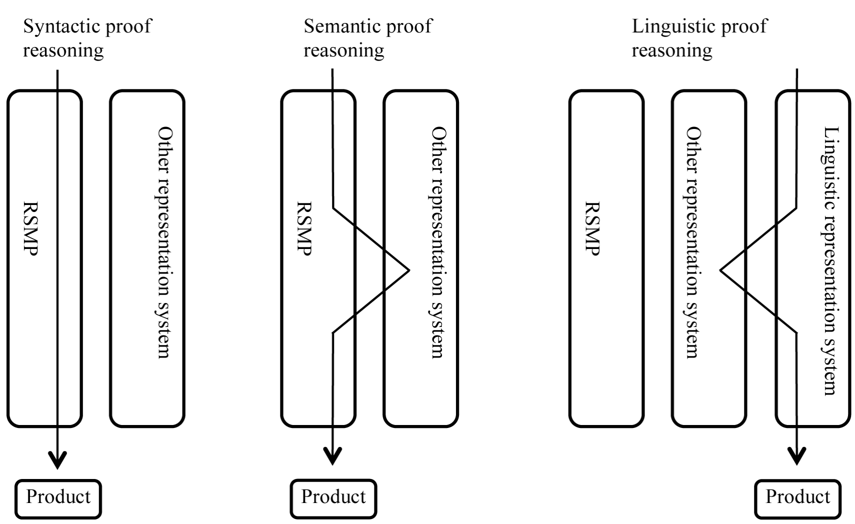

Figure 4. Syntactic or semantic (Alcock & Inglis, 2008) versus Kirk's linguistic proof reasoning.

"knowledge" or "conviction" of the truth-value of a claim for the purposes of deduction. In Goldin's (1998) terms, Oren treated EFP and PP as characters in an inferential representation system that should be linked by the configurations "EFP⇒PP" and "PP⇒EFP" permitted by the structure of deductive proof.

Furthermore, Oren interpreted the conditional forms of EFP and "EFP implies PP" with structures appropriate to the RSMP. Initially, he acted according to the basic conditional proof frame (assume the antecedent and conclude the consequent), which was inappropriate for this particular claim. This reflected Oren's awareness of common proof frames. He used semantic (imagistic) reasoning as he sought to get "from Euclid's to Playfair's" by transforming a transversal configuration into a point and line diagram. At that point, his progress was stymied because Oren's proof plan involved removing a transversal. To foster an appropriate conditional-implies-conditional proof frame, I eventually encouraged Kirk and Oren to draw a diagram of the hypotheses of PP.

To relate the two geometric arrangements, Oren began to explore the interrelationships and invariances within his thought experiment of rotating one of the two lines (line l in Figure 2). He noted that the angle β would not change and examined the relationship between the angle α and the line l. He said, "Cause like the line [l] that we draw [wobbling his hand to imitate rotating the line] is umm, affects, directly affects alpha." Because intersecting lines form angles, Oren imagined varying the line influencing angle measures converse to EFP and Theorem *'s "direction" of implication. According to Kirk's thought experiment of drawing lines through point P, Kirk asserted that the line creating angle sum 180° was unique. Oren considered Kirk's claim, saying, "And there exists only, a unique line. How do we know that the line's unique? ... How do we know that the angle, the line that's generated by [the angle]." Once Oren reformulated the problem from classifying lines to deducing from the property $\alpha+\beta=180$, Oren began to reverse the direction of implication by saying that the angle "generated" the line.

Whereas Kirk seemed to maintain a "directionless" or empirical link between line intersections and angle sums (exacerbated by the fact that the converses of EFP and Theorem * are true in Euclidean geometry), Oren was able to separate the two properties and examine the exact nature of the inferential links between them, thus *reifying the propositions*. While Oren's reasoning was semantic because he relied heavily on his diagrams, Oren reasoned about and applied EFP and Theorem * in a manner appropriate to the direct meaning of such conditional claims within the RSMP. Reifying the propositions (angle sums and line intersections) allowed Oren to distinguish their truth-values, imposing structure among those configurations appropriate to propositional logic (as required by the RSMP).

