Historical Digressions in Greek Geometry Lessons

YANNIS THOMAIDIS

This paper presents an attempt to combine the history of mathematics of ancient Greece with the course on theoretical geometry taught in Greek secondary schools The very fact that this attempt has taken place in Greece is of major importance It is hoped that it will contribute towards understanding the way in which the distant cultural past of a country can influence its contemporary mathematical education.

I will try to explain briefly the position and the role of history of ancient Greek mathematics in education in modern Greece; however, my main aim will be to demonstrate how this history can be used to provide answers to questions and problems that arise during teaching

1. History of ancient Greek mathematics and the subject of geometry

The current situation in Greece with respect to the relationships between ancient Greek mathematics in secondary schools shows some special characteristics It is true that school books, especially those on geometry, contain many historical notes with profiles of the great mathematicians of ancient Greece and information about their work, in an effort to stress the Greek origin of mathematical science. [1] This historical material serves an educational policy with a long tradition in Greek education The roots of this tradition can be traced back to the formation of the modern Greek state, early in the 19th century, when the classical Greek civilization was viewed as the main factor in the formation of the national character of modern Greece

In secondary education, this particular policy was mainly demonstrated through the imposition and domination of ancient Greek language and grammar over all other lessons. With respect to mathematics, particular emphasis was given to theoretical Euclidean geometry and, as the historian Michael Stephanidis said in 1948, it was not a rare occurrence "to start or finish the geometry lesson, for example with the patriotic apostrophes" [2]

The preservation of the Euclidean character of the geometry taught in schools, and the promotion of the work of ancient Greek mathematicians, has since then been a fundamental directive in our mathematical education and the influence is apparent even today. [3]

All this seems to reveal a favourable environment for the development of an interest in the history of ancient Greek mathematics; the reality is different, however. Neither in Greek universities, which prepare the future teachers of mathematics, nor in in-service training schools, has the history of mathematics been taught as a self-contained subject. A few recent exceptions to this rule do not make up for the fact that the overwhelming majority of secondary teachers lack knowledge about the history of mathematics and its educational value.

This lack of a historical education in mathematics may well be the reason that most teachers regard the historical notes in the schoolbooks as "space fillers" and do not bother with their content. There is, naturally, a small number of teachers who try to use them to advantage in their day-to-day teaching, especially in geometry lessons, aiming to promote class discussion. This practice, however, has shown that this historical material, as it is presented nowadays (not linked with the lesson itself and its difficulties) tends to leave students uninterested.

At this state, I should point out that school geometry in Greece consists of a modern version of Euclidean geometry founded on Hilbert's axioms The course is taught to the first two Lyceum classes (ages 16-17) and its aims are as follows:

> Students must (a) understand the role of axioms in founding a field of mathematics, in particular the field of theoretical geometry;
> (b) familiarize themselves with the process of deductive reasoning;
> (c) understanding the meaning of the terms "geometrical construction" and "geometrical locus". [4]

In practice, these aims lead to a standardized form of teaching which, by following the textbooks, culminates in the formulation of definitions, (usually in set-theoretic terms), the detailed proofs of a long chain of theorems, and the solution of a large number of exercises.

Student reaction to this deluge of theory is vividly expressed through various queries, such as:

- Why do we have to prove things that are blatantly obvious?
- What is the use of all these theorems and exercises?
- Why must geometrical constructions be made using only straightedge and compasses? etc

These first negative reactions towards theoretical geometry are usually followed by a passive if not indifferent attitude which certainly does not favour the materialization of the course's ambitious aims.

Trying to find a way around this situation, I thought of making use of the historical notes in the textbooks by linking them to the context of the lessons, especially at those particular points where students seem to need to breach the narrow limits of geometrical theory. This effort did not bear the marks of systematic research: it was made within the framework of day-to-day teaching, in the form of "historical digressions" from the normal route of the lesson.

These digressions, up to now, have consisted of the following topics:

(a) The fifth Euclidean postulate and non-Euclidean geometries.
(b) Geometrical constructions using straightedge and compasses.
(c) Applications of geometrical propositions to astronomical measurements.
(d) Geometrical solution of quadratic equations using the "geometrical algebra" of the ancient Greeks.
(e) The determination of pi (π) by Archimedes through the use of regular polygons

The rest of my paper will present two such digressions.

