History of Mathematics for Teachers: the Case of Irrational Numbers

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In Arcavi, Bruckheimer and Ben-Zvi [1982] we described an approach to the use of history of mathematics for pre- and in-service teacher courses Basically, this approach is characterized by the following

a) *Relevance*
The topics whose history is studied are those which are directly connected to the curriculum the teacher has to teach or will teach. Further, the treatment of the topics is designed in the light of specific teacher needs.

b) *Primary sources*
The materials to be used are, mainly, primary sources and historical documents.

c) *Active learning*
The material is prepared in worksheet form. Participants read the sources and work by themselves (or in small groups) with some guidance, through questions, exercises and problems especially prepared for each historical source

d) *Conceptual history*
The history relates to how a concept evolved, the different approaches by mathematicians in the past, their difficulties, their mathematical creativity, up to the formalization stage. A small dose of facts, dates, biographies and anecdotes enter incidentally.

The description of a course in the history of negative numbers and its implementation are reported in Arcavi et al. [1982] Using the same approach, we developed and implemented a course on the history of irrational numbers, based on what we found to be the needs of the target population.

Needs assessment

Having in mind our requirement of relevance, an assessment of teachers' previous knowledge, conceptions and/or misconceptions on irrationals was undertaken to precede and accompany the development of the course. In the following we shall discuss some of our findings and their role in shaping the course objectives, as well as the course itself.

Previous knowledge of history

The vast majority of teachers (attending a large summer in-service teacher training program related to a national mathematics curriculum for junior high schools in Israel, N = 84) had little or no knowledge of the history of mathematics. This became clear from a questionnaire administered: 83% of them had never learned history and declared themselves to be interested in such a course if offered This suggests that history can be a suitable and attractive vehicle through which many mathematical topics can be learned, relearned (remediation) and enriched, without the feeling of *déjà vu* Also, and more specifically, we asked a smaller group (N = 56) the following question:

In your opinion, when is the first time that the concept of irrationality arose?

a) Before the Common Era (Babylonians, Greeks, etc.)
b) Early Middle Ages (Hindus, Arabs)
c) Between 1300-1600 (Europeans)
d) Between 1600-1800 (Europeans)
e) Between 1800-1900 (Europeans)

This question was answered correctly (item a) by about 70% of the participants. It seemed that the "when" was well known, but few of the teachers also knew "how" This became apparent in a further question, in which they were asked to order chronologically the appearance of three concepts: negative numbers, decimal fractions and irrationals. About 55% of the teachers indicated that decimal fractions preceded irrationals (and an additional 10% did not answer at all). This was not only an indication of lack of historical knowledge about the relatively recent development of decimals, but also that the origin of the concept of irrationality, although associated (by most of them) with the Greeks, is conceived as relying upon decimals, and not connected to geometry (commensurable and incommensurable segments) as occurred historically. The historical origin of irrationals in general, and the connections to geometry in particular, can provide an insightful understanding of the concept as well as teaching ideas for the introduction of the topic in the classroom

Definitions

The following question was asked of 56 in-service teachers: "Do you know of any mathematical definition of irrational numbers? yes/no. If yes, which?" More than two thirds responded yes, quoting one of the definitions from the textbooks they use: "A number that cannot be expressed as a quotient of two integers" or "A number whose decimal part is not periodic and has an infinite number of digits." Only two teachers mentioned, for example, the definition by means of Dedekind cuts.

For the Learning of Mathematics 7, 2 (June 1987)
FLM Publishing Association, Montreal Quebec, Canada

In order not to give the wrong impression, we would add that the above should not be taken to mean that we expect junior-high school teachers to remember the details of a formal mathematical definition (nor was this asked for), but at least they should be aware that

a) the description of, and the introduction to, some concepts as given in junior-high school texts is not a formal mathematical definition, and
b) such a definition exists.

We would also expect them to appreciate that the former is guided by didactical considerations and can be justified in terms of the relative mathematical immaturity of the target student population.

This leads us to the conclusion that it would be a desirable feature of the materials to present the search for a formal mathematical definition of irrationals motivating the necessity for such definition, and thus giving a proper picture of mathematical activity. Students in colleges and universities are often presented with mathematical definitions, divorced from the context which gave rise to them, and which can contribute to the feeling of logical necessity for such definitions. History seems to be a natural means for fostering this feeling, since "The history of mathematics is a bountiful source of such examples showing ways in which previous generations have experimented and discovered the need for formal mathematical structures." [Meserve, 1983]

Recognition

The following question was administered to 60 prospective teachers from various teacher colleges.

