A SET-ORIENTED PERSPECTIVE ON SOLVING COUNTING PROBLEMS

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The purpose of this article is to present and describe one particular perspective on solving counting problems (a set-oriented perspective) that emerged during a study involving post-secondary students. The notion of a set-oriented perspective is presented both theoretically and empirically and is integrated with Harel's (2008a, 2008b) *ways of thinking.* I make the case that such a perspective offers a valuable lens through which to study the learning and teaching of counting and may be an effective factor in helping students correctly solve counting problems.

There is clear evidence in the literature that students at all levels struggle with solving counting problems correctly. Many researchers have identified student difficulties with combinatorial topics and reported low success rates (*e.g.*, Eizenberg & Zaslavsky, 2004), and others speak to the overall difficulty of teaching and learning counting (Annin & Lai, 2010; Martin, 2001; Tucker, 2002). Some researchers have taken strides toward helping students become more successful at solving counting problems (*e.g.*, Batanero, Navarro-Pelayo & Godino, 1997; Eizenberg & Zaslavsky, 2004; English, 1991; Lockwood, 2011; Maher, Powell & Uptegrove, 2011), but there remains a need for further examination of the ways of thinking that students bring to counting problems.

In particular, the current combinatorics education literature has not explicitly highlighted the importance of focusing on *sets of outcomes* in students' solving of counting problems. A set of outcomes is the set of things (objects, arrangements, partitions, *etc.*) a student is trying to count, the cardinality of which is the answer to the given counting problem. I call this focus on outcomes a *set-oriented perspective* on counting, which is a way of thinking about counting that involves attending to sets of outcomes as an intrinsic component of solving counting problems. The point is that by adopting such a perspective, in which facility with outcomes is a natural part of how they approach and solve counting problems, students may avoid common pitfalls such as incorrectly applying a formula, inadvertently over-counting, or confusing issues of order. I elaborate this perspective throughout the remainder of the article, illustrating the ideas with examples from my research.

Sets of outcomes in the literature

While research has not yet explicitly focused on the effects of students' reasoning about sets of outcomes, some work has described counting in terms that are reflective of a set-oriented perspective. For example, in their study of undergraduate students, Hadar and Hadass (1981) intimate that successfully solving a counting problem entails correctly identifying the set of outcomes to count. English (1991) had young students count shirt-pant combinations, and she reports that students' strategies ranged from random to systematic listing. Although English focused on students' strategies generating the lists (and not specifically on how her students conceived of the set of outcomes), her work is an example in which the set of outcomes played an important (albeit implicit) role in students' counting. Also, Mamona-Downs and Downs (2004) had students solve counting problems by creating a bijection between sets, suggesting (again, implicitly) that these researchers recognize sets of outcomes as a fundamental aspect of counting.

Attending to outcomes is also fundamental to computing probabilities. Referring to the fact that the likelihood of an event is determined by the ratio of desirable to total outcomes, Abrahamson (2014) notes that, "Implicit to a successful application of this algorithm is the construction of an event space that includes all possible, unique outcomes" (p. 242). Some research on probabilistic thinking has thus also addressed outcomes, often in terms of counting event spaces, although students' combinatorial reasoning involving outcomes tends not to be the explicit focus of their work (*e.g.*, Abrahamson, Janusz & Wilensky, 2006; Shaughnessy, 1977; Tarr & Lannin, 2005).

A set-oriented model of combinatorial thinking

The idea that a set-oriented perspective might be significant for students emerged during a study in which I interviewed post-secondary students as they solved five counting problems. The resulting model of students' combinatorial thinking (Lockwood, 2013) emphasizes sets of outcomes as a major component of such thinking (Figure 1, overleaf).

