Communications

Word Problems: Applications or Mental Manipulatives

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Along with Gerofsky (1996), Thomas (1997) and Boote (1998), all of whom have written relatively recently in this journal, I am very concerned with word problems and would like to say something about them too At the school level, many non-word problems tend to be technical exercises, which are necessary, but not so exciting. Many interesting and non-standard problems are in the form of word problems It does not mean that all word problems are difficult, but all of them need some understanding of natural language and the ability to translate between different modes of representation: words, symbols, images This is similar to Thomas' main idea, although I do not quite agree with his treatment of solving equations.

In Russia, where I grew up, word problems have been used for decades at all levels of difficulty with purposes ranging from the dissemination of mathematical literacy to that of challenging the most gifted children If you open a Russian elementary, middle or high school problem book, or a collection of recreational or Olympiad problems, you will see a lot of word problems For example, Perelman's famous book *Recreational Algebra* contains many excellent word problems, including the following:

> A team of mowers had to mow two meadows, one twice as large as the other. The team spent half a day mowing the bigger meadow. After that, the team divided One half remained in the big meadow and finished it by the evening. The other half worked on the smaller meadow, but did not finish it that day The remaining part was mowed by one mower in one day How many mowers were there? (1976, p. 39)

This problem is more than a hundred years old We know that Leo Tolstoi, who was very interested in public education, liked it especially because it can be solved in a visual way, without algebra.

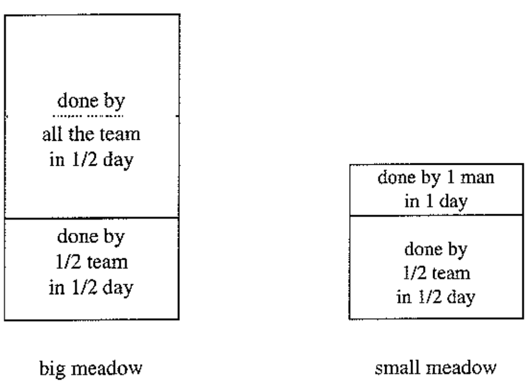

Figure 1

My main intention here is to address the following important question asked by Gerofsky:

> All this leads me to a question for which I have no answer as yet, the question of the purposes of word problems as a genre. (p 41)

Although Russian educators have been using word problems very productively for a long time, as far as I know they never cared to explain rationally why word problems are so useful, because nobody questioned this in their presence They were and still remain guided by tradition, experience, intuition and aesthetic criteria, all of which should not be ignored However, I agree with Gerofsky that, in the present situation, we need to discuss the purposes of word problems

She observes:

> The claim that word problems are for practicing real-life problem solving skills is a weak one, considering that their stories are hypothetical, their referential value is nonexistent, and unlike real-life situational problems, no extraneous information may be introduced. Nonetheless, they have a long and continuous tradition in mathematics education, and that tradition does seem to matter (p. 41)

I believe that her question has no single answer: like many other cultural phenomena (fables, for example), word problems have several purposes. Here I concentrate on two of them and compare them with each other: word problems as *applications* and word problems as *mental manipulatives*.

Word problems as applications

In this case, a word problem provides an application of mathematics to some situation which can occur in everyday life. An example: 'A grocery store sells eight oranges for a dollar. A customer wants to buy seven. How much should he be charged?'

This problem is based on a real event from my life I was buying food in a grocery store, where eight oranges were sold for a dollar. I put (as I thought) eight oranges in a plastic bag and went to the cashier, who counted my oranges and said that there were only seven I asked her to pro-rate the price. She took out her calculator, but did not know what to calculate She called another clerk with a bigger calculator, but he also could not figure it out. He counted the oranges again, found that there were eight, and this settled the matter

I know that stories about relative mathematical incompetence of young people are told in quantities. My question is, what should we conclude from them? Some educators suggest increasing attention to 'real-world problems', perhaps similar to this one. I maintain that 'real-world problems' should not constitute the only or even the main kind of problems used in classroom and I shall present two reasons for this.

One is that study of mathematics should be *systematic*: if study of some topic is undertaken at all, it should result in an ability to solve all problems within a certain range of difficulty, most of which, of course, have no counterpart in everyday life. The other reason, which is still more important in my opinion, will be presented below

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Word problems as mental manipulatives

These problems deal with imaginary situations, which do not need to be met in everyday life Numerical data do not need to be taken from reality. Quantities asked for do not need to be unknown or needed in real situations and quantities given do not need to be easily accessible in everyday life. What matters in this case is intrinsic consistency and interesting mathematical structure rather than consistency with or importance for everyday life. These problems are intended implicitly to introduce children into substantial mathematics, such as number theory, graph theory or combinatorics, but without heavy professional terminology.

Of course, these two purposes for problems do not exclude one other Many problems, actually used in schools and included in textbooks, are mixtures of intent. However, many of the best and most pedagogically useful problems clearly belong to the second kind: they certainly are not 'real-world' Their purpose is to convey a *mathematical meaning*, that is the use of suitable concrete objects to represent or reify abstract mathematical notions. Like animals in fables, 'real objects' in these problems should not be taken literally. They are allegories, *mental manipulatives* or *reifications* which pave children's way to abstractions.