> Indicate the irrationals among the following numbers
> (a) $\sqrt{26}$ (b) $-7/3$ (c) 1.010010001... (d) $22/7$
> (e) π (f) $\sqrt{2}/\sqrt{8}$ (g) $(\sqrt{2}-1)/(\sqrt{2}+1)$

About 60% of the respondents had two or more errors. The most common error was to indicate 22/7 as irrational. This suggested that, for this population, one of the sources of confusion between rational and irrational numbers is the common use of a rational approximation to an irrational as the irrational itself. Although this is the way many practical problems are solved (for example when measurement is involved) the distinction should be very clear — certainly to the teacher.

The objectives of the course

The responses to the above questions indicated the main issues to be addressed by the course materials: the concept of irrationality arising from geometry and remaining connected to it until the irrationals began to be recognized as numbers; irrationals passing to the realm of arithmetic and algebra, rational approximations to irrationals; different kinds of irrational numbers and the "modern" need for a formal mathematical definition of irrationals (and of reals in general).

Therefore our first stated objective was to strengthen (learning, relearning and/or enriching) the mathematical knowledge related to the concept of irrationals.

In addition, history provides a very fertile environment in which other very important teacher-related objectives can be pursued. For example, to bring teachers to appreciate and discuss the difference between the demands of (pure) mathematics as opposed to the demands of didactics, showing where the curriculum has to compromise (as in the case of the definition of irrationals).

A further objective related to the work through questions, exercises and problems around primary sources. The aim of this way of working was the improvement of mathematics reading ability, which is an often neglected objective in teacher preparation programs.

Last, but not least, the historical context may foster the creation of a reasonable image of mathematics and mathematical activity as a human, creative and dynamic endeavour, as opposed to the more common view of mathematics dropping "ready-made-from-the-skies". The history of irrationals has nice examples to illustrate this point. For example, irrationality shaking the Pythagorean world; the discussion in the sixteenth century whether the irrationals are true numbers, etc.

Description of the materials

The course materials consist of a sequence of worksheets in each of which a biographical-chronological introduction sets the historical scene for the source, which is supplied alongside its translation into the readers' language.* The source is followed by leading questions, exercises and problems designed to help understand the text and the mathematics explicitly or implicitly involved. (For a full description of a general framework for the design of learning activities around primary sources, see Arcavi, in press.)

For each worksheet there is an answer sheet with detailed solutions to the questions as well as supplementary historical materials to enrich the answers. In the following we describe the worksheets in the sequence on the irrationals.

The Pythagoreans

In the first part, this worksheet presents a little of the Pythagorean philosophy, as well as some of the topics from their mathematical world, such as figurate numbers, properties of the pentagon, and commensurability of line segments. The latter two topics are needed for the second part of the worksheet [built around Baron, 1974] in which the discovery of incommensurable segments and the subsequent crisis in Pythagorean mathematics are described.

One of the assignments in this worksheet is to restate the proof that the diagonal of a square is incommensurable with its side (compared to: "$\sqrt{2}$ is irrational"). Another exercise deals with the construction of sides of squares of areas 3, 5, ..., 17, which are incommensurable with the side of the square of area 1, but constructed upon it, and this is complemented by the discussion of proofs of the irrationality of $\sqrt{3}$, $\sqrt{5}$, ..., $\sqrt{17}$.

* So far the materials have appeared in two versions: Hebrew and English. The latter is available from the authors on request.

Finally the worksheet presents a second theory on the discovery of incommensurable line segments. This theory [von Fritz, 1945] maintains that the context of the discovery was the side and diagonal of a regular pentagon, rather than the side and diagonal of a square. Teachers are guided to find by themselves that the irrational number involved in this case is the golden section. The answer sheet contains further historical and mathematical information about the golden sections.

Euclid and the Elements
The next stage in our story is the "legitimisation of incommensurability" in geometry. This is illustrated by an extract from Euclid's *Elements*. The worksheet begins with a brief introduction about the *Elements* and the importance of this masterpiece in the history of mathematics. The extract from Book X brings the definitions of commensurable and incommensurable magnitudes, straight lines commensurable in square, and irrational straight lines. The exercises are designed to guide the teachers

— to identify and construct examples of the various definitions;
— to analyze the truth and to provide examples (or counterexamples) of given statements which connect some of the definitions;
— to construct diagrams connecting the definitions;
— to realize that the diagonal of a square of area 1 can be a rational straight line according to Euclid's definition.