In this model, *Formulas/Expressions* are mathematical expressions that yield some numerical value, and formulas/expressions are considered to be different if they differ in form. *Counting processes* are the imagined or physical enumeration processes in which a counter engages. *Sets of outcomes* are the sets of elements that one can imagine being generated or enumerated by a counting process, the cardinality of which may determine the answer to a counting problem. The model also involves relationships between these components (which I elaborate elsewhere: see Lockwood, 2013; Lockwood, Swinyard & Caughman, in press). In terms of this model, the set-oriented perspective emphasizes *sets of outcomes,* with a robust and flexible understanding of the relationship between *counting processes* and *sets of outcomes.* Individuals with such a perspective would recognize that different ways of structuring a set of outcomes might reflect different respective counting processes.

To illustrate these components, consider the Passwords problem:

> A password consists of eight upper case-letters. How many such 8-letter passwords contain at least three E's?

The *set of outcomes* is the complete set of all desirable passwords. This problem has an *at least* constraint, and a potential *counting process* would be to break the problem down into cases, in which the passwords contain exactly three, four, five, six, seven, or eight Es [1]. For any of those cases, the number of passwords containing exactly k Es is found by an application of the multiplication principle: first choosing spots for those Es to go, represented by $\binom{8}{k}$, and then filling in the remaining $8 - k$ spots with any of the 25 non-E letters. This counting process leads to the *formula/expression*

$$\sum_{k=3}^{8}\binom{8}{k}25^{8-k}$$

or

$$\binom{8}{3}\cdot 25^5+\binom{8}{4}\cdot 25^4+\binom{8}{5}\cdot 25^3+\binom{8}{6}\cdot 25^2+\binom{8}{7}\cdot 25^1+\binom{8}{8}\cdot 25^0.$$

Another tempting approach to the Passwords problem reflects a common error. We could first choose three spots to fill with Es ($\binom{8}{3}$) and argue that once the three Es constraint is met, the remaining five spots can be any letters, including E (26^5). The resulting expression of $\binom{8}{3}\cdot 26^5$, however, results in a number that is far too large.

A set-oriented perspective would focus one's attention on the outcomes that are being counted. By being attuned to ways in which the product $\binom{8}{3}\cdot 26^5$ would generate outcomes, a student could discover how this answer causes particular passwords to be counted more than once, thus explaining why overcounting occurred. For example, the password E E E A B E E E is counted multiple times, such as when Es were placed in the first three positions (E E E _ _ _ _ _) in the $\binom{8}{3}$ step of the process and the rest of the password was filled in with ABEEE in the 26^5 step, and then again when Es were placed in the last three positions (_ _ _ _ _ E E E) and the rest of the password was filled in with EEEAB. Without considering outcomes, it can be very difficult to determine why such a seemingly logical counting process is incorrect.

In this article, I also draw on Harel's (2008a, 2008b, 2008c) *duality principle,* which proposes that mathematical knowledge is twofold, consisting both of students' understanding of mathematical content and the characteristics of their knowledge that more broadly influence their mathematical practice. To clarify this duality, Harel (2008c) articulated the notion of a *mental act,* which includes activities like interpreting, conjecturing, explaining, searching, and problem solving (p. 3). Harel proposed that a *way of understanding* is "a particular cognitive product of a mental act carried out by an individual" (p. 4), whereas a *way of thinking* is "a cognitive characteristic of a mental act" (p. 4). As an example, for the mental act of problem solving, a solution to a particular problem represents a way of understanding, but a general problem solving strategy, applicable across a variety of contexts and problems, can represent a way of thinking. An important aspect of Harel's duality principle is that ways of understanding and ways of thinking interact with one another.

