The Story of 0

RICK BLAKE, CHARLES VERHILLE

Eighth Grader	But with zero there is always an exception it seems like
	Why is that?
	Because it is not really a number, it is just nothing.
Fourth Grader	Anything divided by zero is zero.
	How do you know that?
	Well, one reason is because I learned it."
	How did you learn it? Did a friend tell you?
	No, I was taught at school by a teacher.
	Do you remember what they said?
	They said anything divided by zero is zero
	Do you believe that?
	Yes
	Why?
	Because I believe what the school says. [Reys & Grouws, 1975, p.602]
University Student	Zero is a digit (0) which has face value but no place or total value. [Wheeler, Feghali, 1983, p. 151]
	If 0 "means the total absence of quantity," we cannot expect it to obey all the ordinary laws. [Romig, 1924, p. 388]
University Student	$0 \div 0 = 0$ because:
	If you have a zero amount of something (in other words nothing) and divide that by zero amount of something (nothing) you will obviously end up with nothing.
	nothing ÷ nothing ≠ something *but* nothing ÷ nothing = nothing $0 \div 0 = 0$

The comments noted above display the language used to convey some of the common misconceptions about zero.

This paper is about the language of zero. The initial two sections deal with both the spoken and written symbols used to convey the concepts of zero. Yet these alone leave much of the story untold.

The sections on computational algorithms and the exceptional behaviour of zero illustrate much language of and about zero.

The final section of the story reveals that much of the language of zero has a considerable historical evolution; the language about zero is a reflection of man's historical difficulty with this concept.

The linguistics of zero

Cordelia:	Nothing my Lord
Lear:	Nothing!
Cordelia:	Nothing...
Lear:	Nothing will come of nothing; speak again
	.
	.
Kent:	This is nothing, fool.
Fool:	... Can you make no use of nothing, uncle?
Lear:	Why, no, boy; nothing can be made out of nothing

Ideas that we communicate linguistically are identified by audible sounds. Skemp refers to these as "speaking symbols." These symbols, although necessary inventions for communication, are distinct from the ideas they represent. Skemp refers to these symbols or labels as the surface structures for transmitting the associated mathematics concepts which he calls the deep structures. The surface structures are more accessible. They represent the routes to existing deep structures as well as for introducing new ones. Formulating deep mathematical structures depends on:

(a) interaction with and between existing deep structures.
(b) existing surface structure.
(c) how well an individual's deep structure is matched with the corresponding mathematical structure
(d) corresponding connections (attractions) between (a) and (b)

> Symbols are an interface between the inner world of our thoughts and the outer, physical world [Skemp, p. 281]

Mankind has been generous in inventing a copious quantity of colorful "Speaking Symbols" that denote zero. In our society, zero continues to demonstrate unrest. Communications and media sources, sports and common-use language indicate but a few of the current generators of colorful new symbols. The list includes zero, nothing, zilch, oh, nought, ought, empty, goose egg, no, none, origin, shut out, cipher, luv, null, no hitter,.... In ordinary conversation where the idea zero is used, we seldom use the word zero. Telephone numbers, street addresses, time and license plates which include the zero symbol are almost universally read as "oh." Cultural and common use factors influence this preference. This dominant linguistic surface structure "oh" thus denotes two distinct structures: the letter O (another surface structure) and zero (a deep mathematical structure). This infringement and usurpation would not have occurred if different symbols had evolved to denote these structures. This same speaking symbol is used to elicit different structures, one which is itself and the other the number zero.

The contexts where the word zero is used indicates some ways in which the properties and uses of zero have become part of our language. Around these properties we have discovered and invented such a diversity of uses that zero is indispensable. The following expressions indicate some of this diversity: zero-based budget, zero-hour, below zero, visibility zero, zero-in, zeros of a polynomial, zeros of a function, zero slope, absolute zero, zero—the name of a WWII Japanese fighter plane, Zero—a character in the Beetle Baily comic strip, her patience has nearly reached zero.

Yet another way to examine the linguistics of zero is to ask for an elaboration of responses to "What is zero?" Research evidence indicates that the responses include number, nothing, place holder, empty set, identity element, symbol. The range of responses depends on the extent to which an individual's zero structure has evolved.

"No apple"; "Q: How many apples are in the basket? A: None."; "Q: What's in the basket? A: Nothing." are common ways of effectively verbalizing that there are zero apples in the basket. This "zero is nothing" analogy has evolved from the language as one of the most common speaking symbols for zero. This analogy is deeply rooted such that it is common to virtually every conversation involving the idea of zero and is recorded in most articles about zero and publications which include this concept. However, common language is notoriously ambiguous and imprecise and when used as the basis for establishing precise notions, the results may be less than desirable. The desired deep mathematical structure does not evolve, resulting in a flimsy, superficial surface structure with inadequacies that are not apparent until later. Dienes correctly indicates that "because children confuse the number zero with nothing... a great deal of mathematical difficulty arises later on." [Dienes, p. 15] The zero-is-nothing analogy is used in several examples that follow.

(A) If I have eight pennies on the desk in front of me and then I take eight pennies away, *I have nothing left*. This is written as $8 - 8 = 0$.

Teachers of young children are well aware that *"nothing,"* or *"zero,"* has to be introduced very carefully in the early stages of arithmetic.. $0 + 8 = 8$ and read as meaning *"if eight is added on to nothing,* the answer is eight." Now a great deal of imagination is required to *start by having nothing. It is far simpler to start with something, take it away and become aware that there is nothing left...* [Land, p. 35]

(B) Multiplication of a number by 0 gives the number 0. If I have a pocket full of sixpences, put my hand in my pocket but do not take *anything* out of it, *I have nothing in my hand. This obvious and trivial statement* merely serves to illustrate the statement $6 \times 0 = 0$.

Multiplication of 0 by any number gives the answer 0. *If I put nothing on the table and then I put nothing on the table again and then I put nothing on the table again, altogether I have put nothing on the table, so that* $0 \times 3 = 0$. [Land, p. 43]

(C) Division... NB: division by 0 leads to difficulties... If again I start with 12 pennies and now I take out *nothing,* I put my hand in again..., there is no limit to the number of times which *I can do this and there is no answer in simple numbers to the division* $12 \div 0$.

