Didactics, Mathematics, and Communication*

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This is an extended version of some lectures given at universities in the USA. It contains basic deas of the author's view on mathematics and (mathematical) education with a special emphasis on the relationship of mathematics to communication. Most of this paper is a free translation of parts of the book "Mensch und Mathematik: Eine Einführung in didaktisches Denken und Handeln" by R. Fischer, G. Malle and H. Bürger, and of the article "Zum Verhältnis von Mathematik und Kommunikation" by R. Fischer.

How to learn from the learners

Let me start with an example of an interview with four 10-11 years old children which we conducted studying the learning of elementary algebra. We posed the following problem:

> In a stable there are R rabbits and G geese. What is the meaning of the equation
> $$R = G + 4?$$

In the discussion of the interviewer with the pupils during the solution process the following dialogues occurred:

P: "There must be as many rabbits as geese."
I: "Why?"
P: "It says "R = G""
I: "But there is also a "+ 4""
P: "..."
I: "How many rabbits could there be, for example?"
P: "There could be four rabbits, for example."
I: "Draw a picture of the rabbits and the geese which could be in the stable!"

The pupils draw four rabbits and four geese.

I: "Is the equation correct now?"
P: "Yes, for there are as many rabbits as geese."
I: "Why must there be as many rabbits as geese?"
P: "Because it says "R = G"

. . . .

I: "What does the "+ 4" mean?"
P: "It must mean animals of another kind."
I: "Which, for example?"
P: "Hens or goats."

The children were *not able* to solve the problem. They hadn't yet been taught elementary algebra. But what were their difficulties?

- The pupils didn't see the right-hand part of the equation, namely "G + 4", as belonging together, as a unity. They saw "R = G" and then "+ 4, but "R = G" dominated. One could also say: They didn't obey the mathematical convention that "+" binds more strongly than "="
- It wasn't clear to them that "+4" meant a relation between numbers, especially numbers of animals, that there was *no need to interpret "+ 4" in the real world* (for instance as the number of another kind of animal).
- The pupils *couldn't see any sense in this kind of problem* (This could be inferred also from a discussion after the interview.) They didn't know what to do. They were not able to see that a *relationship was expressed by a certain notation* and that their task was to express this relation in ordinary language. There was nothing to calculate, a type of problem they had never met before in mathematics.

Many scientists in mathematics education — didacticians of mathematics as we call them — would now ask themselves how *to overcome* or, better, how *to avoid* these difficulties. (In fact, in the discussion following the interview, suggestions for teaching methods were proposed in connection with all three of the above explanations.) Some people even think that there must be an *"optimal curriculum"* by which the knowledge would slide frictionessly into the heads of the pupils; at least there should be optimal strategies or reactions of teachers. There is also the opinion that a structural correspondence exists between the mathematical content — elementary algebra in our case — and the cognitive abilities of the individuals, since mathematics has been created — or was discovered — by men. These approaches disregard at the least the following two facts:

— A *vast variety of variables* exist which influence the congnitive structure of the individual: his/her learning history, his/her relation to the teacher, the different meanings one can associate with mathematical concepts, etc. For the present, at least, there is no chance of controlling these variables.
— Mathematics — knowledge in general — is a *social construct,* which has its roots not only in individual thinking, but also in the interaction of men. A distance between the structure of "official" and the structures of individual knowledge does and should exist: "should" because otherwise the individual would cease to exist and innovatory knowledge would become impossible.

My intention in undertaking research like that described in the above example differs from that usually pursued in educational research. I don't want to find a *method* of teaching better but rather I am interested in finding a relevant *content* for teaching. The relevant content I can

* I am indebted to W. Nemser for going over a preliminary draft eliminating language mistakes and improving the style.

For the Learning of Mathematics **8**, 2 (June 1988)
FLM Publishing Association, Montreal, Quebec, Canada

find by interviews of the kind illustrated are insights into the *relationship between men or women and knowledge.* For me the study of this relationship is the main task of didactics I'll come back to the importance of this relationship a little bit later.

We can learn about this relation from the learners if we concede to *both parts* of the relationship — learners and knowledge — *equal rights of existence.* This means that we must not interpret the statements of the pupils primarily as mistakes, as failures, as preliminary disabilities, but as the views of human beings whose thinking has not yet been formed by mathematical paradigms.

What do the statements tell us? They reveal, for instance some special aspects of *mathematical notation,* which we sometimes forget because they seem so self-evident and natural. Further — from the above example — we can learn about the *arbitrariness of the separation* between what has to be interpreted in real terms (the variables R, G) and what is part of the formalism (the number 4)

> Another example illustrates this last aspect An interpretation of the equation $x = 3y$, where x means the salary of Mr. Adam and y that of Mr. Bedam, is requested. The pupils insist heavily that both get the same salary The explanation, which we didn't initially understand: If y, the salary of Bedam, is multiplied by 3, then he gets as much as Adam. The multiplication, which is needed — on the formal level — to express a relation, has been interpreted by the pupils as a large raise in the salary of Bedam!