Finally, a theorem on incommensurable magnitudes (Book X, 16) is presented and the assignment consists of rewriting-translating it into modern notation.

Irrationals in the 16th and 17th centuries
The (almost) 2000 year jump can be justified by the sparseness of historical sources that can offer something new relevant to the development of the irrationals. Nevertheless, one should not entirely overlook this long period. Thus, in our workshops, for example, we considered a remarkable Hebrew source by Maimonides (1135-1204), one of the greatest Rabbis of all times, codifier, philosopher, physician and astronomer. In his commentary to the Mishnah (Eruvin), he discusses the nature of the ratio between the circumference of a circle and its diameter. He states:

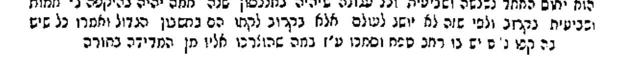


Figure 1

which very briefly means that its essence is unknown and can be conceived only by approximation, and then he gives 22/7 as an approximate value.

The concern about the nature of irrational numbers became more widespread after decimal fractions were developed in the 16th century by Simon Stevin (1548-1620) and others, and this is the topic of the present worksheet.

Arguments in favour and against the acceptance of irrational numbers are produced, to be read, understood and discussed through the questions. For example, the worksheet brings paragraphs from Stifel's *Arithmetica Integra* (1544) (as they appear in Kline [1972] and Tropfke [1902], in which he states that irrationals are useful when "rational numbers fail us". But other considerations "compel us to deny that irrational numbers" are numbers at all. They "flee away perpetually" and they lie "hidden in a kind of cloud of infinity" (when one tries to use decimal representation). The worksheet also brings extracts from Stevin's *Arithmetique* (1585) with Girard's comments (as they appear in Klein [1968]), containing further arguments about the nature of irrational numbers. The answer sheet is enriched by a paragraph from Peletier's article "Are the irrational numbers numbers or not, and of what kind." [Klein, 1968]

R. Bombelli—N. Saunderson
Once the conflict on the nature of irrationals had abated (but not disappeared, in the absence of a mathematical definition), the next stage is that of rational approximations to (algebraic) irrationals. It should also be noted that there were mathematicians who had treated the topic before this period also.

In these two worksheets, two different methods of approximation are discussed respectively. The first, taken from Bombelli's *Algebra* (1579), has been interpreted as using what were later called continued fractions. Some work has to be done in order to put Bombelli's rhetorical and syncopated writings into modern symbolic notation. Thus the worksheet begins with a guide towards the creation of a "dictionary" for that purpose. The next questions are designed to help understand the approximation method for the particular case given ($\sqrt{13}$), then to apply it to another case, and finally to discuss its generality and other related mathematical issues.

The Saunderson worksheet brings the known algorithm for finding square roots and its justification. The paragraph is taken from *The Elements of Algebra* (1741).

After these two worksheets we discussed other stages in the story with the various teacher groups who worked on this sequence. For example:

— the rationality or irrationality of some given numbers (the case of π and Lambert's proof of its irrationality in 1766);
— the existence of algebraic and transcendental irrationals (and also special cases, such as Lindemann's proof of the transcendence of π in 1882; the existence of numbers whose transcendence is not yet proved, and the like)

These topics are not treated in separate worksheets because they require a wider mathematical background than most of our junior high school teachers possess

Dedekind and the definition of irrationals

Having shown how the irrationals entered extensively into mathematics, the story still lacks their final "legitimisation". Thus, the next stage is to bring a formal mathematical definition of irrationals.

In this worksheet we bring the original development, which seems to be easier than its elaboration appearing in many later texts. The reason for this lies, perhaps, in the fact that Dedekind (1831-1916) tells us how he felt the need for a formal definition and how he developed the idea using a geometrical analogy. Then he proceeds to give the definition in a purely formal way, together with the definition of the operations. [Dedekind, 1963] Based on these definitions, one of the exercises is to prove that $\sqrt{3} * \sqrt{2} = \sqrt{6}$.