2. Geometrical constructions using straightedge and compasses

Two characteristics of Euclidean geometry, directly related to its historical origin, are straightedge and compasses constructions and geometrical loci. The geometry textbook for the first year of the Lyceum devotes two chapters to these topics, accompanied by the following historical notes:

> The solution of geometrical problems with the use of straightedge and compasses was introduced by Plato (427-347 B C.) Much later, during the 17th and 18th centuries, some attempts were made to solve geometrical problems using the straightedge or the compasses alone and it was proved that if a problem can be solved with a straightedge and compasses, it can also be solved with compasses alone It has also been proved that there exist problems which cannot be solved by straightedge and compasses One such problem is the division of an angle into three equal parts
>
> Geometrical loci were developed in Plato's Academy in conjunction with the analytical and synthetic methods Apollonius of Perga was later involved with loci problems and around the year 247 B.C. he wrote a book entitled "Place loci" which unfortunately has not been preserved [5]

Students can find varying explanations of how there concerns arose. For example, some extra-curricular geometry books refer to the philosophical implications of the subject, with particular reference to Plato's metaphysical conceptions of the circle and the straight line Here is a typical extract from an older book.

> Plato was under the impression that a straight line symbolizes the eternal genesis and consumption of the Cosmos, since it can extend to infinity and can be divided into an infinite number of equal or unequal parts. Similarly, since the circle has no beginning or end, Plato believed that it represents God (the Spirit).
>
> Since the psyche by means of the body participates in genesis and by means of immortality participates in God, it is symbolized by the intersection of a circle and a straight line For the above reasons the two instruments, straightedge and compasses, were considered to be sacred in ancient times [6]

Leaving aside the metaphysical implications, I have used the historical notes of the lyceum first year text to stimulate a discussion about the possibility of solving geometrical problems by means of the Euclidean tools. My ultimate objective was to emphasize that mathematical creativity is not necessarily restricted by the rather strict bounds of an axiomatic theory.

I started from the book's reference to our inability to trisect an arbitrary given angle. The students were skeptical of this statement, as they knew that any angle can be bisected using a straightedge and compasses. I extended their skepticism after demonstrating the ease with which a right angle xOy can be trisected, by making it an angle at the center of a circle and then constructing the equilateral triangle OAC, as shown in Figure 1.

I then asked the students to try to devise a method of trisecting any given angle, using straightedge and compasses

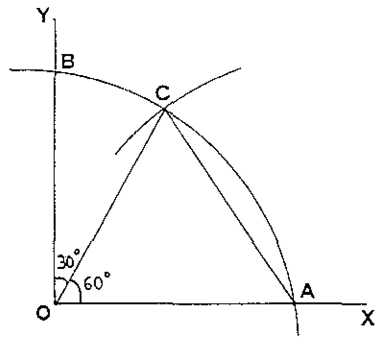

Figure 1

How to trisect a right angle

only. They worked in groups and after a while some of them suggested the construction shown in Figure 2; angle xOy is made an angle at the center of a circle and chord AB is divided into three equal parts

$$AC = CD = DB$$

according to the well known method of dividing a line segment. The speaker of that group claimed that lines OC and OD divide angle xOy into three equal parts.

Most students seemed to agree with this construction, but some of them requested that their classmate prove that angles AOC, COD and DOB are equal. He found no difficulty in proving that angle AOC = angle BOD (taking into account the congruent triangles AOC and BOD) but there seemed to be some difficulty with angle COD

Let me sketch the basic idea in brief, before reflecting on the significance of this particular story and why I am telling it here. Assume we are given a regular n-gon, with the co-ordinates of its corners. Suppose it to be inscribed in the unit circle, and placed so that an edge is cut symmetrically by the x-axis. How can we derive the coordinates of the corners of a similarly placed regular 2n-gon? Connect the point (1,0) on the x-axis with an adjacent corner of the old n-gon, the point (x_{old}, y_{old}). The midpoint of the connecting line has coordinates $((1 + x_{old})/2, y_{old}/2)$. The projection of this midpoint, out onto the circle, is a corner of the new 2n-gon, and will have coordinates

$$(x_{new}, y_{new}) = \left(\sqrt{\frac{1+x_{old}}{2}}, \sqrt{\frac{1-y_{old}}{2}}\right)$$