The question about which one can learn very much from the so-called difficulties of pupils is that of *sense.* It isn't always easy to get through the superficial and outer difficulties towards the kernel, which is often the question of sense. The pupils are not accustomed to talking about sense, *to evaluating the problems* they have been exposed to. In the case of the above interview they laughed helplessly. Only in the discussion afterwards in a more informal setting with coke and frankfurters could they express their feelings: they were very astonished that this should be mathematics! So far they had been confronted with mathematics as a set of calculation methods and what they had always had to do was to calculate! That mathematics is also a tool for *representing relations, thereby a means of communication,* was something new to them

> Let me offer another example illustrating the question of sense. A (real) *classroom situation* The teacher explains the *"modulus of a real number"* (absolute value) by means of examples: $|5| = 5$, $|-3| = 3$, $|2| = 2$, $|0| = 0$, ... and geometrically: the modulus of a real number is its distance from the origin (on the number line). Or: taking the modulus means cancelling the minus sign. The pupils soon have no difficulty doing problems like $|10| = ?$, $|-7| = ?$, $|5-7+1| = ?$ etc. But some of them ask: What is the sense of the modulus? What is the purpose of writing "$|-5|$" instead of simply "5"? The teacher had difficulties understanding the difficulties of the pupils Now, in what does the pupils' difficulty consist?
>
> The pupils' conception of the modulus-sign is that of a *direction for calculation,* such as $+$, $\div$, $\sqrt{}$ or ^{2} (squaring) — it has been so presented to them and all the mathematics they had learned previously has been of this kind But the modulus calculation is trivial and so they can find no sense in this concept. Now, what *is* the sense of the modulus concept? Actually there is little sense in it as long as only concrete numbers and no variables occur The modulus sign is of use if general facts are to be expressed, such as the triangle-inequality or $\sqrt{a^2} = |a|$. This is also an example of how making things "easier" through concrete numbers makes them senseless

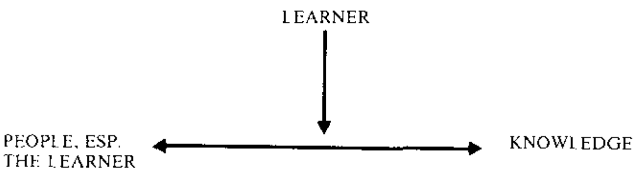
What I want to express through the examples and, above all, by means of interpretation is, first, the *concentration on the relationship between people and mathematics,* both being of equal importance That means the student is *not the observed patient* who shall be cured, he or she is a person with special views, sometimes a special logic, etc., which are to be taken as serious matters A difference between the official and the student's knowledge can be interpreted as a deficit of the student *as well as* a deficit of mathematics, or better — non-evaluating — simply as a difference. This may be seen as only a slight shift as compared with the usual view of researchers in cognitive psychology But the next step, which is a consequence of this view, is a bit more radical: I suggest that this *relationship should become the content* of mathematical education in the classroom — in addition to the ordinary teaching and learning of the subject matter. (By the way, I find this the only way to deal honestly with the theory-practice dilemma in educational research)

I believe that the *school of the future* will have to concentrate more on the relationship between people and knowledge than on knowledge itself. I offer two reasons for this conjecture:

- Knowledge, especially mathematical knowledge, is increasingly available in applicable materialized form, in books, journals and machines. The competence of people in the *flexible handling* of this knowledge is important. For this *a meta-knowledge* of the relationship between people and knowledge is necessary.
- There is an *abundance of knowledge* available today. General advice for everybody is problematic, we must stress the *responsibility of the individual* for the organisation of his learning and for choosing the content. But there should be *general guidelines,* resulting from reflection on the relationship man-knowledge to help the individual to find his way — not only in school, but also afterwards.

At this point I want to make the following remark: In order to study the relationship people-knowledge a *distance* between these entities is necessary. Usually education tends to minimize the distance between the learner and the knowledge, even and especially with respect to the affective, emotional part of the matter I believe a basically new orientation in our philosophy of education must be developed which recognizes the need for a distance between the learner and the knowledge to be learned.

From the above it is clear that knowledge about the relationship between people and knowledge cannot have the degree of generality which we usually associate with scientific knowledge, it has *subjective components,* it depends on the situation and, above all, it is reflexive, which means it is, in a more direct way than other knowledge, *knowledge about ourselves* Therefore, it must, in part, be learned from ourselves, from the *student's own, his fellow student's and the teacher's relationship to knowledge knowledge about ourselves.* Therefore, it must, in part, be learned from ourselves, from the *student's own, his fellow student's and the teacher's relationship to knowledge.* This means: The kind of *empirical research* which yields insight into the relationship people-mathematics should, in part, become a *model for the classroon.* I can describe the learning situation by means of the following figure:



The learner must establish a relationship between himself and knowledge — this means the ordinary learning of a subject matter But, from a meta-point of view, the learner must reflect upon people's, actually his, relationship to knowledge

With the phrase "people's, actually his" I wish to indicate that the interesting relationship has an aspect of *globality,* i.e the relationship between mankind and knowledge, as well an aspect of *individuality.* But there are also aspects which lie in between, e.g the *relationship of a learning group to knowledge.* I believe that these "in between" social formations are crucial for learning in school. More generally school is the place where the "global" relationship — expressed in terms of curricula and syllabuses — confronts the "individual" relationships; *the learning group must cope with this confrontation.*

What does this mean for the role of the teacher? Only a few words on the subject. Mathematics teachers usually tend to *identify themselves with the subject matter,* though, perhaps, only on a surface level. They *are* mathematics, there is no difference between them and mathematics. They present mathematics as if they were totally convinced of its truth and worth.

The goal of their effort is to identify their students with mathematics (to minimize the distance). What is lost through this attitude on the part of the teacher is an *explication of the teachers' relationship to mathematics* (and of the students' too, because the teacher is a model). The teacher's relationship to mathematics need not be a very good one. Perhaps he or she, the teacher, has had difficulties learning the subject matter at college or at the university Often he or she feels *insufficient,* not good enough for mathematics and therefore guilty. But the teacher would find it inappropriate to articulate such feelings in the classroom. So he or she hides behind the subject matter — identifying with it, and, maybe, *carrying over feelings of guilt* with respect to mathematics to the pupils

Another reason for the identification with the subject matter is the feeling of full responsibility for it on the part of the teacher I think that one opportunity for *removing this burden* from the teacher is the *existence of knowledge in material form,* above all in textbooks. Usually decisions about the choice of textbooks are made by a larger community, so the teacher could say to the students: "This is the knowledge which society finds worthwhile. My personal opinion differs in some cases — I'll tell you when." So his or her relationship to knowledge could become clear and become a stimulus for developing and discussing other possible relationships in the classroom.

