Formats of Interaction and Model Readers

ANNA SIERPINSKA

Among the interactionist "sensitizing concepts" in mathematics education evoked in the works of Bauersfeld and Voigt [see Cobb and Bauersfeld, 1995], one in particular has attracted my attention in the context of our research, at Concordia University in Montreal, on the teaching and learning of linear algebra by tutoring and reading a text. I am referring to the notion of *format of interaction*, proposed by Bruner [1985] in his theory of language acquisition and adapted to the study of mathematical argumentation in primary education by Krummheuer [1995]. For us, it proved quite useful in understanding the complicated mechanisms through which mathematical meanings were being established and stabilized in the interactions between the different tutors, their students, and linear algebra texts, as well as in describing (or building models of) these meanings.

Formats and formatting

Bruner introduced the concept of format of interaction as a noun, thus focusing on the results of the interactions between an adult member of a culture and a child he or she is bringing up. These results are certain standards of interactive communication in specific kinds of situations. Bruner's point is (I think) as follows: When learning the mother tongue, the child is not learning to associate words with objects or states or qualities or actions. The child is learning forms of social behavior in various social situations in which speech plays a significant role: greeting a person, inviting, requesting, offering, playing a game, chatting, arguing, etc. At the beginning, the child will participate with inarticulate sounds and mostly gestures. But each achievement of the child in the direction of articulation and syntax will be met with an explicit approval by the adult, meant to send the message to the child that this is what will from now on be expected from him or her: The child's newly achieved behavior will be established as a new standard. For example: Richard has learned to say "cookie". Now he wants a cookie, so he points his finger towards the cookies in the jar and utters inarticulate sounds like "mmm!". The mother refuses to comply with his wish immediately. She says: "No, Richard, tell me what you want first". The standard will change along with Richard's achievements, and, at some point, the mother will not accept anything below a full request sentence complete with a polite "please".

But let Bruner speak for himself:

> Language acquisition "begins" before the child utters his first lexicogrammatical speech. It begins when mother and infant create a predictable format of interaction that can serve as a microcosm for communicating and for constituting a shared reality. The transactions that occur in such formats constitute the "input" from which the child then masters grammar, how to refer and mean, and how to realize his intentions communicatively [Bruner, 1985, p. 31].
>
> [I]n order for the young child to be clued into the language, he must first enter into social relationships of a kind that function in the manner consonant with the uses of language in discourse—relating to intention sharing, to deictic specification, and to the establishment of presupposition. Such a social relationship I shall call a format. The format is a rule-bound microcosm in which the adult and the child do things to and with each other. In its most general sense, it is the instrument of patterned human interaction. A format entails formally a contingent interaction between at least two acting parties, contingent in the sense that the responses of each member can be shown to be dependent upon a prior response of the other. Each member of the minimal pair has a goal and a set of means for its attainment such that two conditions are met: First, that a participant's successive responses are instrumental to that goal, and, second, that there is a discernible stop order in the sequence indicating that the terminal goal has been reached. The goals of the two participants need not be the same; all that is required is that conditions of intraindividual and interindividual response contingency be fulfilled. Formats, defined formally in this sense, represent a simple instance of a "plot" or "scenario" [Bruner, 1985, p. 39].

The way in which we extrapolate this theory to mathematics learning is the following. Before the pupil is able to formulate a mathematical definition or theorem, he or she enters a microcosm of interactions some of which are initiated by the teacher though their further development and shape depend heavily on the student's own initiatives. The format of interactions determines boundaries and rules for what to say, how to say it, when to say it, and who says what in different circumstances, but all these rules have a transactional character: They were obtained through some more or less explicit and direct negotiations between the teacher and the students, and between the students themselves. The path to mathematical thinking leads through getting clued into the mathematical discourse: knowing what are the intentions behind certain typical expressions, what are the common presuppositions. One learns this by interacting with those who can legitimately set the standards of the mathematical discourse (teachers, usually), by making mistakes and being corrected, trying to negotiate one's own ways of saying things and thinking, arguing with other learners. But the social and institutional setting of these interactions (school, the authority of the teacher) imposes limits on how far things can be negotiated and argued.

The phenomena of establishing standards of interaction at school at the primary level have been well illustrated in the works of Bauersfeld, Voigt and Krummheuer [*op cit.*] In a mathematics class, the teacher is, for example, "formatting" the students' interpretation of a picture showing three birds sitting on a branch and two flying away in making them see it as representing the arithmetic sentence 5 - 2 = 3. This is not, of course, the most obvious interpretation, and children come up with a large variety of proposals But it is enough that one or two of them say something close to this sentence and the teacher will immediately repeat it louder, will put it, in a way, in front of everybody in the class, as a model, a standard to apply and follow. Between the teacher and the students, an activity of formatting is going on which establishes how things should be understood and how one should behave in similar situations.

In our research we found it useful to distinguish between the process of formatting and its result, the format. Focusing not so much on the social aspects of the interactions but on their mathematical contents, we became interested in describing *the mechanisms of stabilization* of mathematical meanings through various kinds of formatting, and in analyzing the thus-established formats or *"standards" of understanding* in mathematics.

In order to be able to identify these mechanisms and standards, we needed a research methodology We attempted to adopt some ideas from the methodology of discourse analysis

Discourse analysis as a methodological tool

The interactionist approach, as developed in mathematics education, differs in particular from that of the constructivist and Vygotskian approaches with regard to its position on language. From the Piagetian point of view, language is an instrument for the expression of thought [Piaget, 1959, p.79]. For Vygotsky, language is the principal vehicle for cultural transmission [Vygotsky, 1987]. In interactionism, language loses this status of an object and a tool Bauersfeld underlines the difference between the expressions "speaking (a language)" and "languaging" [1995] "Speaking" is not "expressing" one's thoughts by using one language or another. Speaking is already, at the same time, thinking, knowing, and living in a certain manner The "discourse" that is "held" while speaking is reflective of a whole universe, established by means of communication, in which human groups arrive at certain conventions and build a shared understanding of contexts [Bruner, 1985].