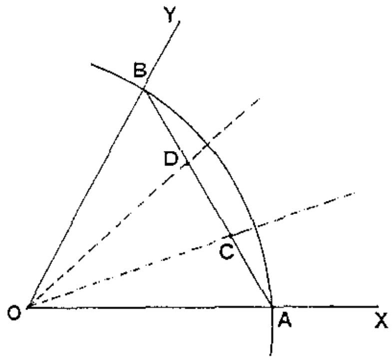

Figure 2

A method of trisection of any angle?

This gives us a pattern for bisecting given arcs under the control of their coordinates on the unit circle. By successive straddling we could approximate sin z and cos z, as close as we wish, by starting from $z_{old} = 90°$ and generalizing the bisection trick. But our first aim is to compute the circumference or the area of the unit circle by n-gon approximation. The placing of the n-gon in relation to the x-axis has been chosen precisely to make this straightforward. The area A_n of the n-gon consists of n copies of the triangular slice whose area is $x_{old} \cdot y_{old}$, and from this expression and our mid-point formula we can easily derive the area of the 2n-gon in recursive form as $A_{2n} = A_n/x_{old}$ (It was from this equation that Viète derived the first closed expression for π:

$$A_4/\pi = \prod x_{2}i = x_4 \cdot x_8 \cdot x_{16} \cdots$$

See Beckmann [1977], Schelin [1983].)

In honour of Archimedes we may start with the regular hexagon giving $(x_{start} y_{start}) = (\sqrt{3}/2,\ 1/2)$. This leads to the following results:

n	6	12	24	48	96	192	384	768
A_n	2.6	3.0	3.106	3.133	3.139	3.1410	3.14145	3.14156

Archimedes stopped at the 96-gon, and with some manipulations and roundings he produced his famous result

$$3\frac{10}{71} < 3.1409\ldots < \pi < 3.142826 < 3\frac{1}{7} \quad \text{[cf Knorr, 1976]}$$

Discussing this easy way to compute pi and the trigonometric functions with my friend Hermann Karcher from Bonn University, we got the idea of increasing the convergence rate of the calculations by a simple trick. From calculus it is known that the area of a segment of the parabola is two-thirds of its circumscribed secant-tangent-parallelogram.

Parallel to the calculations of x_{new} and A_{new} we can derive augmented areas A^*_{new} which consist of A_{new} and 2n parabolic segments added to the edges of A_{new}. From Figure 3 we have

$$A^*_{new} = 2n \cdot (x_{new} \cdot y_{new} + \frac{2}{3} \cdot 2y_{new} \cdot (1 - x_{new}))$$

$$= 2n \cdot y_{new} \cdot (\frac{4}{3} - \frac{x_{new}}{3}) = A_{new} \cdot (\frac{4}{3\ x_{new}} - \frac{1}{3})$$

which completes our calculations in the following way:

n	6	12	24	48	96	192	384
A_n	2.6	3.0	3.106	3.133	3.139	3.1410	3.14145
A^*_n	3.13	3.1411	3.14156	3.141591	3.14159'2533	..'2646	..'2653

and the 768-gon gives the correct value of pi up to ten places, a result which Viète had got from the 393 216-gon by the Archimedean method. Notice that A^*_{12} is as good as A_{192}!

We became curious about this astonishing increase in the convergence velocity, and wondered why Archimedes did not seem to have our idea—after all, he had found the area of the parabolic segment, and the rest was simple enough. I looked at the classical literature and discovered that in the seventeenth century Huygens had got the same algorithm, but derived it from tedious geometrical investigations of chords and tangents [Rudio, 1982]. I then discovered that a little known German teacher, W. Kämmerer wrote a paper in 1930 on the subject in which he translated the Huygens method into the language of parabolic segments, essentially as we had done without realizing it. Finally I learned of Knorr's 1976 paper, in which he makes it likely that Archimedes had known a method as challenging as that of Huygens and ours. Knorr's article is very thrilling. I think it a masterpiece of an historical story and I do not dare to report the details here.