Mathematics as a means of representation

Now let's return to mathematics in a more concrete way I will in the following present a view of mathematics which is in some sense "materialistic", namely *mathematics as a material means or representation* This view, I think, *facilitates the establishment of a fruitful distance* between men (people) and mathematics and thereby the study of the relationship between them. It differes from the view of mathematics as a body of knowledge which must, in all respects, be accepted as an absolute truth. The latter view allows no distance between ourselves and mathematics

Let's start with the simple question: *What is mathematics?* For many people mathematics means *calculating.* What is calculating? I would suggest: Operating according to certain rules. What are the objects used in this operation? *Symbols and sequences of symbols* These are representation of (more or less) abstract issues, relations, etc Mathematical activity — from simple addition, solving word problems up to creating a proof for a theorem — can be viewed as a process consisting of

— representation
— operation
— interpretation.

By "interpretation" I mean not only the interpretation of mathematical models outside mathematics in the "real world" Even within mathematical symbolism there is an interplay between "operation" and "interpretation". Consider the equation

$$(3x + 4)/(7x - 1) = 12$$

One must *see* that it's like

$$A/B = C$$

Now one can perform the opertation

$$A = C \cdot B$$

which means

$$3x + 4 = 12\,(7x - 1)$$

Now on the right hand we have the structure

$$A(B - C)$$

And so on. *Mathematics is a kind of visual art.*

Often only the second aspect in the above list, the operation, is considered, since the first seems to be given by the symbolic apparatus of mathematics and the last becomes efficient only through the second. In my opinion we should put more emphasis on representation and on interpretation, not least because the *computer will increasingly take over the operational work*. Incidentally the development of the following ideas began with the question: What will remain as the important mathematical competences of people if we subtract all that the computer can do better than we can?

Let me give some examples of mathematical forms of representation. All of them had substantial influence on the historical development of mathematics.

- *Different modes of writing numbers* (Egypt, Roman, decimal notation etc.)
- *Graph of a function* (14th century)
- *The symbolism of elementary algebra* (15th — 16th century)
- *Symbolism of differential and integral calculus* (17th century)
- *Matrices* (19th century)
- *Graphs* (for structural relations as important as the function notation for quantitative dependence)
- *Flow charts*
- *Set-theoretic symbolism*. The most universal form of mathematical representation. Almost all mathematical concepts can be represented as sets, mappings, relation or as n-tuples. (For instance: a vector space is a quadruple (V, K, +, ·), such that ...)

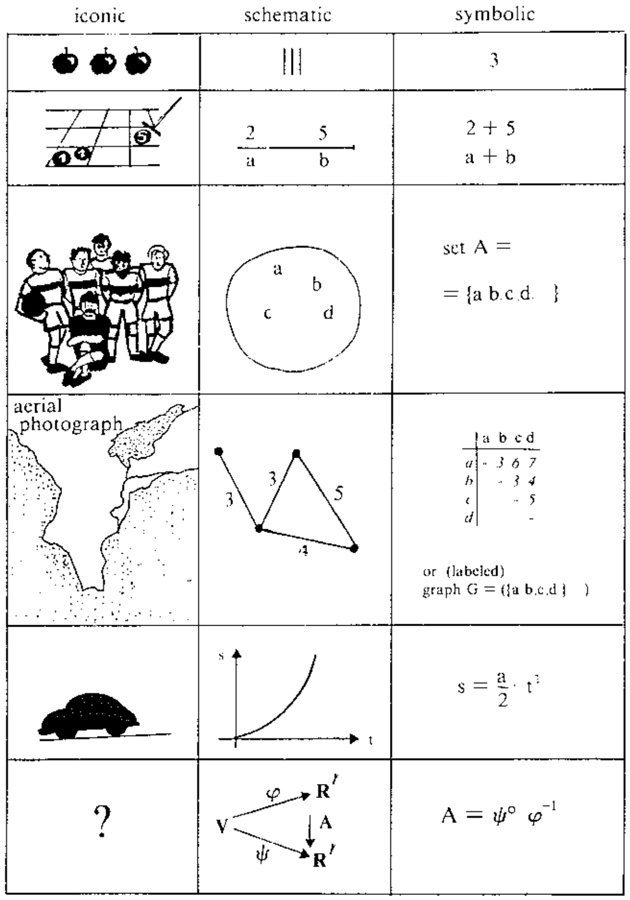
These are all *"graphic" representations*. *Other forms* of mathematical representation of abstract issues are the "calculation of stone" used in prehistoric times and the "abacusses". The most recent form of material representation is of course the computer. Studies in the *history of science* yield arguments for the thesis that the possibilities for material representation substantially influenced the development of mathematics [cf. DAMEROW/LEFEVRE, 1981].

Let's return to the visual aspect of the representations. There is a (psychological) *theory of visualisation* according to which three levels of visualisation exist:

- iconic: like a foto;
- schematic: essential features are pointed out (e.g. a scheme for the circulation of blood);
- symbolic: arbitrary choosing of signs, the sequences are linear (not totally!), higher-order symbols (for relation) exist, operations with "lower-order" symbols are possible, the same symbol can occur several times for a linear argumentation in time, the generalization is more explicit.

The figure from BOECKMANN illustrates this distinction. For mathematical activity a sort of *interplay between the schematic and the symboilic level* is often very important. Or more generally: between different modes of representation. There is no sharp distinction between the "schematic" and "symbolic" levels, in fact all representations have symbolic features, even the so-called "iconic". This relativizes the above distinctions.