The stages in the history of irrational numbers represented in these worksheets, are far from being complete. There are certainly further sheets that could be added, but those represented here give, in our view, the major highlights in the development of the concept, which reflects the general framework described by Harnik [1986]:

— a preliminary stage ("A new concept is born out of necessity. At the beginnings it is often vague and even the inventors of the concept may feel uncomfortable with it");
— a familiarization stage ("The concept is used again and again with increasing confidence until it is fully understood"); and
— an axiomatization stage

Implementation

The materials described were implemented in active learning type workshops and courses and in correspondence courses. The analysis of the implementation experiences was carried out, both in order to correct possible flaws in the materials as a contribution to their subsequent improvement, and also to obtain mainly qualitative, but also some quantitative, feedback as to whether and to what extent the objectives were achieved. In the following we bring some of the results

The teacher's view

One of the ways to evaluate the materials was to ask the teachers what they felt they had learned. We asked the teachers the following question: "What did you learn in the workshop (course) from the point of view of (a) history, (b) didactics and (c) mathematics?" This question was answered extensively by almost all the participants, in each of the implementation experiences.

(a) History
The responses can be classified in two categories: those who enumerated names of mathematicians or the history of the topic (in general, or in quoting specific details), and those who stressed that they became aware of the evolving and dynamic nature of mathematics, as the main contribution of the materials.

(b) Didactics
Many people indicated that they learned "new ways of explaining things". And, also (as was the case with the sequence of worksheets on the negative numbers) that the distinction between pure mathematics and the didactics of mathematics became clearer.

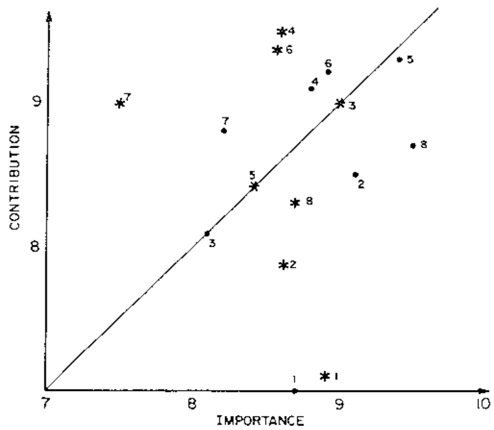
(c) Mathematics
The most common answer under this heading was related to the formal definition of irrational numbers. An oft-repeated comment in this respect was that they learned about "ways of defining concepts clearly on the basis of previous concepts". Many teachers also listed various specific mathematical topics.

Another way of learning about teachers' views was to ask them to characterize the potential of the materials. The following question was asked of those groups who worked on both sequences of worksheets (irrationals and negatives) during either a correspondence course (in-service, N = 15) or a university course (pre-service, N = 13).

> It has been proposed to introduce the sequences of worksheets on the negative and irrational numbers as a course in teacher training institutions. Since you are now familiar with the materials, we would like to have your opinion. Therefore, in the following, we list the objectives of the learning materials. You are requested to mark on a 1–10 scale (1 lowest and 10 highest) both
> — the *importance* of each objective for the training of teachers, and
> — the *contribution* of the sequences to the achievement of that objective.
>
> *Objectives*
> 1 To enhance mathematical knowledge
> 2 To enrich didactical background
> 3 To give training in the reading of mathematical texts
> 4. To enhance the awareness of mathematics as a developing subject
> 5. To discuss mathematical topics in an interesting way
> 6. To learn about the historical development of a topic
> 7 To learn from history about mathematicians and their work
> 8. To enjoy "doing" mathematics

The coordinates of the points in the following diagram indicate the means obtained for the questions on "importance" and "contribution" respectively. The diagonal determines the zones in which the importance attributed by the respondents to a particular objective is respectively greater than, equal to, or less than the contribution to that objective. The two groups who responded to this question are so small that not too much significance should be attached to the absolute results, nevertheless the patterns are of interest

At a first glance, we see that almost all the means for "importance" and also for "contribution" are above 8. These high values, and the small differences between importance and contribution (as exhibited by the fact that nearly all points are close to the diagonal), suggest that, in general, the participants in both groups regarded the course as a serious "contribution" to the achievement of the "important" stated objectives for teacher training



* - means obtained from the in-service correspondence course.