This raised some speculation in class and the students all started looking for a way to prove that angle *COD* was equal to the other two. After some fruitless efforts, one of the students remembered that this topic had been dealt with in the form of an exercise in an earlier chapter of the book; however, what had been proved there was that angle *COD* is in fact greater than angles *AOC* and *DOB* [7].

Following this, the particular trisection method was rejected. [8]

At this point I intervened in order to remind the students that there is another exercise in the book which solves the trisection problem by a method attributed to Archimedes. This exercise had been solved in an earlier chapter as a routine exercise without any particular reference to its bearing on the trisection problem.

This is how the exercise is phrased in the book: "Consider a diameter *AB* of a circle *(O,r)* and a point *C* on the extension of *AB* away from *A*. From point *C* draw a line meeting the circle at points *D* and *E*, such that *CD* = *r*. Show that angle *EOB* = 3 times angle *DOA*." [9]

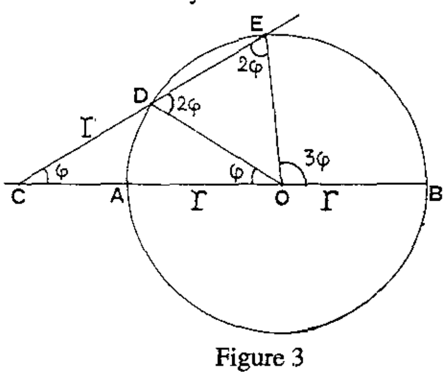

Figure 3

Archimedes method of trisection

by "neusis-construction"

(the insertion principle)

This is a minor deviation from the eighth proposition of the "Book of Lemmas" by Archimedes, a collection of 15 geometrical propositions preserved in an Arabic translation. The eighth proposition itself reads as follows: "If a chord AB of a circle is extended to C, in a manner such that BC is equal to the radius of the circle, and if the diameter ZE is drawn through C, then arc AE will be three times the size of are BZ." [10]

Archimedes's proof of this is shown in Figure 4, where chord EH, drawn parallel to chord AB, seems unnecessary. However if we take into account that this is a geometrical

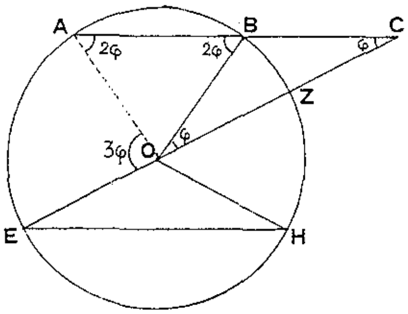

Figure 4

construction problem (of arc BZ, equal to one third of arc AE), then it is most probable that chord EH is a by-product of the analysis of the problem.[11]

I then showed my students how to trisect an arbitrary angle EOB, making use of Archimedes's proposition.

> *step 1*: Angle EOB is made an angle at the center of a circle (O,r) and the diameter passing through B is extended away from A, as shown in Figure 5.
>
> *step 2*: With the help of the compasses we mark on the straightedge two points such that the distance between them is equal to radius r.

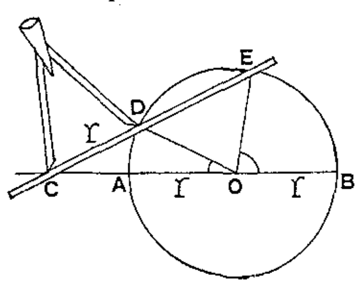

Figure 5

Using the insertion principle for trisecting a general angle with straightedge and compasses

> *step 3*: We then place the straightedge so that it passes through E and we then slide it until the two points marked on it meet the circle and the extension of diameter AB at points D and C respectively.

Angle DOC formed by this construction is equal to one third the size of angle EOB.