Three levels of representation
(according to BOECKMANN 1982)

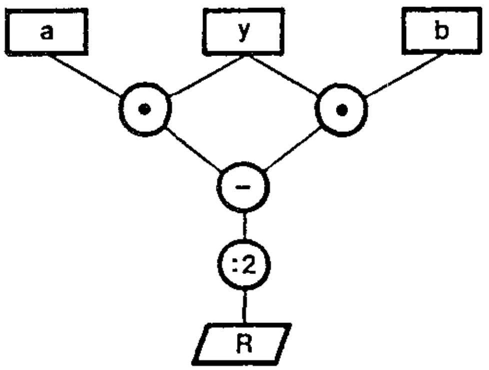


I have introduced this psychological issue to lead our attention to the *question of properties, usefulness*, etc., of mathematical notation (representation). Implicitly mathematicians always deal with this question, they have an intuitive knowledge about it. What I argue for is making this knowledge *explicit, developing it* and making it a *content of mathematical education. To compare and evaluate different modes of representation will become more important than skillfully handling one mode.*

In the following I shall present some *classroom problems* through which the aspect "mathematics is a means of representation" can be emphasized.

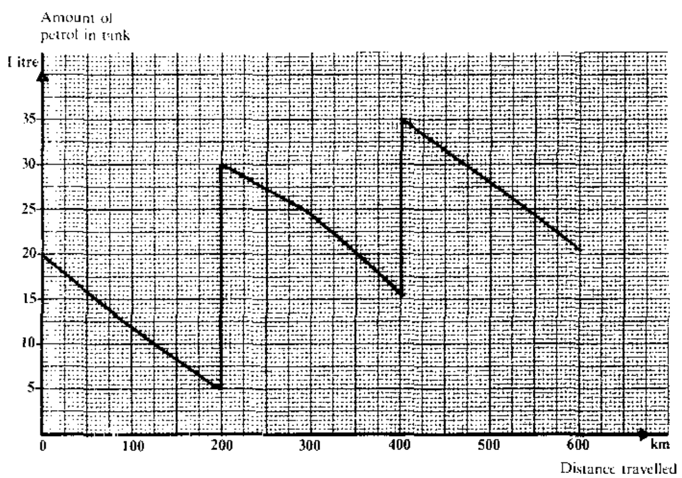
- A number R is calculated according to the scheme on the next page — starting with numbers a, b, y.

Here one can discuss questions concerning how different forms of representation, expecially the one above and the usual algebraic notations, are appropriate for different kinds of problems. Possible answers: calculation diagrams (trees) are appropriate for the *process of calculation*: algebraic notation gives a better *insight into the structure* of the relation; further, operations are possible with algebraic notation.



Are the numbers R and y proportional?

The next example stems from a project at the Shell Centre for Mathematics Education at Nottingham [SWAN, n.d.]. It was conducted subsequent to the dissertation of Claude JANVIER, "The interpretation of complex Cartesian graphs representing situations" [JANVIER, 1978].



The graph shows how the amount of petrol in my car varied during a 600 km journey. Try to answer the following questions:

1 How much petrol does my car hold?
2 How many petrol stations did I stop at during the jouney? Where were they?
3 At which petrol station did I buy the most petrol?
4 How much petrol did I use over the entire journey?
5 Suppose that I had not stopped for petrol, where would my car have broken down?
6 Town driving uses petrol at a faster rate than country driving (this is because you have to keep stopping and starting). Do I live in the country or in a town?
7 Did I pass through any towns on my journey? Where were they?
8 Were the petrol stations situated in the country, in the centre of towns or on the edge of towns?
9 Sketch a graph to show what would have happened if the car had only stopped at one petrol station.

Here the emphasis is on *interpretation*. The following example is concerned with the *notion* "f [...]"

- Let f be a real function. Interpret verbally and by use of the graph of f:

  a) $f(x+1) - f(x)$
  b) $f(2x)/f(x)$
  c) $f(b) - f(a)$
  d) $f(x+h) - f(x)$
  e) $f(c\ x)/f(x)$
  f) $f(x)/(x)$
  g) $f(x+h)/f(x)$
  h) $(f(b) - f(a))/(b-a)$
  i) $(f(x+h) - f(x))/(h \cdot f(x))$
  j) $(f(b) - f(a))/((b-a)\, f(b))$
  k) $[f(x+h)/f(x)]^{1/h}$

Now the notion of *derivative* is used:

- Let $t: i \rightarrow t(i)$ be the income-tax function. What do the following mean:

  a) $i - t(i)$
  b) $t(i)/i$
  c) $t'(i)$
  d) $1 - t'(i)$
  e) $t''(i)$
  f) $\lim_{i \to \infty} t(i)/i$
  g) $\lim_{i \to \infty} t'(i)$

The last two examples are a bit more complex:

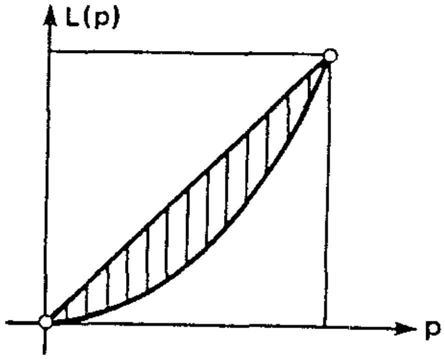
- Let $n(i)$ be the number of adults in a certain country with income i.

  a) What does $\int_0^\infty n(i)\, di$ mean?
  b) Give an expression for the total income of all the people (sum) using an integral!