This attitude towards language implies two fundamental postulates of the interactionist approach. Firstly, the postulate of the discursive and "conventional" (in the sense of "obtained through negotiations and accomplishment of consensus") character of scientific knowledge [Bauersfeld, 1995, p.227; Gergen 1995, p.22]. In particular, mathematics is seen as a special kind of discourse and is thus considered a special kind of way of seeing and thinking. This results in another hypothesis. Since all discourse emerges by means of communication, the mathematics that is learned at school depends very much on the characteristics and the modes of communication developed in the classroom.

In the interactionist approach to research into the teaching and learning of mathematics, therefore, communication in mathematics classrooms and the discourses that develop in them become the prime objects of study. This explains the interest accorded to methods of discourse analysis and the research into ways of analysis which are specific to the objectives of mathematics education research [e.g. Steinbring, 1993]. In this research, structural and quantitative methods have not proved very fruitful, and the encoding and the categorization of acts of individual speech are outside the domain of interest. For a mathematics educator, the primary unit of meaning is neither the word nor even the act of speech but *an episode of interaction*, characterized by a common and identifiable *theme* of conversation, relevant for the teaching of mathematics Hence, what decides about the unity of an episode is its content and not its form

In interactionist research, the discourse analysis becomes rather an "ethnography of communication", where interest is focused not only on the written product but also (and sometimes especially) on the oral product. Furthermore, one passes from "a monologic to a dialogic perspective of discursive facts" and "from a structural and immanent perspective to a communicative-type approach" [Cosnier & Kerbrat-Orecchioni 1987, p.5] The roots of this approach can be found in a stream of interactionist studies focused on conversational analysis which began in the 50's in the United States under the influence of G. Bateson.

Analysis of texts: Some tools

In the research that we have undertaken at Concordia University on the interactions between a tutor, a student, and a linear algebra textbook, the discourse analysis became more complex We were concerned not only with the oral exchanges between tutor and student, but also with the discourses of written text: the textbooks used in the research. What interested us the most was the interaction between these two discourses. We therefore needed some analytical tools to help us study the written texts In the examples which follow, we refer to certain concepts and distinctions that we have developed by applying, to the didactic texts, certain methods proposed by U. Eco in his "Lector in fabula" [1979].

The Model Reader

A text, according to Eco, "è un meccanismo pigro (o economico) che vive sul plusvalore di senso introdottovi dal destinario" [Eco, 1979, p. 52] A text can only come alive through an interaction with the interpretations of the reader. But it also cannot become fully realized without including, in its strategy, some constraints on these interpretations. These constraints are not necessarily restrictive, with a view to reducing the number of possible interpretations. Sometimes, as in the works of Joyce, such as *Finnegan's Wake* referred to by Eco, if the reader does not wish to devote some time to considering alternative interpretations, different possibilities, and divergent associations, he or she will not be able to "fill in the blanks" of the text and the text will remain silent for this reader. When we speak of constraints on the interpretations of a text, we are talking rather of what Eco calls the "Model Reader" of a text.

The Model Reader is not only *presupposed* by the text; it is also *created* by the text. The text assumes the reader possesses a certain competence; for example, an "encyclopedic" competence (knowledge of the different possible meanings of a range of words according to context and co-text). But the text also devises strategies to "produce a competence" [Eco, 1979, p. 56]

The Model Reader, therefore, erects boundaries to the universe of the *interpretation* of the text. These boundaries also imply certain limits on how the text can be *used*. Going beyond these limits forces a change in the universe of interpretation. For example, the Model Reader of a train schedule is, according to Eco, "an orthogonal cartesian operator with an awakened sense of the irreversibility of the passing of time" [Eco, 1979, p. 60]. Of course, such a schedule can be used differently; for example, to evoke memories of a past journey. But, in this case, the actual reader is led to interpretations that have nothing in common with the search for a number of a train and its hour of departure in order to arrive at a particular place at a particular time, as assumed by the Model Reader.

All texts contribute in a certain manner to forming (molding, shaping, educating) their Model Reader, but *didactic texts* can be defined as texts for which the formation of the Model Reader is their main concern and goal.

The didactic and mathematical layers of a didactic text

A didactic text usually includes two types of discourse: the didactic discourse and discourse specific to the discipline being taught. An ordinary linear algebra textbook will therefore have a *didactic layer* and a *mathematical layer*. The mathematical layer is composed of definitions, theorems, proofs, worked examples, exercises and solutions to problems. The didactic layer contains all that is concerned with the explicit formatting, on the one hand, of the behavior of the reader as a learner-user of the textbook and, on the other, of the reader's interpretations of the mathematical layer.

The quantitative relations between these two layers can vary from one text to another. Sometimes the didactic layer is so important that the didactic text becomes a text about another text (the content matter layer) which is cited *in extenso*. In other cases, the didactic layer is nearly nonexistent. We can thus speak of *strongly* and *weakly* didactic texts.

Different formatting strategies of interpretation and use in texts

Textbooks can use different strategies of *formatting interpretation* of their mathematical layers. We could speak of texts that are *open* to erroneous interpretations and of texts that are *closed* to such interpretations. We are using here Eco's concept of *open texts* (testi apierti). According to this author, an open text is one which conceives, *a priori*, the possibility of divergent interpretations (diverging, that is, from that presupposed by the Model Reader) as "an inevitable pragmatic situation". An open text will use this possibility of divergent interpretations as a "regulating hypothesis of its own strategy" [Eco, 1979, p. 58].