It is sometimes convenient to be able to record that something can be continued without there being any limit to the number or any end to the process and for this purpose a special symbol has been introduced, namely ∞. [Land, p. 43]

(D) First, if I had no marbles to divide equally among 3 boys, how many marbles does each boy get? *Nothing divided three ways is clearly just that, nothing.* [Henry, p. 366]

(E) Two of the sixth graders said that '*You can't divide by zero,'* and their *justification was simply that 'my teacher said so.'* Four of them said that *'0' was the quotient of* $6 \div 0$. These responses also seemed based on intuition; however, *two of the children said that they KNEW it was zero 'Because my teacher told me.'* [Reys, p. 155]

(F) How can you divide *nothing* into *nothing* and receive *anything* but *nothing*. [Wheeler, p. 152]

(G) .

Jennifer: ... so you have to say that "8 goes into 48, 6 times" and write:

$$\begin{array}{r} 6 \\ 8/4806 \end{array}$$

Teacher: All right. What do you do next?

Jennifer: The *zero doesn't count for anything,* so you say '8 goes into 8 once.' She wrote:

$$\begin{array}{r} 6\;1 \\ 8/4808 \end{array}$$ [Davis, Greenstein, p. 95]

(H) Teacher: Any number is divisible by 1.
Student: Is zero a number?
Teacher: Yes.
Student: Then it's not divisible by 1.
Teacher: Yes, it is *because zero divided by anything is zero. Okay!*
Student: Okay! [Teacher Intern]

The examples from (A) to (G) above are from published material. (A) to (C) are from one source.[1] We recommend that the reader reflect on these examples; in particular, the parts emphasized by italics.

The linguistics of zero is deeply influenced by what is written. Teachers make frequent use of published material and may not be in a position to question its accuracy. Mathematics texts for teachers, school text series and professional journals frequently publish material about zero that is sparse, incomplete, misleading or incorrect. This is unfortunate since zero misconceptions seem to be common in many of the error patterns displayed in computation. Example (G), involving Jennifer, is one such illustration—others are described in the section on computation involving zero.

Examples (A) to (F) represent common linguistic rationales and justifications used in computation involving zero. The zero-is-nothing analogy leads to a linguistic structure typically involving the words "nothing," "something" and "anything" to establish a valid result. Several additional illustrations of the dialogue treating the mathematical statements $8 \div 0 = \square$, $0 \div 8 = \square$, $0 \div 0 = \square$ as linguistic statements, as well as the linguistic arguments used to establish and justify an answer, are included in the Grouws and Reys article.

The use of the zero-is-nothing analogy likely represents a dominating intrusion of common language which creates a superficial surface structure of zero that is used with some regularity for several years. It is attractive because it is easy to learn and retrieve, intuitively satisfying and it seems to work—it seems to effectively and consistently lead to correct answers.

It is not until division involving zero is encountered that the semantics of the zero-is-nothing analogy leads to a hopelessly tangled rhetoric that is unsatisfactory both logically and intuitively. For this operation there is no logic in using the zero-is-nothing analogy, and intuition consistently leads to incorrect results. Examples D, E, F, H illustrate the linguistic dilemma which is then commonly compounded by resorting to still another linguistic device. By decree (we choose to make it sound better by saying "by definition") we disallow division by zero.[2]

The compounding effect of the decree and the zero-is-nothing analogy is deeper confusion. As illustrated in Example E, students have no conceptual basis for differentiating situations with zero as divisor, dividend, or both (i.e. $a/0$, $0/a$, $0/0$). They become mired in rhetoric or "teacher said" reasoning—the tragedy of this type of situation is that the students have no conceptual basis to verify their thinking. They are dependent on some external authority for verification.

Despite much evidence of the inadequacies of these notions of zero, zero-is-nothing survives intact and continues undaunted. This analogy effectively prevents the teaching of the deep, complex structure of zero which does not lend itself to distinct and powerful concrete pedagogical models. There is a failure to even recognize this depth, breadth and complexity of zero. Edwards suggests, "Nothing, nought, zero, empty, use words which present a single notion to children who are preoccupied with the ideas of nothingness rather than use the symbol 0 as a numeral." [Edwards, p. 2] The consequences of the use of such a shallow surface structure are ignored or unknown.

Arguments involving nothing, something and anything in computational situations reflect a frequent overgeneralization of the following powerful mathematical properties:

(1) $a + 0 = 0 + a = a$ (3) $a - 0 = 0$
(2) $a \times 0 = 0 \times a = 0$ (4) $0/a = 0,\ a \neq 0$

These are commonly explored using a set model. The language of such explorations including statements like "any set," "empty set," "any number" and "zero" are replaced by "anything" and "nothing." In this context, "theory" is used as a variable to represent "number" or "set." The connotation that the shift from

any *set*
any *number* } to — *anything*

empty *set*
zero } to — *nothing*

is mathematically sound is misleading and untrue. The potential precision and consistency which numbers and sets offer is lost when the abyss of common language is allowed to intrude and dominate.

The way to satisfactorily develop the concept of zero is with models that reflect its deep mathematical structure. The ability to use sound judgement in selecting pedagogical models depends on having a sound, deep mathematical structure of a concept. Otherwise, there is no basis for discriminating between apparent choices since the most important factors influencing these judgements are absent.

The linguistics of zero discussed in this section represent spoken words invented to communicate conceptual structures. This communication depends on the soundness of both the conceptual and linguistic structures as well as the correspondence between them. Many problems related to the linguistics of zero that arise when these structures or their correspondence are inadequate have been noted. The remedy is to develop the conceptual structures of zero

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[1] This publication includes soundly presented mathematics using informative and interesting historical and contemporary illustrations. It also is the best illustration of some of our points concerning zero.

[2] Jack Motta, in the Reader's Dialogue of *The Arithmetic Teacher*, suggests that he was taught in this manner and that "the insinuation was that if you did divide by zero, lightning would strike you."

rather than rely on ambiguous surface structures derived directly from common language. This is not likely to occur since pedagogical material suggests that we do not realize the conceptual depth and complexity of zero.