Oren: seeking and finding warrants

In addition to reifying statements, Oren *sought and identified warrants* for mathematical claims. Beginning from the hypotheses of PP, Oren wanted to construct a transversal line. However, he hesitated as to whether he was "allowed" to do so without an express reason. In the case of claiming that there is only one line that produces the angle sum of 180°, he asked, "How do we know that the line's unique?" indicating a desire for a warrant or reason to justify the claim. Later, when Kirk appeared to apply the converse of Theorem *, Oren pointed out "But, no you are using Playfair's Postulate," identifying the implicit warrant by which Kirk's claim could be deduced. Weber and Alcock (2005) observed that professional mathematicians often interpret conditional claims as "warranted conditionals" meaning they assess the claim according to access to appropriate reasons by which the consequent can be deduced from the antecedent. Oren appeared to employ a similar sense of conditionality. Previous research (Alcock & Simpson, 2004; Weber & Alcock, 2009) indicates semantic reasoners (which Oren revealed himself to be throughout the interviews) tend not to exhibit a need for translation into the RSMP because imagistic arguments fully convince them. Oren's competency of seeking and identifying warrants allowed him to avoid that trend. During the first interview Oren did not have access to sufficient warrants for all of his claims. During the final interview of the semester when Kirk and Oren revisited this proof, Oren was able to supply (from the axioms and definitions covered in class) all of the warrants for the proof.

Conclusions

One of this article's essential contributions to the syntactic/semantic framework is to draw attention to the structure students exhibit in their reasoning rather than to assume students "begin" with such structure as they interpret a formal mathematical statement. It is important to note that Kirk's reasoning did not lack standard logical structure because he was not familiar with such structures regarding conditional statements (*i.e.,* it was not a matter of resources). Kirk's professor spent several weeks teaching pertinent logic and proof operations. Even during the same interview, Kirk called upon this "institutionalized" competency when Oren suggested proving "PP implies EFP" by contradiction. Kirk correctly laid out the assumptions for beginning such a proof (assume PP and ¬EFP). However, Kirk never invoked those competencies for interpreting EFP or for proving "EFP implies PP." Kirk's access to formal competencies while he engaged in linguistic reasoning further emphasizes the point that structure is enacted by the student in the course of their proof-oriented activity. It also emphasizes that mathematics educators need to identify ways to foster students' broader use of the logical tools they learn (Stylianides, Stylianides & Philippou, 2004). Further research should explicate when and why mathematics students adopt formalized language use and normative logic, lest they remain an isolated set of skills.

I acknowledge the limitation that my criteria for distinguishing structure focus particularly on language use. This does not fully address issues of delineating representation systems in a broader context. Verbal statements referring to diagrammatic inscriptions seem to blur the lines between representation systems. Attending to the blending of or interactions between representation systems may provide fruitful new insights.

My second contribution to the framework is to incorporate proof-oriented competencies, which may extend or explain our understanding of student's proving "styles" (Weber & Alcock, 2009). I concur with Weber and Alcock

(2009) that students' styles should be fostered rather than selected for by instruction, and emphasis on semantic and syntactic competencies may suggest pedagogically actionable tools for doing so. Further investigations may also identify key competencies Kirk lacked or how he could be encouraged to more appropriately invoke the logical competencies he possessed.

Notes

[1] Cases such as geometry proofs in which theorems are presented with diagrams are less clear. It is certainly significant that Kirk recalled the key statements via a diagram.

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Talk alone will certainly not suffice to engender spontaneous non-egocentric concerns and ethically developed persons. Even more than experiences of insight, words and concepts can be easily grasped at, taken as ground, and woven into a cloak of egohood. Teachers in all contemplative traditions warn against taking fixed views and concepts as reality. We simply cannot overlook the need for some form of sustained, disciplined practice or *pratique de transformation de sujet* to use Foucault's apt term. This is not something that one can make up for oneself—any more than one can make up the history of Western science for oneself. Nothing will take its place. Individuals must personally discover and grow into their own sense of virtual self.

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