Comparing the steps shown above to the solution of the schoolbook exercise, the students were convinced that any angle can be trisected in this way, using only straightedge and compasses. A question then arises: Why does the historical note in the schoolbook say that the trisection of an angle is an unsolvable problem? My emphasizing of this question created some puzzlement in the classroom. None however doubted the correctness of that construction. I then asked the students to re-examine the construction shown in Figure 5 in a step by step manner until they located a point where one of Euclid's limitations was violated.

This activity was accompanied by revision and some discussion of the meaning of the term "geometrical construction with the use of straightedge and compasses" until it was finally made clear that the point in question was the marking on the straightedge of the distance equal to the radius of the circle. This method, which allows the insertion between two curves of a line segment passing through a given point, constitutes a fine yet substantial violation of the constructional limitations of Euclidean geometry. In ancient times this technique, which was termed "neusis", played an important part in the solution of geometrical problems. It demonstrates that ancient mathematicians had no qualms about by-passing Euclid's traditional limitations. In reality, it appears that geometrical research involved the exclusive use of straightedge and compasses only at a sophisticated theoretical level.[12]

These last comments resulted in a wider discussion about the aims and characteristics of geometry courses. Most students agreed that this problem-oriented activity on the trisection of any angle was much more interesting than the accumulation of definitions and theorems which comprises the usual practice in geometry teaching.

I believe that this activity, which was based on the history of a geometrical problem, was found interesting by the students because it allowed them to promote their creativity, and to experience an interchange between acceptance and rejection and the revision of preconceived ideas. In other words, to become acquainted with mathematics as an evolutionary process rather than a definitive, abstract theory.

I will finish my first example by noting that the exploration of the angle trisection problem could be naturally expanded upon to include the subject of algebra, and a similar classroom activity could be planned based again on the historical evolution. The latter is associated mainly with Viète, who discovered the relationship between this problem and the solution of cubic equations, using the triple-angle trigonometric formulae. [13]

It is now time to proceed to our next historical digression, from the geometry syllabus of the second year of the lyceum.

3. Applications of geometrical propositions

As we previously mentioned, the teaching of theoretical geometry has as a main objective the familiarisation of the students with the way that a branch of mathematics is founded and developed axiomatically. In this way geometrical applications are cast aside or, at best, some propositions with important applications appear as routine exercises without any mention of their utility being made.

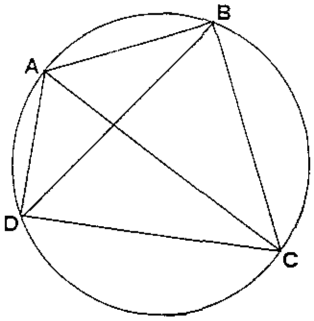

Figure 6

$AB \cdot CD + AD \cdot BC = AC \cdot BD$
Ptolemy's theorem for an inscribed quadrilateral

Ptolemy's theorem is a typical example. It appears in the schoolbook as an exercise in the chapter on similar figures:

> The sum of the products of the opposite sides of any inscribed sides of any inscribed quadrilateral is equal to the product of its diagonals. [14]

On the same page in the schoolbook there is also the following footnote:

> Claudius Ptolemy Greek mathematician, astronomer and geographer who lived in Alexandria during the second century B.C

This is a typical footnote of historical content which aims at emphasizing the Greek origin of a theorem. But, as one of the students asked, why should there be special reference to an exercise which does not seem to be any different to all the others in the textbook?

The book does not mention anything about the fact that this proposition had a key role in the construction of a table of chords, which was essential to the solution of important measurement problems of ancient astronomy. Both the proposition and the table of chords (the equivalent of a modern table of sines), are contained in Ptolemy's great astronomical work called "Mathematical Syntaxis" which, better known as *Almagest*, was the bible for astronomers, geographers and explorers up to the 16th century.

This is the original form to the theorem:

> Let $ABCD$ be any quadrilateral inscribed in a circle and let AC and BC be joined It is required to prove that the rectangle contained by AC, DC and AD, BC. [15]

Using this theorem, Ptolemy proved some propositions that are equivalent to trigonometric formulae. Such a proposition is the one known as "the three chord problem which can also be found in the schoolbook in the form of an exercise:

> Consider two consecutive arcs of a circle (O, R) having a sum less than a semicircle and chords $AB = a$ and $BC = b$.
> If $AC = x$, show that:
> $$x = \frac{1}{2R}\left(a\sqrt{4R^2 - b^2} + b\sqrt{4R^2 - a^2}\right) \quad [16]$$

The validity of this formula can be shown with the aid of Pythagoras' and Ptolemy's theorems (see Figure 7).