The mark of zero

Canadian comedians, Wayne and Schuster, presented a television program which had a skit entitled "The Mark of Zero" during 1983. Being a take-off on the well-known story "The Mark of Zorro," the hero left the mark "0" rather than "Z" as a reminder of his presence. Man has invented many other uses for the mark of zero. Several of the mathematical ideas denoted by the symbol for zero are presented in this section

The conceptual structures of zero are communicated by written as well as spoken symbols. The diverse use of the mark of zero rivals its spoken counterpart because of the diverse use of zero as a mathematical structure. On the surface many of these uses of the symbol for zero are indistinguishable. Access to the mathematical structure of zero is necessary in order to meaninfully interpret the context of the written symbol

ZERO AS A PLACE HOLDER

Historically, the first known use of the mark of zero by the Mayans and Babylonians was as a place holder to signify the empty place. The historical roots of our currently used zero are briefly traced in, "The Origin of Zero." For some time after its invention, zero was used exclusively as a symbol to designate the empty place since, "the discovery of zero was an accident brought about by an attempt to make an unambiguous permanent record of a counting board operation." [Dantzig, p. 31] Historical material suggests that all numeration systems were invented and initially used exclusively for recording purposes. For this purpose, as evidenced by several ancient numeration systems, it was unnecessary to conceptualize a zero in any role. The Hindus are credited with inventing the forerunner of our current numeration system including the zero. This zero, initially known as sunya bindu, was not the number zero. Many centuries were to pass before zero was recognized as a number.

With the eventual discovery of the number zero, the symbol 0 had two distinctly different roles, as a symbol for a number and as a place holder. Today, we unconsciously use zero interchangeably in these roles. Established conventions on typewriter keyboards and the numeric pads of calculators reflect both number and symbolic perceptions of zero. The following test indicates whether one is more accustomed to handling the symbol or number mode.

A TEST

On Zero As A Symbol	On Zero As A Number
$1 + 10 =$	$1 + 0 =$
$10 + 1 =$	$0 + 1 =$
$1 - 10 =$	$1 - 0 =$
$10 - 1 =$	$0 - 1 =$
$1 \times 10 =$	$1 \times 0 =$
$10 \times 1 =$	$0 \times 1 =$
$10 \times 10 =$	$0 \times 0 =$
$10 \div 1 =$	$0 \div 1 =$
$1 \div 10 =$	$1 \div 0 =$
$10 \div 10 =$	$0 \div 0 =$

[Reid, p. 6]

Reid further suggests,

> The symbol is the zero he (the reader) knows; for it is a curious fact that positional arithmetic which depends for its existence upon the symbol zero, goes along very well without the number zero [p. 6]

The hindu "sunya-bindu" provided a way of representing the gaps in empty columns in recording numbers so that one could clearly differentiate numbers like 52, 520, 502, 5002, etc. This notion of zero is a holder of an empty place persists. Wheeler and Feghali received the following response, among many others, to the question, "Is zero a number?"

> Zero is not a number because it has no face value; it is a number used as a place holder which has no value

The "holds the place" notion of zero is taught to young children. This function of zero, that is, the zero in 502 "holds" the tens place, is explicitly taught and emphasized such that this role of zero is perceived to be different (an exception) from the role of the other digits. Yet the "5" and the "2" hold the hundreds and units places respectively. Furthermore, each of the ten symbols in our numeration system "holds" a place whenever it is used to record a numeral. In other words, the placeholder role is not unique to zero.

ZERO AS A VARIABLE

For the Hindus, the dot sunya, their zero symbol, is used in another mode. They still use it to represent the unknown in an equation since their interpretation is that until a space is filled with its proper number it is considered empty.

This function of "filling a space until its proper number is established" is used in some devices which provide digital displays. Calculators, other than those using RPN, customarily display a single zero when turned on. What is the function of this zero symbol? It behaves as the number zero as follows:

↙ $0 + 5 = 5$

Initially Displayed Zero From
Turning Calculator On.

↘ $0 \times 5 = 0$

On the other hand, if after turning the calculator on, thus obtaining a "0" display, any digit key is pressed except "0," that digit replaces or fills the space being held by the zero symbol.

ZERO AS SPACE FILLER I — ROUNDING

In the numeral 604, just as the digit 6 means that there are six hundreds and the digit 4 means that we have four ones, the digit 0 means that there are zero tens The distance to the sun, in premetric days, was commonly recorded as 93 600 000 miles. Do these zeros act as place holders as described above or do they behave like the Hindu variable—the space is filled by sunya until it actual value is known? On the other hand, it may be an easily handled linguistic package that is easily verbalized (ninety-three million) as compared to, say, 93 600 000 (ninety-three million six hundred thousand) Still again it may represent the distance to the sun to the nearest million miles.

The record 93 000 000 miles does not represent the actual distance to the sun (although there may be occasions when it is the actual distance). When we choose to represent the distance in this manner, a number of digits are now recorded as zeros. The actual distance at some instant today may be 92 700 140 miles How does the meaning of the zeros in this numeral compare with those in 93 000 000? There is no visible means of distinguishing zeros used differently In the deeper sense they have very different meanings yet this is not apparent from the surface symbolic structure.

ZERO AS SPACE FILLER II — PETER'S FIRST INVENTION

Herbert Ginsburg's article describes Peter's technique for solving the following division problem: $10\overline{)1071}$. This article discusses Peter's language. Peter's verbalizations do not match the actual computations performed. This suggests that the mental procedures utilized in performing the computation are not the same as the mental procedures used to verbalize the procedure Gaps in the linguistic structure must not be construed as gaps in the mathematical structure.

Peter recorded the following:

$$\begin{array}{r} \underline{0170} \\ 10\overline{)1701} \end{array}$$

When asked about the first zero in his quotient:

> Why did you put a zero there?
> I usually do that so no number goes there
> Clearly he meant that the zero's function is a space filler. [Ginsburg, p. 69]

ZERO AS SPACE FILLER III — PETER'S SECOND INVENTION

In calculating $10\overline{)1500}$, Peter recorded the following:

$$\begin{array}{r} 100 \\ \times\ \underline{10} \\ \overline{000} \\ \underline{100} \\ 1000 \end{array} \quad \text{and then} \quad \begin{array}{r} 50 \\ \times\ \underline{10} \\ \overline{00} \\ \underline{50} \\ 500 \end{array}$$

[Davis, p 38]

Peter seems to use the standard multiplication algorithm, recording 000 and 00 as the first partial product representing $\begin{array}{r}100\\ \times\ \underline{0}\end{array}$ and $\begin{array}{r}50\\ \times\ \underline{0}\end{array}$ respectively. These partial products seem to have the sole purpose of filling a space in order to satisfy the steps of a multiplication algorithm.