Of course, most of the time, a didactic text addresses divergent interpretations only in order to eliminate them (and not to play on them, as in some satirical texts, discussed by Eco). Such is the strategy of one of the textbooks used in our research where the reader is warned about possible errors and where erroneous interpretations are attacked in the exercises.

Textbooks can also use different strategies of *formatting their usage*. Some can leave a wide margin of liberty to a student who wants to use the textbook to learn, others can restrict their possible uses. We could speak about *liberal* and *apodictic* textbooks from the point of view of their use.

Totally apodictic textbooks are rare. But we can conceive of didactic texts that leave no liberty of choice of action to the reader (other than closing the book, of course). That would be the case, for example, with so-called programmed texts, as well as textbooks that use a "problem solving" or "constructivist" approach. The latter suggest sequences of activities and problems without announcing to the student (by name or definition) what the concepts are that he or she is supposed to be deriving by performing these activities. To know what these concepts are and to "construct his or her own knowledge" the student has no other choice but to follow the text page by page and to perform the prescribed activities (this is the case, for example, with Fletcher's book "Linear Algebra Through Applications" [1972])

Neither of the two texts used in our research restricted their usage in such a drastic fashion.

The remaining part of this paper contains a sample of our application of the notions of format and formatting and of the outlined concepts of discourse analysis in our research on individual learning of linear algebra with the help of a text and a tutor.

Study of formatting mechanisms and their consequences in interactions between a tutor, a student, and a linear algebra textbook

Hypotheses and research questions

Our research was concerned with the interactions between a tutor, a student, and a linear algebra textbook in two projects taking place simultaneously: a teaching project (the tutor's task was to help the student learn the rudiments of linear algebra from a textbook) and a research project (the tutor had to elicit, as far as possible, evidence of the student's learning process by asking questions about his or her ways of understanding, putting these ways of understanding to the test, etc.).

A "tutoring situation" is characterized, among other teaching situations in general, by the fact that the tutor, unlike the teacher, is not one "who delivers knowledge to the students". In other words, he or she is not seen by the student as the principal source of knowledge. He or she is supposed to help the student understand and learn what someone else has been trying to teach. In our case, this other source of knowledge was a textbook.

Our research documents (protocols of the exchanges between the tutors and their students and copies of the students' notebooks) can be studied from different perspectives. For example, we can attempt to identify the conceptual difficulties of the students. That would be to take a "cognitivist approach" to the analysis of the documents. But we can also take an interactionist approach and study the emergence of mathematical meanings in the interactions between the tutor, the student, and the textbook, and the schemes and routines of interaction which assure the stabilization of these meanings. We can attempt to describe these meanings as a function of the mechanisms of their emergence and their stabilization.

In the following analysis we take an interactionist perspective and concentrate on one of the mechanisms of the stabilization of meaning, namely the process of "formatting" the behavior of the student as

a) a user of the textbook (his or her behavior as a student), and

b) a reader who can reason mathematically and interpret the mathematical definitions of the text (his or her behavior as a mathematician).

More precisely, our analysis is based on the following hypotheses.

The written material of the textbook presents, to the reader, a format or a "mold" of behavior-as-a-student and of behavior-as-a-mathematician. The student responds to this formatting in a certain way, but his reactions are influenced by the formatting action of the tutor. The student can react in different ways to the actions of the tutor, which in turn can provoke changes in the formatting actions of the tutor, and so on. The behaviors of the student, as a student and as a mathematician, are therefore the result of exchanges and negotiations between the tutor and the student with respect to the text. These behaviors can be seen to have "the potential of a vector field" (to use an analogy from physics) whose vectors are determined by the following forces:

— the text's strategy of formatting
  — its use,
  — the behavior of the reader as a student,
  — the behavior of the reader as a mathematician;
— the formatting activity taking place between the tutor and the student related to
  — the use of the text,
  — the behavior of the student as a student with respect to the text,
  — the behavior of the student as a mathematician with respect to the text.

We propose ourselves to deduce, from a study of this "field of forces" for a given triplet, tutor-student-text, the "potential" of the student as student and as thinker in mathematics. Our intention is to see how the different types of formatting are associated with different types of behavior both as student and as mathematician.

The experimental setting of the research

By defining, for the sake of this research, a "tutoring situation involved in a research project" as "the set of relations uniting the elements of the triplet, tutor-student-textbook, within the same educational project and the same research project", we can say that we had six situations involving three tutors (P I, P II, P III), five students (Endy 1, Endy 2, Sandy 1, Sandy 2, Sandy 3) and two texts (M I and M II). Except for Sandy 3, all the students used only the text M II. Student Sandy 3 used both texts: M I in the first month and M II the rest of the time. For each situation, the observation sessions and the tutoring was stretched out over thirteen weeks at a rate of five hours per week. The students named "Endy" were students "in difficulty" (*en difficulté*) with very basic algebraic concepts, especially with the notions of variable and the solution of equations. Students named "Sandy" (*sans difficulté*), did not have these difficulties.

Textbook M I was a collection of problems (in Polish) destined for students studying Economics [Bazanska *et al.*, 1978]. It was used as an autonomous textbook in the research. Text M II was a recent American textbook addressed to a wider public attending the first year in US universities [Lay, 1994].

In the following I will present a sample of our analyses, the illustrations coming exclusively from the protocols of interactions in the triplets P I + Endy I + M II and P III + Sandy 3 + M I/ M II, where the contrasts are perhaps the most extreme.