I utilized this exercise to give a historical example of the importance of geometrical propositions and their applications, which are not apparent when these propositions appear as routine exercises The selection of this particular proposition was influenced by the fact that I also taught

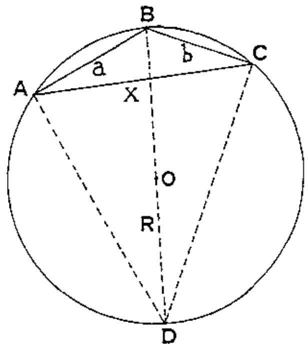

Figure 7

The chord of the sum of two arcs
(A corollary to Ptolemy's and the Pythagorean theorem)

astronomy to this class, a subject that had often become the stimulus for discussions about the applications of mathematics.

I started with a discussion about the reasons why it was important for ancient astronomers to know the lengths of chords of a circle. As an early example of astronomical calculations, I mentioned to my students the proof by Aristarchus of Samos (circa 260 B.C.), of the following inequality:

$$18 < \frac{ES}{EM} < 20$$

where EM is the distance between the earth and the moon and ES the distance between the earth and the sun.

Aristarchus assumed that at half-moon:

angle $EMS = 90^\circ$ and angle $MES = 87^\circ$

and proved the previous inequality through an impeccable but rather long-winded procedure, based on Euclidean geometrical principles. [17]

At this point I asked my students, working on the diagram shown in Figure 8, to estimate the ratio ES/EM using any method they considered easy. As was to be expected, many of them used their knowledge of trigonometry and

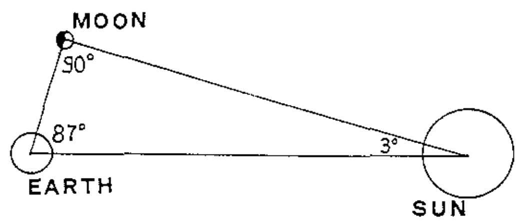

Figure 8

The method of Aristarchus for comparing the distances of the Sun and Moon from Earth through the right angled triangle *MES* found that $ES/EM = 1/\sin 3°$. Then, using the sine-table in the textbook, they arrived at the conclusion that $ES/EM \approx 19$, thus confirming Aristarchus's result. [18]

I attempted to show them that this rapid trigonometric solution was the result of a historical evolution, the origin of which was the application of geometrical propositions to the solution of astronomical problems. Ancients were not familiar with sines; they had, however, developed an equivalent method which was based on the calculation of the chords of a circle. Using this method, we can easily estimate the ratio ES/EM by considering a triangle $M_1E_1S_1$ similar to triangle MES and inscribed in a circle (Figure 9). Then $ES/EM = E_1S_1/E_1M_1$, therefore $ES/EM = 2R/\text{chord } 6°$ and thus the ratio ES/EM can be immediately determined if the radius of the circle and the chord of a 6° arc are known.

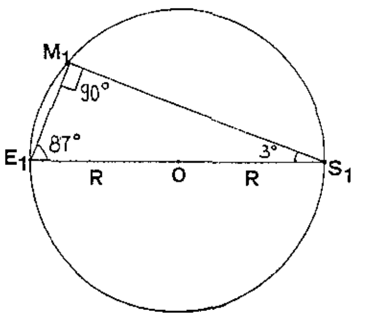

Figure 9

Using chords to obtain the result of Aristarchus

Ptolemy expounds in his Almagest the whole geometrical theory which is necessary for the construction of the table containing the lengths of chords of a long series of circular arcs. Of major importance in this construction are Ptolemy's theorem and the three-chord problem. The latter is used to calculate the chord of the sum of two arcs whose chords are given.

Aiming to reveal the relation between the ancient method and modern trigonometry, I asked the students to work on the following questions:

(a) What is the relation between a chord and a sine?