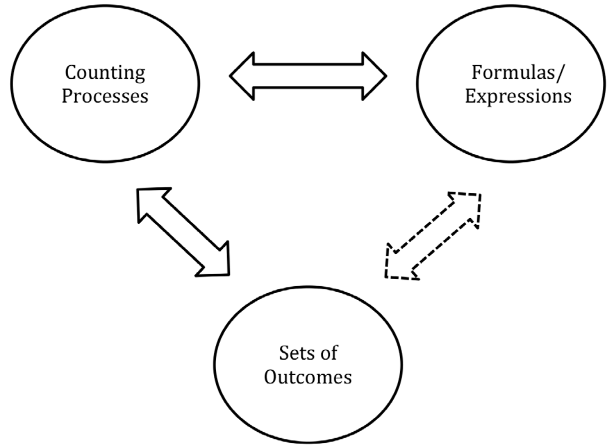

Figure 1. A model of students' combinatorial thinking (Lockwood, 2013).

For the mental act of solving counting problems, ways of thinking represent general characterizations of how a person views counting, and this may affect how that person approaches the solving of counting problems. In this way, the *set-oriented perspective* is a way of thinking about counting that involves viewing an explicit focus on sets of outcomes as a fundamental aspect of solving counting problems. The term *perspective* is intentionally chosen because of its broad connotation, referring to a person's overall estimation of what the act of counting entails. While to some, this way of thinking about counting is apparent (one might argue that counting *necessarily* involves determining the cardinality of a set of outcomes), students do not always seem to view counting from such a perspective. Indeed, the counting activity of many students is not primarily grounded in reasoning about the outcomes they are trying to count. Instead, counting tends to involve attending to key words, simply applying formulas, or matching problems to known problem types. In the sections that follow, I elaborate the idea of a set-oriented perspective, illustrate the perspective through empirical data, and articulate why it might represent an important way of conceptualizing counting.

A set-oriented perspective in students' work

In Harel's language, a student's work on individual problems could be considered ways of understanding: products of the mental act of solving counting problems. These ways of understanding can reflect ways of thinking (or reveal a lack of a way of thinking), and looking at students' work on multiple problems can shed light on their more general ways of thinking about counting. For this reason, in this section I present three students' work across multiple problems, highlighting three aspects of counting that commonly arise for students: using case breakdowns, determining whether or not "order matters," and dealing with issues of overcounting. By presenting a set-oriented perspective in situations involving significant counting issues like these, I hope to emphasize the potential benefit that a set-oriented perspective can confer. A brief discussion of each aspect of counting is provided below to set the context for subsequent discussion.

The *use of cases* is not specific to combinatorics (the strategy is a common problem-solving technique; see Schoenfeld, 1980) but in the context of solving combinatorial tasks, case breakdowns relate naturally to sets of outcomes. A partition of a set is a division of the set into nonempty, disjoint subsets, the union of which is the set itself. In a counting problem, a case breakdown amounts to a partition of the set of outcomes. If each subset can be counted separately, their cardinalities can be summed to arrive at the total cardinality of the desirable set. However, while case breakdowns can be productively related to sets of outcomes, students can struggle with sufficiently articulating when and why they use cases.

In *errors of order* (Batanero *et al.*, 1997; Mellinger, 2004), students either impose order when they should not, or they fail to impose order when they should. Colloquially, this issue of order is often phrased as whether "order matters" or "order does not matter" (*e.g.*, Mellinger, 2004). "Order mattering" precisely depends on whether or not certain outcomes should be considered distinct: whether the outcomes themselves are *arrangements* of things or *sets* of things. This distinction of what is being counted clarifies whether or not certain outcomes should be considered distinct, ultimately helping to determine the cardinality of the set of outcomes. Deciding whether order matters can be difficult for students, and it easily devolves into relying on memorized formulas or vague intuition. Although students do not always think of it this way, the commonly used expression of "order mattering" heavily relies on the nature of the outcomes being counted.

Authors of combinatorial texts (*e.g.*, Martin, 2001; Tucker, 2002) and combinatorics education studies (*e.g.*, Batanero *et al.*, 1997; Mellinger, 2004) urge counters to use caution when dealing with *errors arising from overcounting*. More than simply causing an incorrect answer, overcounting is dangerous because it can occur in subtle ways, surprising students in how difficult it is to detect. Issues of overcounting can often be resolved by identifying an outcome that is counted too many times (as in the Passwords problem) or that should not be counted as a desirable outcome. However, it is not always clear to students to resolve issues of overcounting in this way.