ZERO AS SPACE FILLER IV — TRANSITIONAL ALGORITHMS

$$\begin{array}{r} 23 \\ +\ \underline{05} \end{array} \qquad \begin{array}{r} 46 \\ \underline{32} \\ 92 \\ \underline{1380} \end{array} \qquad \begin{array}{r} 064 \\ 4\overline{)252} \\ \underline{24} \\ 12 \\ \underline{12} \\ 0 \end{array}$$

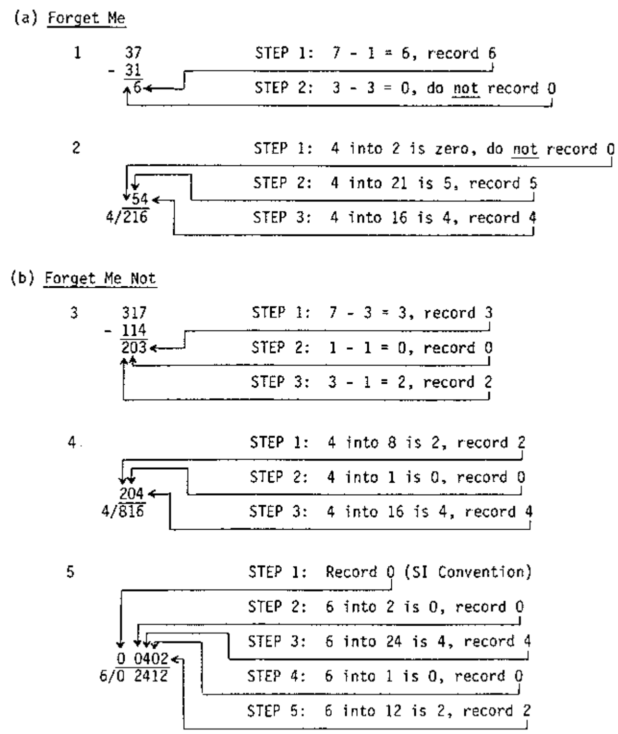
Transitional algorithms, including Peter's division and multiplication algorithm described earlier, frequently record zeros which are dispensed with in the final algorithm form. The examples illustrated above for addition, multiplication and division each include a zero which is not customarily recorded when performing the standard algorithm

ZERO AS CONVENTION

The conventions adopted for the International System of Units (SI) include the following:

> In text and tables, if a numeral value is less than one, a zero should precede the decimal marker [Canadian Standards Association, p 9]

SHOULD YOUR ZEROS BE SHOWING?



(c) Here We'll Write It; Here We Won't

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6    4317      STEP 1:  7 - 4 = 3, record 3
   - 4114
     203       STEP 2:  1 - 1 = 0, record 0
               STEP 3:  3 - 1 = 2, record 2
               STEP 4:  4 - 4 = 0, do not record 0