• - means obtained from prospective teachers in university course

Figure 2

It is also interesting to note that the relative positions of the points representing the same statements in both groups with respect to the diagonal are similar This suggests that the relative importance of and contribution to these objectives is independent of teaching experience. There were objectives to which the materials seemed to "overcontribute" (with respect to the importance attached to them); these were, precisely, the three "historical" objectives, and the evaluation is therefore not surprising.

The overall picture which emerges from the analysis of the responses to the above questions indicated clearly that the teachers felt that they had profited from the materials in the sense of the objectives which guided their design

Teachers difficulties and achievements

Our first objective was to strengthen (learning, relearning and/or enriching) the mathematical knowledge related to the concept of irrationals. As we saw in the needs assessment, many prospective teachers did not successfully recognize irrational numbers. A similar picture emerged from a pre-test given to in-serivce teachers who came to work with the materials. In response to a recognition question in the post-test (after working through the irrational sequence), almost everyone answered correctly. The only real lingering error was that a few still thought that 22/7 is irrational.

Although it seems reasonable to conclude that the materials contribute to the recognition of irrational numbers, it is still surprising that, at the end, there are some that confuse a particular irrational number (π) with one of its rational approximations (22/7) We can speculate whether this is an indication of a general confusion between an irrational and its rational approximation or something particular connected to π and 22/7 Thus, for example, if in the Bombelli worksheet we obtained 3.6060 .. as an approximation to $\sqrt{13}$, did the teachers also regard this value as an irrational number? We thought of the question too late — we hope to investigate it with another population in the future

Another important objective was to improve mathematics reading ability. To check the achievement of this objective, two unseen extracts were carefully chosen according to the following criteria:

- they were taken from primary historical sources and were thus similar to those in the worksheets;
- both were on the same topic: irrationals.

The first was a paragraph from MacLaurin's *A Treatise of Algebra* (1748), dealing with a definition of irrationality, and the second was from Chrystal's *Algebra, an Elementary Textbook* (1886) dealing with a definition of surd numbers In assigning the paragraphs (and the subsequent questions we prepared to test understanding) to pre- and post-test respectively, we consulted expert opinion This indicated that the first paragraph, with the questions that we asked thereon, was "easier" than the second, in addition the definition of surd numbers was not included in the worksheets and is not in common use today Thus we chose the first paragraph for the pre-test and the second for the post-test The questions related to a definition or a property given in the paragraph, or required the respondent to indicate whether a given related statement is true or false.

The responses after the course were more assured (in particular, fewer did not attempt the question). Also, although the second passage was longer, harder and had more questions to be answered, the success rate was decidedly higher. This is a clear indication that the design of the materials (historical source to read: leading questions to work on) is a suitable context in which to practice and enhance the ability to read mathematics

Epilogue

Learning materials on the conceptual history of mathematical concepts designed as guided self-study worksheets in the way described above seem to be a very suitable approach towards the achievements of the main goals in teacher preparation or in in-service programs. In our view further materials of this kind could be developed in order to create a rich resource from which teacher trainers would be able to choose according to their needs.

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It would appear that, on the whole, adult *homo sapiens* has rarely taken it for granted that children could or would just naturally learn by spontaneous imitation. At the same time, the children of *homo sapiens* have not assumed that they would just naturally grow into adulthood. Rather, children have always been aware that they have to validate their status as adults by learning adult techniques from older teachers. It follows that *homo sapiens* has been born on a kind of status machine — a status escalator or a status treadmill, depending on the culture — from which there has rarely been any socially acceptable escape Thus, throughout the historic course, *homo sapiens* has been a "status seeker", and the pathway he has had to follow, by compulsion, has been education. Furthermore, he has always had to rely on those superior to him in knowledge and social status to enable him to raise his own status. On the other hand, it is not clear that adults have always assumed that children would naturally wish to be adult Over and over again, the data shows that children have had to be urged up the ladder by rewards, punishments, and other even more complex devices. From the point of view of the adults this is absolutely necessary, for otherwise the children would remain dependent and disgracefully deviant in other ways. From the standpoint of the child, he must climb the status ladder or suffer the consequences of dependence or deviance It is likely, meanwhile, that this compulsion and the inner conflict involved leave a lasting impression on the child, so that as a mature adult these memories can provide a fertile soil for social change, for if conditions arise that seem to provide an opportunity to eliminate the sources of the pains of childhood growth, adults may be happy to take advantage of the situation and push for change, often, perhaps, not knowing the real sources of their readiness.

Jules Henry