The following three cases of students' set-oriented ways of thinking and ways of understanding help to further articulate the notion of a set-oriented perspective, and also facilitate a deeper discussion about how the ideas presented in this article relate to Harel's duality principle [2].

Kristin: an underdeveloped set-oriented perspective

In Kristin (all names are pseudonyms), we see a student who did not demonstrate a robust, consistent set-oriented perspective. There is a tension for her between viewing counting as primarily involving rules, key words, and intuition, and seeing counting as fundamentally counting the set of outcomes. When Kristin does appeal to outcomes, her reasoning about cases, order, and overcounting is much stronger and more reliable than when she does not consider the set of outcomes.

In two different instances on the Passwords problem, Kristin's work does not reflect an attention to outcomes. In the excerpt below, Kristin had correctly noted that she should use cases (although she went on to count the cases incorrectly). However, when pressed about why she chose cases, her reasoning seems to reflect that her decision-making is guided not by considering outcomes but is based on memorized rules and key words:

> I: Can I ask what made you think to do cases?
>
> K: 'Cause it says at least three, so I know I can have up to eight [...]
>
> I: Can you say more about why you added?
>
> K: I added them because [...] when it says "or" I always think of add. And for "and" I always think multiply [...] So cases I always know, add them.

As she continued to work on the Passwords problem, Kristin explained why she was using combinations, again demonstrating that her understanding of why order did not matter was not grounded in the set of outcomes:

> K: I'm doing the combination ones because I'm pretty sure order doesn't matter with combination [...] I don't want order to matter.
>
> I: Okay, and how come?
>
> K: I'm not sure about that one (laughs). I just kind of go off my gut for it, on the ones that don't specifically say order matters or it doesn't matter.

Kristin's language does not suggest a set-oriented perspective toward counting, but rather discloses a view that counting involves intuition-based guessing. She lacks facility with reasoning about outcomes, which would allow her to use the fact that disjoint cases partition the set of outcomes or that the nature of an outcome could help resolve issues of order. Her work on the Passwords problem thus does *not* reflect a set-oriented perspective.

However, Kristin's work on other problems showed instances in which she did consider outcomes. On the Groups of Students problem (In how many ways can you split a class of 20 into 4 groups of 5?), she had arrived at an initial answer of

$$\binom{20}{5}\cdot\binom{15}{5}\cdot\binom{10}{5}\cdot\binom{5}{5}$$

This answer is incorrect, but her decision to use combinations to select people in the groups is correct. As I investigate why she used combinations, we see that a more meaningful, set-oriented explanation for why order matters emerges. In addition to explaining order, Kristin also makes an insightful comment about wanting to avoid overcounting:

> K: You can pick person 1, 2, 3, 4, and 5 in one group, and that's going to be the same as picking 2, 3, 1, 5, and 4. So it's going to give you the same group, so you don't want to double count, which is what permutation would give you. So I'm using combination for that matter.

I: Cool and that example you wrote there, does that help you make sense of whether to use permutations or combinations?

K: Yeah. So if I write it out, I usually do mess up on whether to use a combination or permutation, because I'm lazy, and I don't write out an example. But when I do write it out, then I can think, okay well would this be the same group as this group, and in this case they would.

In the first six lines of the excerpt, Kristin nicely explains why order matters and, in doing so, she also articulates her awareness of overcounting. Her set-oriented explanation of why order matters is much more convincing than simply going "off my gut" as she had previously described. Phrasing the situation in terms of outcomes provides a way of reasoning about order that she did not previously articulate. The rest of the excerpt is also revealing. She indicates that sometimes she chooses not to think about counting problems in such a way (she attributes this to being "lazy"). While at times her work reflects a set-oriented way of understanding a given problem, there is evidence that the set-oriented way of thinking does not yet fully characterize her view of what solving counting problems entails. It is not that she is incapable of more meaningful reasoning about why order matters, for example, but she does not always bring that kind of explanation to bear as she counts. In terms of the set-oriented model, much of Kristen's work is characterized solely by the relationship between counting processes and formulas/expressions, and only occasionally does she incorporate sets of outcomes into her work. Kristin's work on multiple problems reveals an underdeveloped set-oriented perspective.