7              STEP 1:  6 into 2 is 0, do not record 0
               STEP 2:  6 into 24 is 4, record 4
     402
  6/2412       STEP 3:  6 into 1 is 0, record 0
               STEP 4:  6 into 12 is 2, record 2
```

SIGNIFICANT FIGURES

In science, engineering, etc., where calculations are performed on data collected by measuring, it is necessary to determine the number of significant figures in a measurement. The number of digits which express a measurement and which have meaning are called significant figures.

In general, all recorded digits are significant except zeros used to locate the decimal mark. These zeros may or may not be significant. For example:

(a) 3 — 1 s.f.	(d) 0.04 — 1 s.f.
(b) 3.2 — 2 s.f.	(e) 0.040 — 2 s.f.
(c) 3.20 — 3 s.f.	(f) 400 — 1, 2 or 3 s.f.

The examples above illustrate several different situations. The only instance which is uncertain is the last example. It is not clear what role each zero plays in recording the 400. There are three possibilities:

(1) Neither zero is significant.
(2) The zero in the tens place is significant, the other is not.
(3) Both zeros are significant.

In this section, several contexts in which the mark of zero has different meanings have been presented. For many of these contexts, there are no distinguishing features, leaving them subject to misinterpretation even with knowledge of the corresponding mathematical structures. In some cases, it is not even clear when the symbol is to be recorded. In constrast, nonzero number symbols do not have such a range of meanings and difficulty and there is no confusion about their being recorded

Computational algorithms

> ...without a zero, reckoning is impossibly hobbled [Reid, 1964, p. 4]

Little progress was made in computation until the advent of our modern numeration system. The mystique with which reckoning has held in earlier numeration systems was a reflection of the complex computational algorithms which required trained professionals to perform. Persons skilled in this area were regarded as endowed with supernatural powers. The reckoning ability of the common man was restricted to that which he could do using his fingers.

Centuries after its discovery, man was still struggling with the mastery of zero in computation. This historical development parallels many of the problems encountered by children in school today. Zero has been identified as a "trouble maker" by much of the research on computational errors. [Harvey and Kay, 1965; Trafton, 1970; Cox, 1975; Bradford, 1978; Verhille, 1982; Ashlock, 1982] Here are several of the zero errors associated with the algorithms.

Errors specific to subtraction

1. Reversal — subtracting the smaller digit from the larger:

   Example:
   $$\begin{array}{r} 80 \\ -39 \\ \hline 50 \end{array}$$

   Trafton [1970] found, on the average with grade three students, that this error occurred three times as often if there was a zero in the minuend as opposed to a nonzero digit.

2. Difficulty with multiple regrouping:

   Example:
   $$\begin{array}{r} 4 \\ \not{5} \\ \not{6}04 \\ -\ 217 \\ \hline 297 \end{array}$$

   Verhille [1982], in a study with grade five students, found 6 of 15 skill levels for subtraction with relatively higher percentage of systematic errors. He noted that, "all of these levels have a common feature, that is, the minuend either has a zero or one appears as a result of regrouping." [p. 20]

Errors specific to multiplication

1. $n \times 0 = n$:

   Examples:
   $$\begin{array}{r} 603 \\ \times\ \ \ 7 \\ \hline 4291 \end{array} \qquad \begin{array}{r} 40 \\ \times\ 40 \\ \hline 00 \\ 164 \\ \hline 1640 \end{array}$$

2. Difficulty multiplying by multiples of ten and a hundred:

   Example:
   $$\begin{array}{r} 42 \\ \times 20 \\ \hline 84 \end{array}$$

   Verhille [1982] found, for multiplication, the greatest percentage of systematic errors in the skill level associated with problems of form 30×80.

Errors specific to division

1. Omitting zero in the quotient:

   Examples:
   $$\begin{array}{r} 85R3 \\ 6/\overline{5103} \end{array} \qquad \begin{array}{r} 21 \\ 6/\overline{1206} \end{array}$$

Many of the systematic errors identified in error analysis research seem to center around zero. This identification suggests that zero is an exception and compared to other numbers, has a deeper structure.

Zero has an influence on how we perform algorithms and on which algorithms we use. In some cases these changes, including the linguistics associated with the algorithm, are more complex and in others, simpler When students are taught an algorithm in many cases what they learn is an algorithm rhyme, the linguistics associated with performing the algorithms, rather than any mathematical structure, deep structure, associated with the algorithm. Here are some examples which illustrate the influence of zero on computational algorithms

MULTIPLICATION ALGORITHM

Easy: $\begin{array}{r} 50 \\ \times\underline{20} \end{array}$ vs $\begin{array}{r} 57 \\ \times\underline{23} \end{array}$

Note how complex the algorithm rhyme is for 57×23: 3 times 7 is 21, write down 1 and carry 2. 3 times 5 is 15, plus 2 is 17, write down 17 Now 2 times 7 is 14, write 4 under the 7 and carry 1 2 times 5 is 10 plus 1 is 11, write down 11 Write down 1. 7 plus 4 is 11, write 1, carry 1. 1 plus 1 is 2 plus 1 more is 3, write down 3 and write down 1.

Compare this to an algorithm rhyme for 50×20: 5 times 2 is 10, write down 10 and two zeros

SUBTRACTION ALGORITHM

Easy: $54 - 0$ vs $54 - 6$

To subtract 6 from 54, need to rename, one from 5 is 4, 14 minus 6 is 8, bring down the 5. However, all that is needed to compute $54 - 0$ is the rhyme, "any number minus zero is that number" so the difference is 54

Easy: $\begin{array}{r} 54 \\ -\underline{20} \end{array}$ vs $\begin{array}{r} 54 \\ -\underline{26} \end{array}$

The zero in the subtrahend in the first example guarantees no renaming and the use of the rule $n - 0 = n$ makes the rhyme for $54 - 20$ much easier than that for $54 - 26$.

Hard: $\begin{array}{r} 5004 \\ -\underline{2197} \end{array}$ vs $\begin{array}{r} 5224 \\ -\underline{2197} \end{array}$

There is multiple regrouping in both examples, but the renaming in the first example is more difficult because of the zeros. Here is an algorithm rhyme for this problem: Can't take 7 from 7, must rename, no tens, no hundreds, take 1 from 5 leaving 4 Rename the first zero as ten, take 1 from it, leaving 9 Rename the second zero as 10, take 1 from it leaving 9. Now have 14. 7 from 14 is 7, write down 7 9 from 9 is 0, write down 0 1 for 9 is 8, write 8. 2 from 4 is 2, write 2

With the emphasis on "no tens," the most complex part of this rhyme is renaming the 4 as 14. This complex renaming may be one of the reasons students "borrow" directly from the 5.

DIVISION ALGORITHM

Easy: $100\overline{)472.5}$ vs $137\overline{)472.5}$

Again, the use of a simpler algorithm rhyme; when dividing by 100, move the decimal point to the left two places, which makes $472.5 \div 100$ the much easier problem.

Hard: $6\overline{)12218}$ vs $6\overline{)738}$

Omitting the zero in the quotient in the first example is one of the common errors in division. Compare these algorithm rhymes for $6\overline{)1215}$.

6 into 1, won't go
6 into 12 goes twice, write 2
6 into 1, won't go, write 0 — $\begin{array}{r} \underline{203} \\ 6/1218 \end{array}$
6 into 18 goes three times, write 3

6 into 1, won't go
6 into 12 goes twice, write 2
6 into 1, won't go, — $\begin{array}{r} \underline{2\ 3} \\ 6/1218 \end{array}$
6 into 18 goes three times, write 3

6 into 1 goes 0 times
6 into 12 goes twice, write 2
6 into 1 goes 0 times, write 0 — $\begin{array}{r} \underline{203} \\ 6/1218 \end{array}$
6 into 18 goes 3 times, write 3

6 into 1 goes 0 times, write 0
6 into 12 goes twice, write 2
6 into 1 goes 0 times, write 0 — $\begin{array}{r} \underline{0203} \\ 6/1218 \end{array}$
6 into 18 goes 3 times, write 3

The four rhymes are very similar Three of them are correct, granted that we usually do not write zero as the left most digit A student operating only with a surface structure, the algorithm is the rhyme, must remember when to write zero after the line "6 into 1, won't go," and when not to.