Tutor P I was a mathematics student who had just obtained his Bachelor of Science, had received no teacher training but had had two years of experience as a tutor in mathematics in a university Math Help Centre. He was under the impression that the research involved finding an efficient way to teach linear algebra.

Endy 1 (26 years old) was an adult student who had returned to university from the work force; he had a history of failure in secondary school mathematics.

Tutor P III was a teacher of many years' experience. He believed that the goal of the research was to understand the difficulties encountered by students in linear algebra.

Sandy 3 (20 years old) was a CEGEP student (*Collège d'Enseignement Général et Professionnel*: post secondary, pre-university level in Québec) studying Administration Sciences; he had just finished his first calculus course.

Samples of analyses

The formatting strategies of the texts

<u>The Model Readers of M I and M II</u>

In its preface, textbook M II already defines, to a certain extent, its Model Reader. By stating that the textbook is designed for students "with the maturity that should come from successfully completing two semesters of college level mathematics, usually calculus", the text eliminates a certain number of interpretations. This condition on the reader implies, among others, an algebraic interpretation rather than an arithmetic one of the term "solution" when used in an algebraic context such as "the solution of an equation". An algebraic interpretation of "solution", in such an expression, refers to a system of values for the variables of this equation which, when substituted into the equation, make a true proposition; for example, the vector v is a solution of the equation $Ax = 0$ if $Av = 0$. On the other hand, an arithmetic interpretation of the term "solution" would refer to the result of an operation; for example, in the expression "A times x equals zero", "zero" is understood as "the solution". A certain reader competence is assumed concerning the notion of variable, especially one that is not reducible to that of an unknown but that encompasses the idea of an arbitrary representative of a class of objects. These concepts form part of the presuppositions about the Model Reader and they are not negotiated in the text Possible misunderstandings caused by a lack of these concepts are not addressed in this textbook

Text M I is a collection of problems: Each chapter begins with a list of definitions, theorems (without proof) and worked examples followed by a large number of exercises and problems. (It supplements a textbook which contains the proofs and explanations.) It is mainly intended for part-time students in their first year of studies in Economics These students do not have to attend regular classes: They study by themselves and only consult a professor two times a month Therefore the Model Reader of text M I is an autonomous student, interested in applications of mathematics.

The didactic and mathematical layers of our texts

In text M II, the didactic layer is an essential part of its strategy. It is a "didactically garrulous" text. We examine a sample of the didactic discourse of this text.

> With Example 2 as a guide, we are ready to describe an efficient procedure for transforming a matrix into an echelon or reduced echelon form A careful study and mastery of the procedure now will pay rich dividends later in the course [Lay 1994, p.16].

In this excerpt, the text attempts to format how the textbook should be used: It recommends the student to study carefully a certain part of the mathematics layer (negotiation between the author and the reader in this case can be understood in an almost literal sense as bargaining).

But didactic action is also present in the mathematical layer at the level of content; i.e in the choice of exercises and problems. Certain questions directly address the correction of common misunderstandings. For example, Exercise 1, page 34 [Lay, *ibid.*], asks the reader to decide upon the truth of a series of statements some of which reflect common wrong convictions of students.

Text M I has a very thin didactic layer. It contains hardly any explanations, introductions, or suggestions on how to use it. The reader is not told which concepts are more important and which are less: They are given equal prominence. There is no explicit guidance on finding links between concepts. The text neither warns the reader about conceptual difficulties nor does it consciously address, in its choice of problems, possible erroneous interpretations. The didactic layer is there (implicitly) in the organization of the mathematical text and (very slightly) in its layout.

One could say that, compared to M I, text M II is strongly didactic.

Strategies of formatting interpretation and use in textbooks M I and M II

Text M II, which warns its reader of possible errors and addresses erroneous interpretations in its exercises, can be classified as an "open" text, in Eco's sense of the word. On the other hand, divergent interpretations do not play any role in the strategy of text M I. The exercises require and reinforce certain well-defined interpretations, but do not explicitly discuss any divergent interpretations.

As stated above, neither of our texts was "apodictic" in the formatting of its use. Each left an important margin of liberty to the user and this was effectively taken into consideration by the tutor-student partnerships in our research

The formatting activity between tutor and student

Formatting the use of text

We shall illustrate the variety of possible types of use of the texts and of the formatting of this use between the tutor and the student with two pairs of examples which contrast with each other in certain aspects.

In one pair of examples we can observe the formatting of use in two *didactically different* texts:

> Example I.1: Here P I and Endy 1 are working from a *strongly didactic* text on linear independence in M II [Lay, *ibid.*, p. 63)]
> Example I.2: Here PIII and Sandy 3 are working from a *weakly didactic* text on linear independence in M I [Bazanska *et al*, p.186].

In the other pair of examples the contrast is between the formatting of use of the same text M II—application of linear systems to network analysis—by two different tutor-student pairs:

> Example II 1: P I with Endy 1.
> Example II 2: P III with Sandy 3.

EXAMPLES I

The students and tutors are working from texts M I and M II on the notion of linear independence. Text M II is strongly didactic: The definitions are preceded by an introduction where a link is made with what the reader has already learned His anxiety is assuaged by telling him that this is not really a new concept, only a change of point of view in the study of homogeneous systems M I, weakly didactic, states its definitions of linear independence after a section on scalar product without marking the change of subject even typographically. There is no introduction nor explanation.

Example I.1: Restrained use. The learning of a technical short cut

Endy 1 reads the introductory paragraph in M II, followed by the definition and the example verifying that a system of three vectors in R3 is linearly independent, without any interruption from the tutor and without asking any questions or making any comments He then re-reads the text from the definition on and makes comments that suggest he has not understood this definition. He thinks, for example, that to verify that three vectors are independent you have to add them together and see if their sum is zero.