Anderson: a developing set-oriented perspective

In another student, Anderson, we see evidence of how a set-oriented perspective might help a student identify and correct overcounting, and we also gain insight into how a way of understanding one problem might feed into the development of a set-oriented way of thinking for subsequent problems. In the excerpt below, Anderson was in the process of evaluating the two expressions described above for the Passwords problem: a correct case breakdown, and the incorrect answer of $\binom{8}{3}\cdot 26^5$. As he talked through the latter expression, he justified to himself why it might make sense. Initially his reasoning about the expressions was not rooted in a discussion of sets of outcomes, but rather in evaluating the counting process that had been implemented:

A: Where it multiplies by 26^5, I think it's trying to simplify the remaining parts of the problem, saying that we don't really care what's in the other 5 letters. They could be Es; they might not be Es. It doesn't matter, though, because we've already accounted for three of them, which means that any combination, Es or not, of the other five letters, would already be taken into account [...] Um, yeah so I guess, uh, this solution does make sense to me, or at least how they got there.

Anderson then used a calculator to examine both expressions, and he found that the value of the expression $\binom{8}{3}\cdot 26^5$ was much larger. To investigate the discrepancy, he examined the number of 4-letter passwords that contain at least 3 Es, rather than 8-letter passwords that contain at least 3 Es. In handling this smaller case, Anderson began to articulate particular outcomes for the first time on this problem, and this proved to be helpful. Ultimately, Anderson identified a particular outcome—the all Es password—that was counted too many times by the incorrect solution. I interpret that in Harel's language, Anderson's realization suggests a set-oriented way of understanding the Passwords problem:

A: Oh, there we go, that's where the difference is [...] Yes there are 26 different ways to arrange it so that the first three letters are Es, and then the last one can be any of the 26 letters. And then there's another way to arrange it so that the first two and the last letter are Es, and the third letter is any letter between A and Z, except if the third letter is an E, it's exactly, it's the exact same case as if the E was the last letter in the first case, which means it's counting multiple passwords twice.

Now consider Anderson's subsequent work on the Test Questions problem:

> Suppose an exam consists of 10 questions, and you must answer five questions. In how many ways can you answer five questions if you must answer at least two of the first five questions?

This problem has a similar common overcounting error, in which one arrives at the incorrect answer of $\binom{5}{2}\binom{8}{3}$ by first choosing two of the first five questions to answer, and then answering any three of the remaining eight questions. Anderson worked on this problem after the Passwords problem, and his previous work seemed to have helped him attend to sets of outcomes more quickly. Again, Anderson had been evaluating a correct answer and one that overcounted, and to explain this discrepancy he almost immediately drew the following conclusion:

A: So, the issue is the same; this one [$\binom{5}{2}\binom{8}{3}$] has overlap, because for example, we answer the first two of the five questions, right? This one's saying, hey we have eight more questions, how many different ways can we answer them? So let's say we answer the third question, then the ninth and the tenth question. Well there's also the case where there's a combination with 5 choose 2 where, let's say we answer the first and third question. We still have 8 problems left, why don't we answer the second question, the ninth question and the tenth question? We have answered exactly the same questions, 1, 2, 3, 9, 10, but we're counting them as two different combinations, which, they're not.

Anderson is thus able to detect the overcount, and his explanation of why it occurred is clear and convincing, rooted in arguing why a particular outcome—the set of questions 1, 2, 3, 9, 10—was counted too many times by the erroneous counting process.