- For each income i let $f(i)$ be the relative portion (with respect to the number N of all adults) of people with income i (i.e. $f(i) = n(i)/N$)

  a) How big is $\int_0^\infty f(i)\, di$?
  b) What does $\bar{i} = \int_0^\infty i\, f(i)\, di$ mean?
  c) Interpret the function
  $$F: i \rightarrow \int_0^i f(x)\, dx$$
  d) Interpret the inverse F^*.
  e) Interpret the function
  $$L: p \rightarrow 1/\bar{i} \int_0^p F^*(y)\, dy \text{ (LORENZ function)}$$

Solutions to the last question: If people are ordered according to their income — from the poorest to the richest — then for the portion p of all people, beginning with the poorest, the portion of their total income with respect to the total income of all people is $L(p)$



L is a convex function, it is the identity mapping if, and only if, all people in the country have the same income. Therefore the shaded area in the figure can be viewed as a measure of the inequality in the income-distribution (GINI-Index, see LÜTHI [1981])

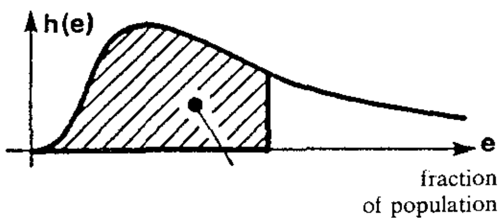
With respect to the last three examples one could argue that the derivative and integral are *not appropriate,* the variables are discrete One should rather take

$$f(x+1) - f(x) \text{ resp. } \sum_{j=1}^{\infty} f(j)$$

This would be simpler and even more precise!

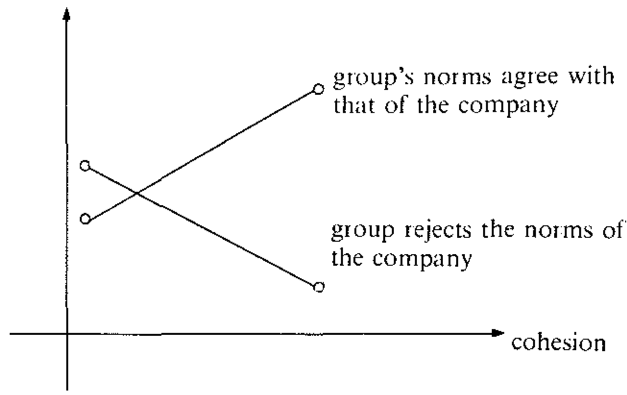
But: integral as an area or derivative as the slope of a tangent are worthwhile for their *representational aspects*

A continuous image of the distribution of people is more impressive than a table or the graph of it. The continuous drawing of the LORENZ-functions induces the idea of the GINI-Index.



> *More generally. The importance of some infinitesimal concepts today, expecially in economics, is based more on their representational than on their calculational value*

In various fields mathematical notations are used *without reference to their full mathematical meaning* I found the following example in a book about group dynamics (BRADFORD 1984). Here relations are expressed which are only partially quantitative, the graphic must be understood properly For instance, it does not mean that the relations between cohesion and productivity is "linear" in a mathematical sense. But it means that it is — in each case — monotonic. Further it means that, if the standards of the company are appreciated and cohesion is high, productivity is higher than in a group which rejects the standards and has low cohesion. Mathematical symbolism here clarifies the statements without allowing operations within the mathematical formalism. I am sceptical about "applications" in these fields which go further.



I think that a combination of mathematical notation and informal interpretation is a desirable mathematical competence, rather than merely the skilful handling and sound analysis of *one mode of representation*

We — in Klagenfurt — have exposed children to problems of the following kind: a situation is *verbally described,* the task of the student is to describe the situation graphically, with mathematical symbols if useful. We have found that children are rather helpless with such problems, where the helplessness increases with age. We think that this is a consequence of their understanding of mathematics.

I think we should stress the role *of mathematics as a means of representation and communication.* This means, by the way, that mathematics thereby enters the neighborhood of the *arts.* New criteria arise for assessing such tasks. But I won't discuss this topic here.

Open mathematics

The above-postulated emphasis is connected with a different understanding of the applications of mathematics in comparison with the traditional one, where mathematics above all is used for algorithmic problem-solving. I call this new handling *"open mathematics"* and will try to describe what I mean.

In classical applications of mathematics there is a strong tendency towards a *complete grasping of reality:* if all data are available, if the pre-suppositions for the operations are fulfilled, then one should get a *unique and complete solution.* A paradigmatic case is that of initial value problems in classical analysis: the whole path of a particle is determined by the initial condition and the differential equation. I call mathematical models of this type *"closed models".* "Closed solutions", "closed formulas" belong to this view It is clear that no closed model matches reality totally —

but there is the fiction that it does. A special case of this fiction is called LAPLACE's demon.

Closed models can be opened and therewith achieve a better fit through *feedback procedures*. The computer through its potentiality for rapid calculation has greatly extended the facilities of feedback procedures. The underlying mathematics can sometimes be much simpler than in a closed model. Consider for example *central heating*. Constant temperature can be obtained by a simple mechanism which *only uses the monotonicity* of the function.

> temperature of the water $t_w \rightarrow t_a$ temperature of the air in the room,

if the outside temperature is fixed. If one were interested in a "closed model" one would have to establish complicated partial differential equations and solve them.

A further step must be taken towards what I call "open mathematics": *men or women must play a role in the interaction between a mathematical model and reality,* they must be part of the feedback process.

This type of application is greatly needed in *business, economics, planning,* etc. Only subsystems can be mathematized, their combination makes decisions and evaluations by people necessary. Unique solutions don't exist. Models yield *different representations* of a situation, they offer an aid for structuring. These *partial models* are used mostly in a communication process between those who in the end must make decisions.

One difference between closed and open models consists in the fact that for closed models the boundary between the mathematical model and so-called "reality" is rather clear, sharp and precise. For respectable classical applied mathematics it is important to know the boundaries of the model. For instance in inferential statistics: it is important to know the hypothesis, the process of gathering of data must be controlled, the mathematical result should not be confused with its interpretation. On the other hand in *open models* — I think that some modern work in *descriptive statistics*, namely exploratory data analysis, corresponds to this view (TUKEY 1977) — *the boundaries are not so clear*. Data are not so controlled, hypotheses are not presupposed but rather generated. One cannot in all cases discriminate between a statement which is mathematical and one which is derived from other assumptions or beliefs. This is dangerous if truth is the highest judge; but it is a possibility in many instances where closed models fails, or their assumptions are not fulfilled, or would lead to a very narrow view of the situation.