The text itself, and the interventions of the tutor accompanying the reading of the example, seem to be guided by the intention of reducing understanding the notion of linear independence of vectors to practice in a technique of testing whether a set of vectors in R^n is linearly independent and so, finally, to the solving of homogeneous systems. The test concerns the existence of free variables: If there are free variables then the system is dependent, if there are no free variables then the system is independent. The didactic option chosen, therefore, is to show the student a short cut; hoping that this will help him solve problems more efficiently even though he may not understand all the theoretical subtleties of the concept.

In the following transcript, P I suggests some mnemotechnical associations to make sure that the student takes this short cut: If a variable is free then others *depend* on it, therefore we can conclude that the set of vectors is *dependent*.

> *June 20, P I + Endy 1 + M II*
>
> 1266. P I: Can you see why [the vectors] v_1, v_2, v_3 are linearly dependent? They depend on x_3, right? [x_3 was a free variable in solving the corresponding homogeneous system.]

After Endy's reading of the example, P I interrogates him on the definition. Endy 1 finally formulates in his own words a definition for dependent vectors (1285) and it is a definition that makes sense. But before that, his definition for independent vectors (1277, 1283) refers to the verification technique ("only the trivial solution", "no free variables"). This answer is accepted by the tutor (1278, 1284) much to the surprise of the student (1279).

> (Continued)
>
> 1274. P I: What does it mean, linearly independent, and what does it mean linearly dependent?
> 1275. Endy 1: Independent?
> 1276. P I: A set of vectors is said to be linearly independent when. ?
> 1277. Endy 1: When there's a trivial solution.
> 1278. P I: That's it.
> 1279. Endy 1: Oh, that's it?
> 1280. P I: When is the trivial solution?
> 1281. Endy 1: The trivial solution is when Ax equals to zero ..
> 1282. P I: Okay?
> 1283. Endy 1: And there are no free variables.
> 1284. P I: Okay. When does it have a linearly dependent solution?
> 1285 Endy 1: When there are weights so that you can make.. um... weights you could put in front of the vectors so that the equation equals to zero.

We have here a classic example of the formatting process: The student reacts in a certain way in his interaction with the tutor and this reaction is accepted and evaluated as correct. Thus a standard of behavior is established. From now on the student will have no excuse for not behaving in conformity with this standard.

But what is this standard in our case? Has Endy 1 learned a new concept? It would appear rather that he has learned a new word, a new technical term for an old concept: that of the distinction between homogeneous systems with free variables and homogeneous systems with a unique solution.

Example I.2 Free use. The construction of a technical "short cut".

Sandy 3 reads the definitions of dependent and independent systems in M I. His first reaction is one of surprise. Dependent systems must be very rare: "It seems to me that it can happen only in very special cases, very special cases ... that the sum of some vectors multiplied by some numbers gives zero. It can happen only exceptionally". This hypothesis is challenged by the tutor and the student embarks on a search for examples of dependent vectors. By doing so, of course, he must verify whether the vectors he has chosen are dependent or independent. He must therefore solve systems of homogeneous equations, but this solving technique plays the role of a tool only; it has not replaced the concept. He eventually comes to see, much later in the course, that the matrix technique of solving systems of equations is a useful short cut for verifying linear independence, but it was necessary for him to realize this himself, since neither the text nor the tutor would have shown him this.

The standard that can be seen being established in the interaction between P III and Sandy 3 in this episode is not specific to the concept of linear independence: it is more general for this pair. It is produced by the reaction of the tutor to the hypotheses put forward by the student. They are taken seriously and they are verified. This is precisely what has to be done according to this standard: to question the new concepts, to question their meaning and their value, and to accept them only after being convinced of the opposite.

After the session, Sandy 3 makes this comment:

> *February 4, P III + Sandy 3 + M I*
>
> 374. Sandy 3: [*at the end of the session*] I think that if I hadn't done all this here, hadn't dirtied all this paper, then I still wouldn't know, I mean I wouldn't suspect .. I am not saying that I am fully convinced now, but let's say I suppose at this point that this can have some applications, [that it goes] deeper than just a curiosity.

This comment emphasizes the utility, as an aid to understanding, of investigating the meaning of a definition by searching for examples and non-examples of the defined notion. This can sometimes lead, as in the episode quoted above, to the discovery of properties of the object that are not evident from the definition.

Immediately after the definition, text M II chose to give a short cut method that showed, by calculation, whether or not a given object satisfied the conditions of the definition. But nothing in the strategy of this text prevented the reader from actually working on the definition as Sandy 3 did. The didacticity of a textbook can be circumvented by the way in which it is used, as will be seen in Examples II.

EXAMPLES II

The didactic goal of section 1.3 of text M II is to equip the Model Reader with a feeling for the utility of the theory of the solution of linear systems, and especially the Gaussian elimination method. The section opens with the words, "It's time to see some linear equations in action", followed by a half-page introduction on the applications of linear algebra in network analyses.