Harel (2008a, 2008b, 2008c) makes a strong case for the interactive relationship between ways of understanding and ways of thinking, and the case of Anderson exemplifies this. Anderson's set-oriented way of understanding was beneficial for him on the Passwords problem, and this success might have contributed to a broader set-oriented perspective that affected his work on the Groups of Students problem. I contend that Anderson's set-oriented perspective was reinforced because of his positive experiences with sets of outcomes on multiple problems and that this reinforcement could continue to positively affect his subsequent counting activity. The case of Anderson thus shows a student who was developing a set-oriented perspective. In terms of the set-oriented model, Anderson's work on these problems may be characterized as developing the relationship between counting processes and sets of outcomes.

Zach: a robust set-oriented perspective

Throughout his interview, Zach's work suggested a fundamental grasp of sets of outcomes and how they related to his counting processes. His work demonstrates a set-oriented perspective as a consistent way of thinking about solving counting problems.

The Cards problem states:

> How many ways are there to pick two different cards from a standard 52-card deck such that the first card is a face card and the second card is a heart?

Zach correctly arrived at an answer of $3 \cdot 12 + 9 \cdot 13$ and his language below suggests a set-oriented perspective. Unlike Kristin's explanation of cases that hinges on the key word "or," Zach's understanding of the addition is rooted in outcomes and summing up possibilities:

> Z: I for sure deliberately choose a non-heart face card for this first card […] that leaves me with 13 possibilities for what I can grab with my second card. That's case A, maybe. And then B, if this happens to be a heart face card, which there are 3 choices there, that would leave only 12 hearts remaining for this guy, just another 36, so 153 […] Then the addition is because these are 117 cases that win, here's another 36 cases that win, so overall there are 153 that win.

When asked why he went to cases, Zach's response reflects an awareness of a need create disjoint cases in order to avoid overcounting. He seems to realize that the set of outcomes could be organized according to disjoint subsets, demonstrating a connection between cases and the set of outcomes that "win":

> I: What makes you think cases, and what do you mean by cases?
>
> Z: In order to know what I was doing on the second draw, I had to be pretty specific about what I was going to do on the first draw, and that led to two different scenarios that were for sure having no overlap. Because in this case I have a face card that is a for sure not heart, and this one is a for sure heart, so none of these pairs of cards are going to be these pairs of cards. But I had to differentiate them, because if I just said 12 choices at the beginning I would really have no way to proceed in making my decision on that second one.
>
> I: […] Why is the fact that there's no overlap between them important?
>
> Z: Because I'm adding these two together as 117 distinct ways that I can satisfy that, and 36 more distinct ways, otherwise that addition symbol wouldn't be very good. I would be inadvertently counting the same thing twice, which would make this number be larger than it really ought to have been, if there were some overlap between them.

In the Groups of Students problem, Zach stated that the "order of the groups doesn't matter." When asked what he meant, he appealed to a particular grouping of students, saying "A Group 1 with ACDEG, this is not distinct from GCDEA, where I swap the place of any two students […] You just need to put them into a group." Zach's phrase "this is not distinct from" suggests that he did not want to count both ACDEG and GCDEA in the total answer. His argument used two particular potential elements of the set of outcomes, and he determined that because the outcomes were not distinct from each other and should not both contribute to the total, order did not matter. While this response is similar to Kristin's reasoning about ordering people in groups, it was Zach's primary way of articulating why order did not matter. Unlike Kristin, the sets of outcomes were prominent for Zach, and he favored reasoning about them over relying primarily on memorized formulas or intuition. According to the model, Zach regularly drew on sets of outcomes and had a flexible relationship between counting processes and sets of outcomes. His work across these problems demonstrates that he had a robust and consistent set-oriented perspective toward counting.