If students rely only on algorithm rhymes to perform computation, the deep mathematical structures needed to support the algorithm are replaced by a surface linguistic structure. If they forget a line in their rhyme, or get mixed up, they have nothing to fall back on

Operating at a surface level, students have no insight into the power of zero in computation. They have no deep structure to help them in deciding when to use a simple algorithm or how to modify the problem to take advantage of zero Questions like:

$\begin{array}{r} 452 \\ -\underline{199} \end{array}$ or $\begin{array}{r} 799 \\ +\underline{614} \end{array}$

lend themselves nicely to modification to use the power of zero Both can be simplified by using equal addends, i.e.:

$\begin{array}{r} 452 + 1 \\ -\underline{199 + 1} \end{array} \rightarrow \begin{array}{r} 459 \\ -\underline{200} \end{array}$ and $\begin{array}{r} 799 + 1 \\ +\underline{614 - 1} \end{array} \rightarrow \begin{array}{r} 800 \\ +\underline{613} \end{array}$

Zero is indeed an exceptional number It can be a frustrating number, for it is involved in many of the computational difficulties students experience. But zero is powerful in simplifying computation. There are some very strong patterns involving zero which lead to short cutting the standard algorithms Students must develop a deep structure for the algorithms. Computation should not be just a algorithm rhyme.

Zero the exception

Much of the language about zero is a result of its behaviour "Zero is a problem wherever and whenever it appears. It confounds us much of the time and so receives very special attention all the time." [Newman, p. 379] Compared to the other numbers, it took man a long time to see the need to express "not any" as a symbol and much longer as a number.

> Zero is the only number which can be divided by every other number, and the only number which can divide no other number. Because of these two characteristics, zero is almost invariably a "special case" among the numbers. Zero is enough like all the other natural numbers to be one of them, but enough different to be a very interesting number: the last, and the first, of the digits. [Reid, p. 14]

Dantzig [p.35] captured the essence of what many writers alluded to about zero:

> Conceived in all probability as the symbol for an empty column on a counting board, the Indian sunya was destined to become the turning point in a development without which the progress of modern science, industry, or commerce is inconceivable .. By proving the way to a generalized number concept, it played just as fundamental a role in practically every branch of mathematics. In the history of culture the discovering of zero will always stand out as one of the greatest single achievements of the human race. A great discovery! Yes. But, like so many other early discoveries, which have profoundly affected the life of the race, ... not the reward of painstaking research, but a gift of blind chance.

In this section a detailed examination of the exceptional behaviour of zero in division is presented. Also, several other areas of mathematics curriculum which exhibit interesting and exceptional zero behaviour are noted.

Centuries after man invented the symbol sunya to indicate an empty space he was still fumbling toward the mastery of zero as a number which could be added, subtracted, multiplied and divided like any other number. "Addition, subtraction, even multiplication seem to have caused relatively little trouble." [Edwards, p. 4] But an acceptable analysis of division involving zero was to cause troubles well into the 19th Century.

Early Hindu writing reflect man's misconception and confusion involving zero as a number. In 628 AD, Brahmagupta [Romig, 1924] set down the following rules of operations involving zero.

$$0/0 = 0, +a/0 = +a/0, -a/0 = -a/0$$

No indication was given as to why a non-zero number divided by zero was left in this form.

About 850 AD, Mahavira wrote, "A number multiplied by zero is zero, and that number remains unchanged which is divided by, added to, or diminished by zero." [Smith, 1951, p. 162] This statement indicates an identity concept of zero for addition, subtraction and division. This conception of division involving zero was changed some 300 years later in the writing of Bhaskara [Gundlack, 1969] who wrote that a definite number divided by cipher is a submultiple of zero. He used the same notation that Brahmagupta used, namely:

$$3 \div 0 = 3/0$$

but went on to indicate that fractions with a denominator of cipher were termed "infinite quantities."

Most other writings of this period either entirely avoid the question of division involving zero or declare the result of such division to be meaningless. [Gundlack, 1969; Romig, 1924]

In 1657 Wallis declared zero to be no number but introduced the symbol ∞ as $1/0 = \infty$. In 1716, Craig declared that zero must be infinitesimal and of absolutely no value so it cannot be used as a divisor. In the same time period, Berkeley states that "zero is no number" while Landen calls it a mere blank or Absolutely Nothing." [Romig, 1924]

The first argument that division by zero is unacceptable was given by Ohm in 1828. He states, "if a is not zero, but b is zero, then the quotient a/b has no meaning, for the quotient multiplied by zero gives only zero and not a." [Newman, 1956, p. 293; Romig, 1924, p. 388] In the 1880's, texts on elementary algebra used this argument to exclude division by zero [Newman, 1956]. "In 1832 Wolfgang Bolyai de Bolya stated that "$1/0$ is an impossible quantity" but that "if z tends towards 0, then $1/z$ tends towards infinity and $1/z = \infty$." [Romig, 1924, p. 388] "In 1864 De Morgan described $1/0$ as "the extreme infinite"." [Edwards, p. 3]

As recent as the 1950's division by zero was acceptable. In a text for high school students, Oliver, Winters and Campbell presented a discussion of angles of 90 degrees. They indicated that $\tan 90° = \infty$ where "∞ means a number infinitely great or infinity," [p. 32] and that "... $n/0 = \infty$ where n is any number." [p. 33]

Much unrest persists today in division involving zero. Basic computational facts involving zero are customarily presented using a few examples to demonstrate a powerful pattern. The attractiveness of the patterns, the product, overshadows and oversimplifies the processes used to establish them. All other facts may be established via primitive counting procedures which tend to be automatically associated with quantity or magnitude. This association rubs off on to the natural numbers creating a meaning problem for zero as a number. The strong affinity for number as quantity is enhanced by the emphasis on concrete aids to teach number and number operations. This quantity attribute of number as reflected in the following response to "Is zero a number?" recorded by Wheeler and Feghali is not uncommon.

> No, zero is not a number; it is only a symbol which is used to represent that there is nothing; *numbers represent quantity*

This identification of numbers with quantity leads to the zero-is-nothing analogy which is discussed in the linguistics section. This analogy is reflected in many arguments presented with respect to division by zero. These include the following recorded in the research of Reys [1974], Reys and Grouws [1975], Grouws and Reys [1975] and Wheeler and Feghali [1983].

Eighth Grader	Because when you divide zero by zero, you can't get any number except zero. When you divide nothing by nothing,

	you can't get something
	Did someone tell you that?
	No, I just knew it because you can't get something for nothing.
Eighth Grader	So, zero divided by zero is zero?
	Yes.
	How could you show me?
	Because you don't have a number at all to multiply or divide so it will have to be zero, because no other number will work
University Student	How can one divide nothing into nothing and receive anything but nothing?
University Student	One; since the numeral zero has no value, zero divided by zero would equal one zero.