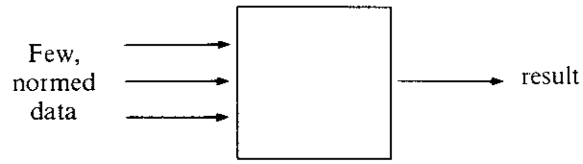
The figure in the next column may illustrate the relationship between "closed" and "open" models.

One desirable attribute of models used in this connection is *transparency*. Let me give an *example*. There is the formula

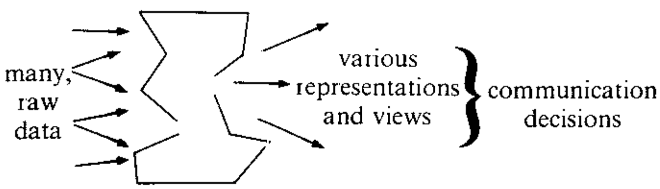
$$S = R_o + R_oq + R_oq^2 + \dots + R_oq^{n-1} = R_o\,(q^n-1)/(q-1)$$

for calculating the sum of an annual rent. It is very elegant and quick, compared with the alternative of calculating the interest step by step and then adding. On the other hand, in practice, the calculation of interest is done in the trivial way! Why? It is more transparent, one has at any time an overview of the situation, one gets intermediate results, etc. Using a computer there is almost no advantage in using the "closed formula".

Closed Model



Open Model



An appreciation of transparency is, I think, in contradiction to the evaluation system of traditional mathematics. It is a criterion of quality for a result to be non-trivial, not transparent, surprising. But for "open mathematics", the use of mathematics primarily as a means of communication, transparency is one important criterion.

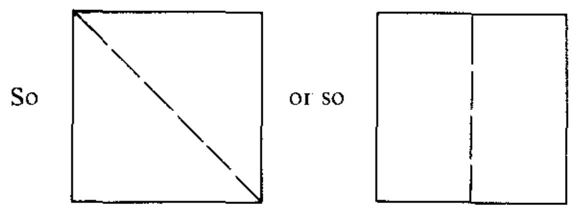
Finally I present a list of *arguments in favour of "open mathematics"* in schools:

- Many theories taught in schools are *too complicated* (overdeveloped) for daily needs. *Parts* of the theories can be used as tools for representation and communication.
- *Closed models often don't fit* because of the high complexity of the "real world".
- Therefore *teamwork is needed*. This requires communication, especially about abstract issues. This can be supported by mathematical representations.
- *More people want to and should influence political decisions*. For such mass-communication, mathematics as a means of representation is needed.
- Closed mathematics can be done by *computers*.
- For the *development of closed mathematics* "Open Math" is needed.
- O.M. always was essential for *mathematical research*.
- O.M. yields a *more realistic picture of mathematics*: not truth for all times, but an offer of theories, concepts and models for representation. It corresponds to *modern views of the nature of science*: perspective, socially constructed truth.

The development of theories and concepts

The previous sections show, in some sense, the importance of the relationship people-mathematics with respect to what is traditionally called "Applied Mathematics". But (pure) mathematics itself cannot be separated from its relationship to people, at least if *mathematics is not reduced to mere facts and skills*. If one claims to gain — or generate — insight into theories and concepts, the latter being the elements of theories, one has to take into consideration the relationship to people. "Theory" comes from the Greek and means "view" and views are always views of people. Theories and concepts always arise *along social processes of communication*. They serve purposes, within or outside mathematics. If they are treated as mere facts or skills, they'll lose all that and become unintelligible.

The following example may show the dependence of concepts on objectives and how quickly we forget this fact. *Is it possible to divide a square into two congruent parts?* Most people would answer: Of course!



or otherwise. But, to which part does the boundary belong? Some people say, after brief thought, the upper part of the boundary belongs to the left, the lower part to the right or vice versa. Now, to which part does the midpoint of the square belong?

One can show through mathematical arguments that it is really *impossible to divide a square into two congruent parts*. There are people who won't believe this, they argue that one can draw a line, one can cut a square of paper into two equal parts. Mathematicians, of course, accept the proof and conceive the case as a paradox. But what are the reasons for the paradox and for the difficulties in understanding?

I suggest the following: Those who will not believe that a square can't be divided have *another concept of squares*, or of geometric figures in general, as well as *of "congruent" and of "dividing"*. The conceptions underlying the proof mentioned are: a square is a certain set of points, "congruent" means that a point-wise isometry exists, "dividing" means finding two disjoint subsets, the union of which is the whole square. These conceptions, based essentially on the point-set model of geometry, are results of a *process of "exactifying"* more intuitive conceptions. Mathematicians now tend to the following point of view: Precise studies have shown that a plane *is* a set of points, congruence must mean the existence of certain mappings, etc. So we have *discovered* that a square really cannot be divided into two congruent parts. We should be glad that after previous ignorance we now know what the reality is.

But another interpretation of the case exists: The fact that such an obvious task, dividing a square into equal parts, cannot be accomplished within the point-set-model is an *argument against this model!* It may have its advantages, but it also has disadvantages, at least it is not appropriate for problems of this type. We know that there are alternatives: We could define "dividing" in another way, for instance allowing the parts to have boundary points in common, or the intersection to be a set of lower dimension. There are also more radical alternatives: points, lines, planes, etc., as elements of a (modular) lattice. Here lines or planes are no longer sets of points. We see the original question "Can a square be divided into two congruent parts" has very different answers.

One of the essential aspects of the development of mathematical concepts is that of *exactifying*, which I already mentioned. I find it interesting that, though exactness is one of the causes for learning difficulties, the roots of exactifying can be found in the *efforts to improve communication, especially teaching*. This was the case in the 19th century, when CAUCHY, WEIERSTRASS or DEDEKIND exactified calculus and Martin OHM did the same with elementary algebra. There is a similar case in the 20th century, when the "New Math" movement emphasized rigour, hoping that rigorously defined concepts were more understandable.