A priori, one can think of many ways to use this type of text: "Example" followed by "Solution". The "Solution" can be read immediately, or the example can be solved first without consulting the textbook, after which the "Solution" can be read or not; the problem in the example can be either extended and generalized, or just treated in the form presented in the text. In each case, the formatting action of the tutor can be weak or strong. But the possibilities become multiplied when we take into account the specificities of the examples. If the example is, as in the fragment of text in question, an example of an application, it normally contains three parts: the modeling, the calculation in the model, and the interpretaton of the calculation in terms of the situation modeled The reader can read some of these parts, or all of them, or none of them and do the work himself, comparing or not the results of his work with the book. In our research, with P III + Sandy 3, the modeling was done without consulting the book, the calculation was done with the aid of a computer software and the interpretation was read immediately and discussed (Example II 1). With P I and Endy 1, the modeling was read, the calculation was done by hand, and the interpretation was not read nor was the situation discussed any further

Example II.1 The modeling
Sandy 3 had refused to be guided either by the text or by the tutor, despite the attempts of the latter to impose his strategy The student was captivated by the problem. He had produced a rough list of equations modeling the intensity of circulation of traffic in the situation represented in the book by a diagram (a fragment of a town plan). The system had five equations in five unknowns. The student had tried to disentangle the system for a few minutes before finally realizing that he was going round in circles. He then exclaimed:

March 7, P III + Sandy 3 + M II

214. Sandy 3: Wait, can't we make ourselves a ... what's its name. a... a. .
215 P III: A matrix?
216 Sandy 3: Yes, a matrix! A very easy one!

The student re-wrote his system in a more organized form in order to extract the matrix. The matrix was then entered into Maple (a Canadian-born computer algebra system) and the command "gaussjord" was used to reduce it The student was already in the habit of using the software, therefore the Gaussian elimination method appeared as a useful solving tool for a problem which had now become the student's *own* problem (as opposed to a problem imposed by the text or the tutor) The didactic goal of the text had been attained. But to arrive there, the student had to disentangle himself from the didactic actions of both the text and the tutor. It was easier with the text: All he needed to do was cover it with a sheet of paper. It was more difficult with the tutor, but with some "Quiet!"s and some "Hush!"s the student succeeded there too.

Example II.2 The calculation
The didactic goal of this part of the text had been lost by the way in which it was used by P I and Endy 1. The tutor had told the student to read the part of the solution where the situation was modeled; i.e. put into equations, despite, it would seem, the student's interest in studying the situation for himself He then told him to copy the system and to solve it—for himself. The solving of the system took nearly all of the session (more than one hour) The example, therefore, had become an exercise in Gaussian elimination, an activity with which the student still had many difficulties. But the elimination method was not seen as a deliverance, as had been the case with Sandy 3.

The activities of Sandy 3 and Endy 1 during the reading of the same text were perfectly complementary. Sandy 3 worked on the modeling and left the solution of the system to the machine Endy 1 left the modeling to the text and worked exclusively on the solution of the system.

In none of the cases analyzed here, was the textbook example extended to generate other problems or generalizations. This only happened once between P III and Sandy 3 during the course of the sessions (in the context of applying the concept of linear transformation to the modeling of a situation involving population dynamics—see Sierpinska, 1995), and never between P I and Endy 1.

<u>The formatting of the student's mathematical behavior</u>
We will only give one example to illustrate formatting of this type. The example involves the concept of "solution" with which Endy 1 had great difficulty.

Example 3 Formatting the interpretation of the concept of a solution
The two textbooks used in our research assumed a reader capable of interpreting the word "solution" in an algebraic manner where the context was algebraic The student Endy 1 did not satisfy this condition: as an actual reader he differed from the Model Reader postulated by the textbook, M II. But the tutor did not realize this immediately He began to suspect something when he heard Endy 1 use the expression "$Ax = 0$" in an "ungrammatical" way with respect to the way in which it was presented in the textbook. For example, Endy 1 states: "the columns of matrix A are linearly independent if and only if the equation Ax equals zero". The Model Reader would say instead: "the columns of matrix A are linearly independent if and only if the equation $Ax = 0$ has only the trivial solution, $x = 0$". The way in which Endy 1 was using the expression "$Ax = 0$" was as a sentence whereas in the textbook it was used as the subject of a sentence. When Endy 1 continues, "So A times this vector must be zero", he probably believes what he is saying is equivalent to what the textbook is saying: "Zero is the only solution of the equation", but the equation for Endy 1 is "A times x".

The tutor tries to "correct" Endy's mistake He says that $Ax = 0$ is "nothing" (line 356 below) by itself, meaning that it is not a complete sentence This is a correction at the level of categories of expressions. He questions Endy 1 on the meaning of the words, "trivial solution", "vector", "zero vector", "number zero". As Eco would claim, he invites the student to "open the dictionary" [Eco, *ibid.*, p. 50] at the words appearing in the statement. He explicitly formats, in his interactions, the way of talking about linear independence. But he does not enter into an explicit negotiation of the meaning of the concept of a solution in algebra and its difference from the concept of a solution in elementary arithmetic. Perhaps he believes that correcting the use of mathematical expressions helps to correct how they are understood This belief would be in tune with Wittgenstein's assumption that the usage of words forms a "mold" for their meaning

June 23, P I and Endy 1

346-8. P I: Okay, suppose instead of the vectors we begin with a matrix, with the columns, let's say, A_1, A_2, A_n. When are the columns of the matrix linearly independent?

349. Endy 1: This is what got me before! When are the columns . independent? When $Ax = 0$.

350. P I: What does it mean, $Ax = 0$? You are always saying, "when $Ax = 0$" But $Ax = 0$ is the homogeneous system You have to do something with it.

351. Endy 1: When are the columns of a matrix linearly independent? When there is only the trivial solution

352. P I: When there is only the trivial solution. Phrase it better If and only if SOMETHING has the trivial solution. What is the THING?

353. Endy 1: If and only if what thing has the trivial solution?

354. P I: The system, right?

355. Endy 1: Which is then $Ax = 0$. That's $Ax = 0$.

356. P I: So you have to do something with $Ax = 0$ It's nothing. You are saying they are linearly independent "if and only if $Ax = 0$" What does it mean, "$Ax = 0$"? [You have to say] "If and only if $Ax = 0$ has only the trivial solution".