The material already presented indicates that historically and linguistically division involving zero has exceptional qualities Yet, there is more The mathematical structure of division involving zero is accessible only through a deep concept of zero using the inverse operation All other approaches leave loose ends, are more complex or are essentially the inverse operation This point is delightfully demonstrated in Herbert Schwartz's [1971] article, "The Experts and the Simpleton " To be effective, the inverse relationship must be firmly established This relationship is a mathematical structure of much greater depth than the partitioning or measuring processes with manipulative materials commonly used in other division situations. Thus it is no simple task to use the inverse relationship Premature stress on this approach may lead to memorization attempts, intuition, or outright avoidance Below is part of an interview with a grade eight student who does not have this inverse relationship well developed. [Reys and Grouws, 1975, p 599]

What is 8 divided by 0?
"Zero" (writes $8 \div 0 = 0$).
Can you write the multiplication sentence?
(writes $8 \times 0 = 0$)
Let's see. Here (interviewer points to first sentence, $12 \div 3 = \square$), the twelve went on this side of the 'equals' sign; here (pointing to the second sentence, $0 \div 4 = \square$), the zero went on this side of the 'equals' sign.
"Oh, do you want me to turn them around?"
I want you to write it, so that this multiplication sentence goes with the division sentence.
"Oh, you do." (Writes $12 = 3 \times \boxed{4}$
$0 = 4 \times \boxed{0}$
$8 = 0 \times \boxed{0}$)
"But that wouldn't be right (pointing to $8 = 0 \times \boxed{0}$) because zero times zero is zero "
So what are you going to put in the box?
"Zero. I mean no number works "
No number works?
"Wait a minute; you have me confused. Zero times zero is zero so the answer is zero."
But you are looking for something to put in the box so that when it is multiplied by zero it is eight. What works?
"Nothing "
Why?
"Zero works here (pointing to division sentence) but it doesn't work there (pointing to multiplication sentence)."
Are you sure?
"No I really like to write it as $8 \times 0 = 0$ "

This interview suggests that the development of the mathematical structure of the inverse relationship is in process and the eighth grader will use it as long as the outcome is consistent with his beliefs. But he does not yet realize that the inverse relationship is a structurally stronger and a more powerful mathematical process than whatever influences him otherwise.

Division involving zero requires the mathematical analysis of three cases as follows:

CASE 1: $0 \div a, a \neq 0$

$0/a = \square \rightarrow 0 = a \times \square$

There is one number, zero, which makes this statement true so,

$0/a = 0$

Mathematically we say that $0 \div a$ is uniquely defined and is meaningful.

CASE 2: $a \div 0, a \neq 0$

$a/0 = \square \rightarrow a = 0 \times \square$

There is no number which makes this statement true. Thus, $a \div 0$ where $a \neq 0$ is not uniquely defined and, therefore, meaningless.

CASE 3: $a \div 0, a = 0$

$0/0 = \square \rightarrow 0 = 0 \times \square$

All numbers make this statement true but there is no way of determining which number to choose as the answer Thus, $0/0$ is indeterminate.

There are no instances prior to this in the mathematics curriculum where consideration of the question of uniqueness is necessary. As demonstrated above, the mathematical analysis of division involving zero is otherwise incomplete. Mathematically, the behaviour of zero is division is exceptional because:

1. A different, conceptually deeper approach is necessary to analyze this behaviour, compared to other numbers.

2. Three cases must be analyzed
3. Each case results in different outcomes, two of which are startling contrasts with previous computational exercises.
4. For the first time it is necessary to use the notion of uniqueness
5. The new language needed because of (1) to (4)

Compared to previous computational situations, division involving zero represents a quantum leap in complexity.

The analysis of zero behaviour in other mathematical contexts demonstrates similar types of complexity as does division involving zero Some of these are noted in the following.

To analyze zero as an exponent in a^0, it is necessary to develop a different approach from a^n, $n \neq 0$ For example, a^2 and 2^3 are expressed as

$$a^2 = a \times a \text{ and } 2^3 = 2 \times 2 \times 2$$

How, then, do we express a^0 or 2^0? Like division, we must resort to a different, conceptually deeper approach to arrive at $a^0 = 1$ and $2^0 = 1$. "Many children find it completely mystifying that $10^0 = 1$ and $3^0 = 1$ as well." [Salisbury, p. 24]

Zero concepts are powerful mathematically in solving equations. When $a \cdot b = 0$, where a and b are real numbers, then one or both of a and b must be zero. No other real number has this property It was not until 1631 that the Englishman, Thomas Harriot, came up with the ingenious idea of using this property of zero for finding the roots (zeros) of a polynomial. [Dantzig, 1959, p. 186] We teach students this approach to solve equations like the following quadratic:

$$
\begin{aligned}
x^2 - 5x &= -6 \\
x^2 - 5x + 6 &= 0 \\
(x - 3)(x - 2) &= 0
\end{aligned}
$$

$$
\begin{array}{ll}
x - 3 = 0 \qquad & x - 2 = 0 \\
x = 3 & x = 2
\end{array}
$$

They become capable of using this approach to solving this and similar problems yet they are commonly unaware of where this procedure comes from or why it exists. Student who attempt to solve $x^2 - 5x = -6$ and similar problems by factoring $x^2 - 5x$ and setting each factor equal to -6 demonstrate "a lack of understanding of an important mathematical concept, the special restrictions on two factors when their product is zero" [Meyerson, McGintz, 1978, p. 49]

As a factorial, we define $0! = 1$, which is intuitively distressing since $1! = 1$.

Several of the ways in which the properties of zero are used in mathematics have been presented in this section. The examples used were selected because of their variety in exhibiting the nature of the behaviour of zero as compared to other numbers. These require the use of the language related to diverse and deep concepts and processes within the mathematical structure as well as the invention of new language to accommodate evolving structures. The historical evolution of zero properties and corresponding language, as demonstrated with division involving zero, is particularly extensive where the recorded evolutionary process has exhibited considerable unrest over centuries. There can be little doubt that due to the breadth and depth of this behaviour zero holds a prestigious position in mathematics and is justly referred to as "the most practical invention in the history of mathematics." [Reid, 1964, p. 137]

The origins of zero

The story of 0 presented thus far reveals a complex structure that confronts us today This structure had evolved as a result of many mathematically powerful uses of the concepts and properties related to zero Because much of this conceptual framework has a long, difficult evolution significant language developments about zero occurred as a result of its use in early numeration systems and its progression to common usage

Early numeration systems

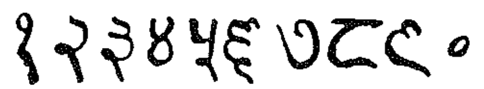
> Yet zero, first of the digits, was the last to be invented; and zero, the first of the numbers was the last to be discovered [Reid, p. 1]

The origin of zero will not likely ever be known, yet its early use as well as the principle of position has been traced to the numeration systems of both the Babylonians (3500 BC) and the Mayan Indians (3300 BC). These numeration systems demonstrate what is the first historical awareness of the use of a zero symbol as a place holder Early Babylonian tablets indicate the appreciation of the need for a zero by leaving a blank space between symbols Later tablets displayed the use of a zero symbol (≷) to signify an ususual power—an empty place. Convention permitted the use of this symbol as a place holder within the numeral but not at the end.

The Mayans used the symbol  to represent zero in their positional system. Their numeration system seems to have been used primarily in relation to their calendar rather than for computational purposes. Its development ceased with the destruction of this civilization of Central and South America