(b) Which trigonometric formula is "hidden" behind the three-chord problem?

Many students found the correct answer to the first question by observing Figure 9, where $\sin 3° = E_1M_1/E_1S_1 = \text{chord } 6°/2R$. Therefore, for $R = 1$,

$$\sin x = \frac{\text{chord } (2x)}{2} \quad \text{(a)} \quad (0° < x < 90°)$$

Also, from the same figure, $\cos 3° = S_1M_1/E_1S_1 = \text{chord } 174°/2R$ and thus, for $R = 1$,

$$\cos x = \frac{\text{chord } (180 - 2x)}{2} \quad \text{(b)} \quad (0° < x < 90°)$$

The answer to the second question seemed to be more difficult and I restated the three-chord problem, using the unit circle and appropriate notation (Figure 10). Starting from Ptolemy's theorem, the students first found the "third" chord,

$$AC = \tfrac{1}{2}\,(AB.CD + BC.AD)$$

and then, substituting sines and cosines for chords according to (a) and (b), arrived at the familiar trigonometric formula:

$$\sin (x + y) = \sin x \cdot \cos y + \sin y \cdot \cos x$$

The validity of this formula had been shown a few days earlier, in a totally different manner which involved the analytical definition of trigonometric functions in school algebra, i.e. with the help of coordinates and the unit circle.

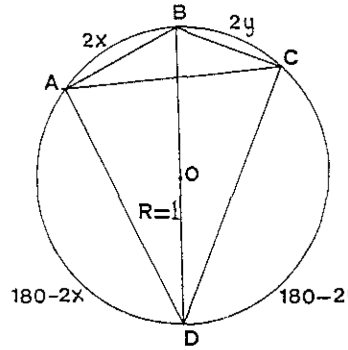

Figure 10

Derivation of addition formula
$\sin (x + y) = \sin x \cdot \cos y + \sin y \cdot \cos x$
using Ptolemy's theorem

This new and rather unexpected geometric approach via Ptolemy's theorem offered the opportunity for a brief discussion about the interaction between geometry and algebra. This interaction is never made explicit at school, especially since (as happens in Greece) the teaching of geometry aims to retain a Euclidean purity and rejects all references to applications.

This last class discussion also served as an introduction to the next historical digression. This was on the subject of "geometrical algebra" of the ancient Greeks, and in particular to what is known as the "geometrical solution of quadratic equations".

Conclusion

I will round off my paper by trying to answer the question which I believe arises every time that the use of historical subjects in the teaching of mathematics is discussed:

> What influence did these historical digressions have on the students, and to what extent did they contribute to the better understanding of the lesson?

My aim was certainly not to investigate the subject of "better understanding". Such an effort requires systematic research and monitoring of the student's perception of geometric and other mathematical concepts. Yet, summing up my classroom experiences, I can make some final observations.

It is my belief that the historical digressions were very effective as a supplementary teaching element, particularly at those points where the established lesson procedure generated negative reactions if not indifference on the part of many students. Exciting their interest and curiosity, the historical work showed that theorems and schoolbook exercises, rather than being the objects of a cold theoretical development, could become the stimulus for discussion and participation in creative activities.

Naturally, all this concerns the particular field of Euclidean geometry as it is taught in Greek secondary schools, associated with some historical material that has been incorporated in the schoolbooks for quite different reasons. I therefore think that what is needed is to organize, in future, another set of historical digressions which will cover a wider range of subjects and aim at a more systematic approach to history in mathematical education.

Notes

[1] A brief analysis of historical notes in the Greek mathematical schoolbooks of the last 30 years can be found in: Thomaidis, Y & Kastanis, N [1987] A diachronic investigation of the relationship between history and the teaching of mathematics. *Euclides C* 16: 61-92 (in Greek)

[2] On the ideological function of school mathematics in Greece, see Toumasis, Ch. [1989] Trends and characteristics of secondary school mathematics in new Greece, in relation to socio-economic changes and the developments of the mathematical science (1836-1985). Unpublished doctoral dissertation in the University of Patras, Greece (in Greek) Particularly chapter 5, where reference to M Stephanidis is made

[3] The features of school geometry in Greece have frequently become a subject of study in the last few years See for example

Kastanis, N [1986] "Euclid must go! We are not going to betray our country!" (an historico-didactic examination of the contradictions within school geometry in Greece) *Mathematical Review* 31: 3-18 (in Greek).