There are many other interactions betwen the tutor and the student where the usage of the term "solution" is formatted and practiced. This formatting is always at the level of the manner of how to express oneself. For example, the student's attention is drawn to the synonymity of certain expressions: "p is a solution of $Ax = b$" means the same thing as "$Ap= b$". And he is encouraged to imitate the schema of this synonymity. The student adopts this schema but is surprised that the meaning of "solution" can be reduced to something so simple This astonishment might perhaps indicate that he has not really understood the concept of a solution as "satisfying" a condition, but only as an element of algebraic syntax

The "correction" of Endy 1's conception of the notion of a solution is done therefore by formatting the syntax. In spite of the corrections, the error recurs. One might say that the action of the tutor was not efficacious. But later in the sessions the student is able to correct himself when he makes the same error. Perhaps, therefore, we can say that in the intuition, in automatic usage, the old concept persists, but it has been corrected at the level of conscious reasoning.

Some conclusions

In the preceding examples we have concentrated on the phenomenon of formatting as a mechanism for stabilizing meanings in the interactions between a tutor, a student, and a textbook on linear algebra. Our intention has been to communicate the idea that different types of formatting are associated with different types of student and mathematical behavior.

Our conclusions are based on the analysis of a much more extensive set of protocol samples than we could include in this present paper, but these limited examples already show essential differences in the formats of interaction of P I + Endy 1 and P III + Sandy 3 that are pertinent to our research.

In the interactions between P I and Endy 1 the most important thing is to learn the methods and techniques of verifying whether or not an object satisfies a given definition. Interactions between P III and Sandy 3 put the effort into the negotiation of the meaning of definitions and theorems. P I tries to "mold" Endy 1's thinking into a structural form, tying it to the corresponding notation. The mold supplied by P III contains the flexible use of several systems of concepts in proofs, each with its different notation. Endy 1 learns a certain number of calculation techniques. Sandy 3 learns how certain non-mathematical problems can be reduced to mathematical problems The routine of interaction most often used to maintain and stabilize the meanings established between P I and Endy 1 is that of the oral revision of definitions and techniques. The routine that stabilizes the meanings between P III and Sandy 3 is that of argumentation and the verification of hypotheses.

We say the tutor "formats" or "molds the behavior" of the student but, in fact, the examples suggest that the formats establish themselves through the interaction. Tutor P III would not be able to force student Sandy 3 to ask himself questions about the existence of objects satisfying the conditions of a definition: The student had to be interested in them himself in order to put forward hypotheses that the tutor could then challenge. If Sandy 3 had less confidence in his mathematical abilities, or if he were only interested in learning efficient techniques for solving typical exercises, and if the textbook reinforced this attitude, the "challenge" format would have little chance of developing.

We have spoken about the relations between the formatting done by the text, implicit in its Model Reader, and the formatting done by the use of the text by the tutor and the student.

We are tempted to say that, on the whole, a didactic text does not reach its didactic goal (to construct a certain Model Reader) with its real reader unless the latter frees himself from the didactic formatting strategies of the text and takes the initiative of interpretation into his own hands. Or, paradoxically: "a textbook does not reach its didactic goal unless it is not followed to the letter" Of course this is not what really happens. In their reading of the example of an application to a traffic network, both Endy 1 and Sandy 3 did a part of the solving by themselves. But it is only Sandy 3 who, by himself, did the part that was pertinent to the goal of the text, i.e. modeling the problem and putting the system of conditions into a matrix format. The modeling showed him the usefulness of a linear model. The matrix format appeared as a technical liberation, as would the invention of a suitable robot. On the other hand, Endy 1 was just led to do the robot's job, without having its technical powers or its speed

We have here two formats of interaction with a text that couldn't be more different. We also see that, in the two cases, these formats are established not by the initiative of one particular element of the tutor-student-textbook triplet, but by the particular type of interaction between all three of them.

The formats that develop depend, to a certain extent, on the individual characteristics of the student and the tutor. One might suppose that the fact that P I had little teaching experience and P III had a lot played an important role. On the other hand, Endy 1 did not satisfy the initial conditions of the Model Reader of the textbook being used—these conditions were apparently satisfied by Sandy 3. So the formatting strategies of the text were destined to have different effects on Endy 1 and Sandy 3. Textbook M II was prepared with a completely different student in mind. P I's task therefore was much more difficult than that of P III. The former had not only to cooperate with the text with respect to its goal of molding the reader but he had to bring the student to the point where this molding might begin.

Tutors P I and P III had also quite different standards of understanding in linear algebra. For P I, for example, geometric visualization and modeling activity were not seen as necessary for understanding linear algebra. What was important for him was skill in algebraic calculations, the solving of a system of equations, and the manipulation of the expressions involving synthetic structural algebraic notation in formal proofs (e.g., if $Av = b$ and $Aw = b$ then $A(v + w) = b$). On the other hand, his student, Endy 1, looks for the meaning of the concepts through geometric visualization and applications and has great difficulties with algebraic calculation and its structural notation. Noting this, the tutor directs all his formatting activity towards algebraic competence. Perhaps, though it cannot be said for certain, he would achieve better results if he tried to help the student develop what is already his strong point instead of forcing him to do activities that he detests and in which he sees no purpose.