Let us have a closer look at the "process of exactifying" a concept. As a rule, the original, intuitive version of a concept is richer in various aspects than the more formal, more general abstract version. Sometimes the aspects which have been lost during the process of exactifying are *retrieved by new concepts*, if necessary. I will illustrate this with the *concept of function*. One of EULER's definitions of function was that of an "analytical expression", another that of a "curve freely drawn by hand". An aspect narrowly connected with these views is the *smoothness* of a function, which later on was dropped (DIRICHLET), but immediately retrieved by concepts such as "continuity", "differentiability", etc. Another lost aspect is the *"length"* of a curve of a function, retrieved by the concept of "rectifiability". The aspect of the *"law of dependence"* was dropped — at least if one defines function as a special subset of the cartesian product. G. FREGE with his "logical concept of function" tried to retrieve this aspect. Another aspect of function is associated with its roots as geometrical mappings, especially translations and rotations. If one thinks of a real "performance" of such a mapping f, then there are *"intermediate positions"* between the argument-point x and the value point $f(x)$, a curve connecting both. The exact definition of a function "forgets" these intermediate positions and sees only the pair $(x, f(x))$. Now, the topological concept of *homotopy* retrieves the aspect of "intermediate positions": In complex function theory, for instance, one must explain the fact that a curve C_o can be continuously brought to the curve C_1 within a certain set $G \subseteq \mathbf{C}$, the curves given by functions

$$f_0, f_1\colon [0,1] \to G$$

$f_o(0) = f_1(0) = A$, $f_o(1) = f_1(1) = B$. The solution: it means the existence of a family $f_t (0 \leq t \leq 1)$ of functions with $f_t(0) = A$, $f_t(1) = B$, and the mapping $(t, s) \to f_t(s)$ being continuous.

Similar reasoning can be applied to concepts such as probability or vector.

What is the conclusion I draw from these examples? Above all, the following: *Exactness is not an absolute value*, it is a variable, which can be handled according to the problem situation and according to the participating people. Thereby we gain a *new freedom,* perhaps comparable with that mathematics achieved through leaving behind its connections with natural sciences and establishing itself as a formal science at the beginning of this century.

Stimulated by the history of calculus, I have designed a course in calculus for 17 year old pupils. This course takes into consideration levels of rigour by following a series of questions which mirror the process of exactification.

> *Question 1:* How can one study functions? (NEWTON, LEIBNIZ, EULER)
> *Question 2:* What is a limit? (CAUCHY)
> *Question 3.* What is function? Or: Which functions are allowed? (CAUCHY, BOLZANO, RIEMANN, WEIERSTRASS)
> *Question 4:* What is a (real) number? (MERAY, WEIERSTRASS, CANTOR-HEINE, DEDEKIND)
> *Question 5* What does existence mean? (DU BOIS-REYMOND, KRONECKER, BROUWER)

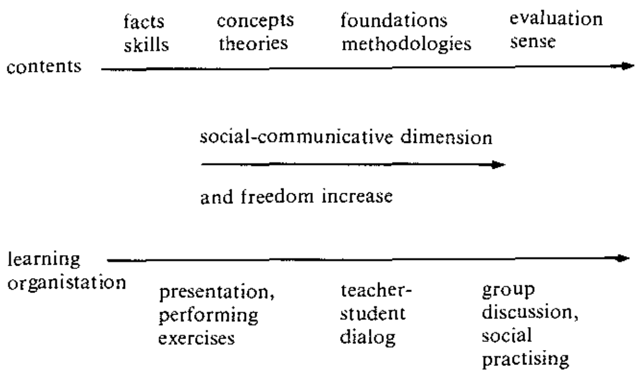
It is remarkable that — step by step — more and more fundamental concepts are called into question, concepts which seem trivial and unproblematical from a "naiver" point of view, meaning a "lower" level of rigour.

Finally, in this section, I want to present an idea about the connection between levels of mathematical content and the *social organisation of the classroom.* A rough division of mathematical content yields the following "hierarchy":

- facts and skills;
- concepts and theories, proofs;
- foundations and methodology;
- evaluation and the question of sense.

Each level from the second onwards has the previous one as its object of consideration (What is a fact, a theory, etc., depends, of course on the state of knowledge of the interested person.) What is important for me here is: *As the "height" of the level increases, the social-communicative component and the freedom with respect to different possiblities also increase and the relationship people - mathematics becomes more and more important.*

What does this mean for the social organisation of learning? The more the social dimension of the content increases in importance, the more the learning organisation should be social. The following picture shows this relation:



By the way: I think that one reason why some efforts in "New Math" failed in schools was that the contents were higher — at least on the second level — than the learning organization.

Mathematics and communication

My personal overall goal for mathematical education is to *improve* communication. Thereby I understand communication in a broad sense: it means speaking about subject matters as well as defining the relation between people. (According to Paul WATZLAWICK the latter occurs even if we think we communicate only about "matter".)

In the following I present some *general reflections* on the relationship between mathematics and communication, not-well-ordered speculations, some of them having the character of paradoxes.

> *Mathematics regulates the social [corporate] life of people in various respects. It is simultaneously a means and system of communications. Thus it establishes a connection between the individual and society.*

The rise of mathematics, especially of arithmetic, is often associated with the development of economics: exchange of commodities, the introduction of money and the administration of communities (taxes) fostered, with increasing complexity, the use of numbers and operations with them — up to modern mathematical economics. By use of mathematical concepts we *determine the relation between* people and hence communicate.