Discussion

It is important to note that having a set-oriented perspective on solving counting problems is neither necessary nor sufficient for correctly solving a problem. Students may correctly solve problems without appealing to outcomes. For example, in a problem like the Pin Number problem (How many 4-digit pin numbers are there, with repeated digits allowed?), one could simply reason that there are ten choices for each of the four digits and correctly answer 10^4 without appealing to outcomes. Conversely, students who display a set-oriented perspective may still answer problems incorrectly. For example, on the Cards problem Brandon paid explicit attention to outcomes, saying, "I've got 12 choices here for this first slot, and then for each one of these cases, it's going to correspond with a distinct outcome where we've got 13 hearts." However, this is an untrue statement and yields an incorrect answer of 156. Brandon recognized that he was counting certain kinds of outcomes, but he did not see how some outcomes were being overcounted, and even a set-oriented way of understanding this problem did not enable him to answer it correctly.

It is, therefore, possible for students to answer problems correctly without a set-oriented perspective, and for students with such a perspective to answer problems incorrectly. However, I argue that there are many counting situations in which a set-oriented perspective can uniquely facilitate meaningful reasoning about a counting problem. We saw this most pointedly in Kristin and Anderson's work. Without referring to outcomes, Kristen struggled to explain why she used cases or why order mattered. When she did turn to particular outcomes, her explanations were more convincing and sound. For Anderson, simply describing an incorrect counting process gave no insight into why the process was incorrect. When he turned to outcomes, though, he could both identify an overcount and justify why it had occurred. The examples I have presented suggest that there are potential benefits of a set-oriented perspective toward counting, and it is a promising perspective by which students may reason meaningfully about key combinatorial issues.

As Martin (2001) writes, "Counting is hard. [...] There are few formulas and each problem seems to be different" (p. 1). Because of such challenges, it can be difficult for students to know to which aspects of a problem they should attend. When students count, they often resort to looking for key words and to memorizing and applying formulas. While limited success with such strategies is possible, an overreliance on them can lead to a perspective that counting is primarily a matter of fitting a problem to a formula; counting is no longer about determining the cardinality of a set of outcomes.

I acknowledge that some of the issues I have addressed, such as students' reliance on memorized formulas or key words, reflect the broader, perennial issue in mathematics education of students favoring meaningless procedures over conceptual understanding. The hope, however, is that the notion of a set-oriented perspective offers a particular way to address this tension in the context of solving counting problems, providing some insight that is specific to combinatorics. Indeed, one way to frame this article is to ask, in the context of counting, how can we help students be more conceptually and less procedurally focused? I propose that one answer is to draw their attention to sets of outcomes. In terms of the set-oriented model, the aim is to reinforce the relationship between counting processes and sets of outcomes, and to help students integrate the set of outcomes as a fundamental aspect of their combinatorial thinking and activity.

Concluding remarks and next steps

Especially in light of students' documented, ongoing difficulties with counting problems, the potential value of a set-oriented perspective is powerful. Regularly reorienting students toward a set-oriented perspective can help students fundamentally view counting outcomes as an exercise in organizing, structuring, and determining the cardinality of sets of outcomes. Harel's duality principle gives pedagogical insight into how instructors may foster a set-oriented perspective in their students. The student work presented in this article is consistent with Harel's description of the interactions between ways of understanding and ways of thinking, and we saw (particularly in Anderson's case) that ways of thinking and ways of understanding can reinforce each other through positive experiences. Teachers could introduce a set-oriented way of understanding across a number of different problems by drawing students' attention to outcomes. Multiple experiences could then be developed to emphasize broader, set-oriented ways of thinking about counting that students can bring to any counting problem they encounter. Elsewhere, I have suggested ways in which K-12 teachers can emphasize outcomes (Lockwood, 2012). Future research plans include designing specific instructional interventions to elicit the set-oriented perspective for post-secondary students.