The Babylonian and Mayan numeration systems represent our first awareness of the use of a zero symbol as a place holder. But neither of these civilizations appear to be the forerunners of the dominant numeration system currently used. Its historical roots, including both the principle of positon and zero, have been traced to the Hindus Near the beginning of the Christian era, some unknown Hindu began the practice of recording a symbol, an enlarged dot, in order to keep a permanent record of the empty column. "Thus, after all the other came zero, the first of the digits." [Reid, p. 5]

Preserved Hindu writings, including the Bakhshali manuscript probably dating from 300-400 AD, indicate the use of the word "sunya" (empty, void) combined with the word "bindu" (dot) to form "sunya-bindu" (the dot making a blank) With time, the symbol was transformed from a dot to a small circle. The Hindu Sanskrit numerals, including zero, were as follows.



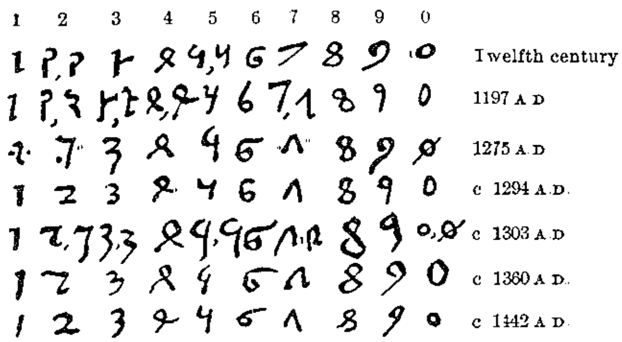
Sanskrit

Arabic

[Newman, vol. 1, p. 454]

The Arabs transmitted the Hindu concepts of zero and the principle of position to the world. They translated the Hindu "sunya-bindu" to their "as-sifr" or "sifr" which means empty or vacant in Arabic. The arabic numerals are shown above. They adopted a dot as the symbol for zero because the Hindu zero symbol closely resembled the Arabic symbol for five.

The name and symbol for zero continued to be transformed following its introduction to Europe by the Arabs. The development of the numerals used in Europe from their first introduction to the beginning of the printing process are illustrated below. [Newman, vol. 1, p. 454]



The development of the printing process resulted in greater standardization and stabilization of the numeral's use.

When the system of numeration was introduced in Italy, the Arabic "sifr" was translated into Latin as "zephirum" or "sephyrum." In addition, over the next century, it was "zeuero," "ceuero," "zepiro," culminating finally as "zero." During this same period, "sifr" became "cifra" with the systems introduced into Germany which was eventually transformed to "cipher" and used as such in England and Europe. Eventually zero was used since the word cipher came to be synonymous with art of reckoning.

The move towards common usage

The initial function of all numeration systems is their provision of a means of recording the cardinal numbers. If our needs of a numeration system had remained this limited, any of the ancient systems would have been adequate. But history indicates that the world felt the need for a better numeration system with far greater potential than record keeping. Many ancient systems do not permit performing the simplest arithmetical computation. The invention or importation of some form of abacus for computational purposes was common among societies with non-positional systems. Yet, "it would seem that the first time anyone wanted to record a number obtained on the counting board, he would automatically have put down a symbol of some sort, a dash, a dot, or a circle, for the empty column—which we today represent by zero. But in thousands of years, nobody did.

Not Pythagoras.
Not Euclid.
Not Archimedes.

For the great mystery of zero is that it escaped the Greeks. They were the first people to be interested in numbers solely because numbers are interesting. [Reid, p. 3] The Greeks achieved great accomplishments in mathematics. To understand why zero as a number escaped them requires an understanding of classical beliefs. The Greeks were obsessed with the actual, that which was real, that which existed as opposed to the abstract. They treated applications of numbers with contempt since their interest was in learning the secrets of numbers, not in using them. Thus computation was of no interest to them. The beliefs of this society undoubtedly had considerable influence on its great mathematicians.

The Pythagoreans were distressed when they discovered irrational lengths such as $\sqrt{2}$ and suppressed the notion since it disrupted the system of whole numbers and saw them as an "impiety against the Divine itself." [Newman, J., p. 2325] Proclus recorded, "It is well known that the man who first made public the theory of irrationals perished in a shipwreck, in order that the irrepressible and unimaginable should ever remain veiled." The idea of a continuum as opposed to discrete numbers was foreign to the classical world belief. Thus the existence of ideas such as irrationals and fractions was impossible for the classical mathematician to accept as numbers. Zero and negative numbers were not possible since a line could not be zero or less in length since it is then no longer a line—such a notion is meaningless. They questioned whether unity (one) was a number; for how can unity, the measure, be a number. [Newman, p. 98]

In Europe the Roman numeration system was commonly used until the eighteenth century. In this system, rules developed for computation were complex and the art of reckoning was held in awe. Supernatural powers were attributed to those who possessed this art. Trained experts were needed to use the crude yet complex calculating devices man invented to perform even simple computations. Thus improvement in the art of reckoning was stifled for centuries by a crude numeration system and a primitive computing device. It is suggested that "even when compared with the slow growth of ideas during the Dark Ages, the history of reckoning presents a peculiar picture of desolate stagnation." [Dantzig, p. 29] In this regard the Hindu who initiated the process of reckoning using both ideas of position and zero orchestrated an event of world renown. Other systems were not capable of creating an arithmetic which could not only be mastered by persons of average intelligence but by young children. Today such skills are taken for granted as is the tremendous power of the numeration system that provides straightforward computational procedures.

In Europe the transition to the positional system required several centuries. Its superiority is obvious today, but adherence to the Roman system persisted and a protracted struggle between the "abacists," the defenders of past traditions, and the "algorists," the advocates of reform continued from the eleventh to the fifteenth century. In 1259, Florentine bankers were forbidden to use the "infidel symbols." "In 1300, the use of the new numerals was forbidden in commercial papers because they could be more easily forged than the Roman numerals " [Reid, p. 5] It has been suggested that zero was the major difficulty since it could easily be forged to become a six or a nine by the addition of a single stroke Religious prejudice against these numerals existed but was not universal in the church. Despite such extreme reactions, the system spread and by the commencement of the sixteenth century the Algorists prevailed and by 1800 it was fully accepted across Europe. Shortly thereafter, no trace of the abacus was evident in Europe It was reintroduced to Europe in the nineteenth century by the French mathematician, Poncelet, as a great curiosity of 'barbaric' origin Poncelet became familiar with the abacus as a prisoner of war in Russia.

With the rapid spreading of the Hindu concepts in Europe, the art of reckoning became easily mastered by the average person. The system soon was recognized and identified by the masses with its most unique feature, the "cifra," that is, its zero, and the art of reckoning was called "ciphering," identifying it with the name of this one symbol The resulting confusion in the meaning of the word cipher was eventually resolved by the adoption of the word zero.

Conclusion

Every aspect of zero presented in this paper supports the notion that zero is exceptional. It not only took man centuries to invent the digit zero, it took added centuries to discover the number zero and still more centuries to accept and use it. Yet, after all this, evidence indicates that we still do not recognize its significance and importance. The uniqueness of zero is more general and deeper than that of any other number and yet pedagogically zero is treated superficially as a trivial and obvious notion.

> The point about zero is that we do not need to use it in the operations of daily life No one goes out to buy zero fish It is in a way the most civilized of all the cardinals, and its use is only forced on us by the needs of cultivated modes of thought. [Whitehead, 1911, p 43]

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