Of course mathematics is *not intimate communication* between two, three or a few people, it is the general rule system, it is a *system of mass-communication.* But mathematics is also a *means of communication* that makes it possible to communicate about complex and abstract affairs, to behave properly within the rules, to use the freedom they admit, sometimes to communicate about the rules. It offers the possibility of governing complex situations.

As a system it requires the individuals to submit themselves to the rules holding for all people; *as a means* it eventually gives the individual the possibility of grasping mankind's (or God's) ultimate rationale. This double nature of mathematics — means and system — establishes a connection between the individual and society in both direction.

> *Mathematics is materialized communication about non-material matters.*

The element of mathematics are relations, structures, not the things themselves. In other words: The objects of mathematics are always (at least) *one grade more abstract* than the things which they concern, to which they are applied. They are "theoretical concepts".

On the other hand mathematicians *not only think* about abstract matters, about numbers, equation, points and lines — like philosophers — but they put their thoughts into pictures, sketches and — above all — into symbols and sequences of symbols, with which they operate. This *materialization of the abstract* is essential for mathematics, it determines its potentialities and its limits

Mathematics is precise, exact, unique. Thus it facilitates and hampers communication. The character of compulsion renders freedom possible

The exact formulation of definitions, theorems, etc., facilitates communication, since the meaning of mathematical statements is clearer than anything else. In its historical development exactness was a product of teaching efforts On the other hand the exactness hampers communications: everything is so rigid, fixed, well defined that there is no place for the innovative ideas of individuals There is no room for contradiction to generate communication *Exactness improves communication so much that it becomes superfluous*

This exactness gives mathematics the character of compulsion: One must believe what mathematics says. On the other hand this character of compulsion makes individual freedom possible: personal force is not necessary, the discipline of mathematics orders things It is a collective disciplining (of thought) which allows more freedom for the individal

Mathematics is also ambiguous and arbitrary

In contrast to its exactness and uniqueness, mathematics also bears the character of ambiguity and arbitrariness And that in the following respects:

- In its *relation to reality*. What mathematical statements mean in reality often is not so clear and needs interpretation (f i in statistics)
- In its *representation*. How mathematical matters are represented — which symbols are used, how formulas are written, how a theory is represented — there are various possibilities.
- With respect to the overall *orientation of its development* There is much freedom in the question of what belongs to mathematics, what is "good" or "bad" mathematics Evaluations are not usual in official scientific discussion All is "equally valid"

I think there is a connection between "local"— what to call an unknown — and "global" arbitrariness And: in mathematics the coexistence of rigid compulsion and boundless freedom is remarkable. Because it is so rigid on the one hand, it gives so much freedom on the other.

Mathematics requires meta-communication and hampers the latter's development.

How should the above-mentioned ambiguity and arbitrariness be handled?

In some sense they are only external, superficial phenomena. For instance, for solving a problem it is not a matter of indifference which kind of representation one chooses.

A simple example: To solve the equation

$$x(x-1) = 31$$

it makes no sense to go to the equivalent equation

$$x - 1 = 31/x \qquad x \neq 0)$$

it is better to go to

$$x^2 - x - 31 = 0$$

in order to apply the formula.

One needs a *meta-knowledge* which exceeds the knowledge which has respectively been made explicit. In the case of the example: the knowledge consists of the rules for transforming an equation into an equivalent, the meta-knowledge consists of heuristic strategies Of course the meta-knowledge can be made explicit too, then the borders change and a new meta-knowledge becomes necessary *To communicate the meta-knowledge implies meta-communication* Teaching mathematics means, to a large extent, to meta-communicate.

But, this meta-communication is not very successful in mathematics. This can be seen from the low prestige of "global" meta-fields of mathematics: didactics, history, the philosophy of mathematics, which are partially connected with the areas of ambiguity and arbitrariness I mentioned in the previous statement. In his article "The nature of current mathematical research", Morris KLINE write: "Expository articles, critiques of trends in research, historical articles or books, good texts at any level, and pedagogical studies do not count." (KLINE 1977, p. 41)

What is the difficulty in developing meta-communication with respect to mathematics? I see the following: Mathematics is itself, to a high degree, communication, and it tends to *mathematize its meta-communication* – which is possible to a great extent But meta-communication is more than just a new level of communication about a lower level: it is also a *determination of relations between people, with individuality of personal interests, with individuality of the particular situation* And this cannot be achieved through the general forms of mathematical communication, this requires special forms and languages depending on persons and groups In *other words: Meta-communication must, in content and form, bear a subjective and situation-dependent character* And thus it is in contradiction to mathematics

A question for the future: *Is a kind of mathematics possible which supports complex processes of communication, the participants being equally objects and subjects of this communication?*

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Continued from page 19

marks come together This is a process of narrowing down the gap. In a three dimensional maze one is given every so often the opportunity to look at the maze from the top of a stairway This is what a structured proof does. As soon as one goes downstairs back into the maze, finding the way through remains a challenge In proving theorems, generic examples can help meet this challenge by bridging the gap. Mathematics teachers should present theorems as a challenge and a proof as untangling the way through the "maze". The Stimulating Responsive method which was presented in this paper is a rewarding way to do it. Can it be applied in the daily teaching of mathematics? Can it be applied to *all* mathematical theorems and their proofs? These questions are addressed elsewhere [Movshovitz-Hadar, to be published].

If teachers decide to take the initiative and design the teaching of mathematical theorems through the Stimulating Responsive method, will it significantly affect achievements and attitudes of students? What role can the electronic media, computers, and television play? Teaching-material development and empirical evidence are needed before these questions are substantiated However, trying to teach ideas along the Stimulating Responsive line should not await this moment

Acknowledgement

I wish to thank my colleagues Orit Zaslavsky, Menachem Finefold and Uri Leron for their enlightening comments

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With apologies to Kant, while semantics without syntax is blind, syntax without semantics is empty. Mathematics is not the manipulation of symbols according to prescribed rules: mathematical activity can be both purposeful and meaningful to human beings.

David Pimm