Textbook M II is, to a certain extent, an "open" text, i.e. a text in which the possibility of divergent interpretations has become "the regulating hypothesis of its own strategy". But this "openness" does not guarantee a standard of interaction between the tutor, student, and text in which the questioning by the student of the meaning of definitions and statements is taken seriously by the tutor and is negotiated by mathematical reasoning (and not, for example, just by pragmatic reasoning). Such was the standard established between P III and Sandy 3. But the "openness" of the text is not a necessary condition for the establishment of such a format of interaction. The same format of interaction was maintained between P III and Sandy 3 with respect to a text (M I) that we have called "closed". Would this format stand a better chance of being established between P I and Endy 1 if they worked with M I, not because it is closed but because it is less didactic? P I would not have been able just to leave the student to read the explanations and the worked examples: the meaning of the concepts would have had to be constructed between them.

Our preceding comments no doubt give the impression that the standards of behavior developed between P III and Sandy 3, both from the pedagogical and the mathematical point of view, are considered to be better than those developed between P I and Endy 1. Even if this is believed privately, we should, as "good interactionist researchers", repudiate such a simplistic conclusion. We are supposed to be studying and trying to understand the phenomena of teaching, not to be judging them and saying which formats are "good" and which are "bad". In fact, the value of a particular formatting can only be judged in terms of the objectives and the expectations of the participants of the interaction. A format like the one developed between P I and Endy 1 can prove very effective for a student who is primarily interested in passing the final exam and for whom the linear algebra course is only a prerequisite for further non-mathematical studies. The "challenge" format between P III and Sandy 3 is more pertinent for someone who is pursuing a profession in which mathematics has to be used creatively. This discussion does not provide all the answers, of course, since other, wider, perspectives can be taken of the value of mathematics and of the general objectives of a university education. All judgement implies a certain ideology. Interactionism proposes a philosophical attitude towards ideologies: To discuss without taking sides.

Acknowledgements

1 The research refered to in this presentation was funded by SSHRC, grant number 410-93-0700

2. I wish to thank my collaborators in the research on the teaching of linear algebra mentioned in the paper: Astrid Defence, Tsolaire Khatcherian, and Luis Saldanha and the students who took part in the research: Amira, Chris, Pierre, Piotr and Sandy.

3. Special thanks go to Astrid Defence who had the patience to read several versions of my text and make valuable comments. Astrid Defence also translated parts of the paper which were originally written in French for the purposes of a presentation at the Grenoble seminar in mathematics education, February 23, 1996

References

Bauersfeld, H. [1995] "Language Games" in the Mathematics Classroom: Their Function and Their Effects, in P. Cobb & H. Bauersfeld (eds.), *The emergence of mathematical meaning. Interaction in classroom cultures*, Lawrence Erlbaum Associates, Hillsdale, New Jersey

Bazanska, T., Karwacka, I., Nykowska, M. [1978], *Zadania z Matematyki. Podrecznik dla Studiow Ekonomicznych*, Panstwowe Wydawnictwo Naukowe, Warszawa

Bruner, J. S. [1985], The Role of Interaction Formats in Language Acquisition, in J. P. Forgas (ed.), *Language and social situations*, Springer-Verlag, New York

Cobb, P., Bauersfeld, H. (eds.) [1995], *The emergence of mathematical meaning. Interaction in classroom cultures*, Lawrence Erlbaum Associates, Hillsdale, New Jersey

Cosnier, J. & Kerbrat-Orecchioni, C. (eds) [198], *Décrire la conversation*, Presses Universitaires de Lyon

Eco, U. [1979], *Lector in fabula, la cooperazione interpretativa nei testi narrativi*, Bompiani, Milano

Fletcher, T.J. [1972], *Linear algebra through applications*, Van Nostrand Reinhold, Co., London

Gergen, K.J. [1995], Social Construction and the Educational Process, in L. P. Steffe and J. Gale (eds.), *Constructivism in education*. Lawrence Erlbaum Associates. Hillsdale, New Jersey, 17-40

Krummheuer, G [1995], in P Cobb & H Bauersfeld (eds), The emergence of mathematical meaning: Interaction in classroom cultures, Lawrence Erlbaum Associates, Hillsdale, New Jersey, 229-270
Lay, D C [1994], *Linear algebra and its applications*, Addison-Wesley, Reading, Massachusetts
Piaget, J [1959], *The language and thought of the child*, Routledge and Kegan Paul Ltd, London
Sierpinska, A [1995], Mathematics: "in Context", "Pure", or "with Applications"? A Contribution to the Question of Transfer in the Learning of Mathematics, 15 1, 2-15
Steinbring, H [1993], Problems in the Development of Mathematical Knowledge in the Classroom: The Case of a Calculus lesson, *For the Learning of Mathematics* 13 3, 37-50
Vygotski, L S [1987], Thinking and speech, Plenum Press, New York

If one had to recopy everything one had read (in order to express one's debt to the past), books would become alarmingly obese Even more important, this repetition would make them not very informative. The day that every text copies or summarizes that part of the library that concerns it we will enter the age of the thesis, of the newspaper and of stuttering Much as they detest each other, the press and academia have this repetitiveness in common Theses and popular magazines—the same duplication

On the other hand, honesty consists of writing only what one thinks and what one believes oneself to have invented My books come only from me. "My glass is not big, but I drink from it." That's my only quote. Don't you laugh at learned articles in which each word is flanked by a number, whose corresponding footnote attributes that word to an owner, as though proper names were soon going to replace common nouns? Common nouns belong to everyone, and in an honest book the ideas come from the author

One word on that word *author*, which comes to us from Roman law and means "the guarantor of authenticity, of loyalty, of an affiliation of a testimony or an oath", but primitively it means "he who augments"—not he who borrows, summarizes, or condenses but only he who makes grow *Author, augmenter* everything else is a cheat. A work evolves by growing, like a tree or an animal

Michel Serres, interviewed by Bruno Latour