Thomaidis, Y [1988] Geometry lessons and non-Euclidean geometries (a didactic and historical analysis stemming from the actualities of Greece) Occasional paper no 11 Group for the History of Mathematics (in Greek).

Toumasis, Ch. [1990] The Epos of Euclidean Geometry in Greek Secondary Education (1836-1985): How a school mathematics subject can function as ideology. Occasional paper no 33. Group for the History of Mathematics.

[4] Pedagogic Institut [1989] *Guide-lines for the syllabus and the teaching of gymnasium and lyceum subjects during the 1989-1990 school year* 2nd Issue. Mathematics O E D B Athens (in Greek) pp 50-51

[5] Papamichael, D. & Skiadas, A. [1989] *Theoretical geometry for the first form of lyceum* O E D B , Athens (in Greek) The extracts come from: Chapter 8 "Simple geometrical constructions-Characteristic points of a triangle", p 136 and Chapter 9 "Geometrical loci and constructions", p 151

[6] Panakis, I. [1970] *The geometry of triangle-geometrical loci and constructions* Vol 2 Gutenberg, Athens (in Greek) pp. 11-12.

[7] Papamichael, D. & Skiadas, A [ibid] chapter 7 "Circle theorems-inscribed angles" p 132, exercise 2.

[8] This method can provide an approximate, yet accurate, solution to the trisection problem. This solution was published by the great German painter A Dürer in 1525 in his study entitled "Underweyssung der Messung mit dem Zirkel und Richtscheyt" (Instruction for measurements with compasses and straightedge). See Kaiser, H. & Nöbauer, W [1984] *Geschichte der Mathematik* G.F Freytag, München: 137-138

[9] Papamichael, D & Skiadas, A [ibid] p 133, exercise 7

[10] Stamatis, E [1974]. *The complete works of Archimedes* Vol 3 T E E , Athens (Ancient text with modern Greek trans) p 26

[11] This subject is examined in detail in: Knorr W R [1986] *The ancient tradition of geometric problems*. Birkhäuser, Boston: 185-186. According to Knorr, the original form of the problem was: "An arc BH is to be divided at Z so that arc BZ is one third of BH" Knorr presents a reconstruction of the analysis, in which chord EH plays a determinative part.

[12] About the "neusis" method and the plethora of other techniques employed by ancient Greek geometers to solve problems, see Knorr, W R [ibid] Particularly p. 13, note 17, and p 345 for the importance of straightedge and compasses constructions.

[13] See e.g. Boyer, C B [1968] *A history of mathematics*. Wiley, New York, p 341. For a review of the historical evolution of the trisection problem, see Dedron, P. & Itard, J [1978] *Mathematics and mathematicians* Vol 2, The Open University Press: 167-180

[14] Varouchakis, N. et al [1989] *Theoretical geometry for the second form of lyceum* O E D B Athems (in Greek) p 39, exercise 7

[15] Thomas, I. [1941] *Selections illustrating the history of Greek mathematics* Vol. II, From Aristarchus to Pappus Harvard University Press, Cambridge, Mass: 422-425.

[16] Varouchakis, N et al [ibid] p 60, exercise 12

[17] Aristarchus' geometrical proof, with a translation of the ancient text and modern notation, can be found in Thomas. I [ibid] pp. 8-15

[18] The true value of the ratio ES/EM is approximately 400 This high deviation is due to the fact that the real value of angle MES is 89°51' and not the 87° that was estimated by Aristarchus Of course this by no means diminishes the great value of Aristarchus' work that lies in the purely geometrical method he used for the solution of astronomical problems. See Papathanasiou, M [1980] Aristarchus of Samos *Mathematical Review* 20, 91-120 (in Greek)

The education of the child must accord both in mode and arrangement with the education of mankind as considered historically; or in other words, the genesis of knowledge in the individual must follow the same course as the genesis of knowledge in the race

Herbert Spencer