Developing a set-oriented perspective may help students to find more meaningful ways of understanding and articulating issues that arise as they solve counting problems, and students may gain much needed traction as they count by grounding their counting processes within sets of outcomes. The cases I have presented in this article suggest the important role that sets of outcomes (and a set-oriented perspective) can play in combinatorial enumeration, and more work is needed to follow up these ideas. Next steps involve developing more systematic tests of the effectiveness of a set-oriented perspective and more detailed examinations of the range of set-theoretic concepts that are typically employed in combinatorial enumeration. Such investigations include deeper elaborations of the sets of outcomes component of the set-oriented model, as well as explorations into pedagogically reinforcing the natural relationship between set theory and counting.

Notes

[1] There is another common response in which 8-letter passwords with exactly zero, one, or two Es are subtracted from the total number of 8-letter passwords.

[2] For each of these students, the episodes provide snapshots into the nature of their perspectives toward counting. These episodes show evidence of local, but not necessarily permanent conceptual shifts in their thinking. To make claims of more permanent growth we would need to study the students over a longer period of time. Nonetheless, even the brief episodes demonstrate empirical evidence of underdeveloped, developing, and robust set-oriented perspectives.

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From the archives

Editor's note: The following remarks are extracted from an article by Efraim Fischbein (1982), published in FLM3(2).

The concept of formal proof is completely outside the main stream of behavior. A formal proof offers an absolute guarantee to a mathematical statement. Even a single practical check is superfluous. This way of thinking, knowing and proving, basically contradicts the practical adaptive way of knowing which is permanently in search of additional confirmation. In principle, the formal structure of the adolescent's thinking possesses all the basic ingredients necessary for coping with both formal and empirical situations. Despite this, the current ways of trying and evaluating are mainly adapted to empirical contents. In solving a problem the mathematician proceeds, at the beginning, in the same way as the "empirical" scientist. He analyses the given situation, he tries to identify some general properties, some invariant relations or dependencies, *etc*. But at a certain moment this search process stops and a new situation appears: the mathematician has found a complete proof for his solution or theorem. Such a proof is the absolute guarantee of the universal validity of the theorem. *He believes in that validity.* This is a new situation in relation to natural mental behavior. Naturally, intuitively, we continue to believe in the usefulness of enlarging our field of research, of accumulating more confirmation. To think means to experiment mentally. Mental experience is the duplicate of the practical trial-and-error goal-oriented process. Therefore this ideal, the perfect proof, has no meaning for the natural empirical way of thinking. In order to really understand what a mathematical proof means the learner's mind must undergo a fundamental modification.

Of course he can learn proofs and he can learn the general notion of a proof. But our research has shown that this is not enough. A profound modification is required. A new completely non-natural "basis of belief" is necessary, which is different from the way in which an empirical "basis of belief" is formed. *The concept of formal, noninductive, non-intuitive, non-empirical proof can become an effective instrument for the reasoning process if, and only if, it gets itself the qualities required by adaptive empirical behavior!*

In other terms: *formal ways of thinking and proving can liberate themselves from the constraints of empirical knowledge if they become able to include in themselves those qualities which confer on the empirical search its specific productivity.* We refer to the global, synthetic, intuitive forms of guessing and interpreting.

It is not enough for the pupil to learn formally what a complete, formal proof means in order to be ready to take complete advantage of that knowledge (in a mathematical reasoning activity). A new "basis of belief", a new intuitive approach, must be elaborated which will enable the pupil not only to understand a formal proof but also *to believe* (fully, sympathetically, intuitively) in the *a priori* universality of the theorem guaranteed by the respective proof. As in every form of thinking, we need, in addition to the conceptual, logical schemas, that capacity for sympathetic, direct, global acceptance which is expressed in an intuitive approach. After learning a formal proof we have to reach not only a formal conviction—but also the internal direct agreement which tells us: "Oh yes. It is obvious that the described property *must* be present in every object which belongs to the given category." *The feeling of the universal necessity of a certain property is not reducible to a pure conceptual format. It is a feeling of agreement, a basis of belief, an intuition—but which is congruent with the corresponding formal acceptance.*

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