For example, coins, nuts and buttons are clearly distinct and countable and for this reason are convenient to represent relations between whole numbers The youngest children need some real, tangible tokens, while older ones can imagine them, which is a further step of intellectual development That is why coin problems are so appropriate in elementary school. Pumps and other mechanical appliances are easy to imagine working at a constant rate. Problems involving rate and speed should be (and in Russia are) common already in middle school Trains, cars and ships are so widely used in textbooks not because all students are expected to go into the transportation business, but for another, much more sound reason: these objects are easy to imagine moving at constant speeds and because of this are appropriate as reifications of the idea of uniform movement, which, in its turn, can serve as a reification of linear function Thus, we can move children further and further on the way of *de-reification*, that is development of abstract thinking.

Let us remember that formal and abstract thinking, which is essential for success in modern civilized technological society, does not come as a straightforward result of physiological maturation or social adjustment. Children do not develop abstract or formal thinking as naturally as they learn to run or jump or speak their mother tongue.

> Knowing how to solve 'school' problems is, of course, not an end in itself In school, pupils are taught primarily scientific information and scientific thinking. It would be impossible to create, confirm and use scientific information if every separate deduction had to be compared each time with reality or with available information on reality. (Tulviste, 1991, p 122)

Observations of this sort (and similar ones given by Boote (1998) in his article in this journal) make me conclude that training students in solving *non-real-world* problems, where data should be accepted at face value, as abstract hypotheses, rather than statements about reality, with conclusions to be deduced from these data, is very important for developing the ability of formal reasoning

The necessity to apply deliberate and organized efforts to develop abstract thinking of children is especially visible in a country like Russia, where historically many people were not educated. For example, most Russians were illiterate at the beginning of this century and I believe that Vygotsky's theoretical ideas should be viewed in connection with a powerful movement towards national education, which took place in Russia in the end of the nineteenth and the beginning of the twentieth century This pathos of enlightenment can also be felt in writings of Perelman, who published several excellent popularizations of mathematics, including *Recreational Algebra*.

When I came to the U S nine years ago and started to teach, I found that many university students had a very poor grasp of solving word problems. When I started to read some American educational literature, I found a (to me) strange approach to word problems, which was quite different from that which I had been used to in Russia. Some seem to think that problems solved in mathematics classrooms should be as close as possible to everyday life.

I believe that this approach can be traced back to the well-known American psychologist and educator Edward Thorndike, whose influential book *The Psychology of Algebra* contains a chapter called 'Unreal and Useless Problems', which starts as follows:

> In a previous chapter it was shown that about half of the verbal problems given in standard courses were not genuine, since in real life the answer would not be needed. Obviously we should not, except for reasons of weight, thus connect algebraic work with futility. (1926, p 258)

Although Thorndike tried to apply a strict scientific approach, such words as 'genuine' on the one hand and 'unreal and useless' on the other have strong evaluative connotations, like those used/abused in political propaganda.

Let us apply Thorndike's approach to the problem quoted at the beginning. Leo Tolstoi's problem: is it possible for several men mowing one meadow not to know how many of them there are? Of course not. Thus, according to Thorndike's criterion, the problem is unreal and useless and, if given to children, will produce a sense of futility. However, I can assure you that it does not.

There is an important similarity between children's play and mathematics: in both cases creative imagination is essential. This idea is not new. For example, Martin Gardner (1959) wrote:

> Perhaps this need for play is behind even pure mathematics. (p. ix)

However, some seem to follow Thorndike rather than Gardner For example, Zalman Usiskin's (1995) article, reprinted by the journal *Mathematics Teacher* on the 75th anniversary of the US National Council of Teachers of Mathematics, says:

> Algebra has so many real applications that the phony traditional word problems are not needed. (p. 159)

The explanation, as I see it, is that there is a strange, but widespread, theory at work, which I shall call the *no-transfer* theory. According to this theory, children *cannot* transfer their skills and knowledge from classroom to life outside school and, because the purpose of schooling is simply the better preparation for that life, the curriculum should be filled with those problems which people solve in everyday life. In addition, children apparently cannot be interested in anything that is not related to everyday life.

Here is just one example. The following problem may be used almost everywhere around the globe without objection:

> Sally is five years older than her brother Bill. Four years from now, she will be twice as old as Bill will be then. How old is Sally now?

yet it is declared locally unfit for the following reason:

> First of all, who would ask such a question! Who would want to know this? If Bill and Sally can't figure it out, then this is some dumb family. (Smith, 1994, p. 85)

As an example of an opposite, much more productive approach, let me quote Perelman again. His second chapter, called 'The language of algebra', consists of 25 sections, each devoted to a problem through which he teaches. One of them, called 'An equation thinks for us', starts as follows:

> If you doubt that an equation is sometimes more prudent than we are, solve the following problem:
>
>> The father is 32 years old, the son is 5 years old. How many years later will the father's age be ten times the son's age?

An equation is made and solved, but the answer is negative: -2. What does this mean? Perelman explains:

> When we made the equation, we did not think that the father's age will never be ten times the son's age in the *future* - this relation could only take place in the *past*. The equation turned out to be more thoughtful and reminded us of our omission.

I believe that this comment is really instructive: there is a moral to the tale that makes it worth the telling, and it constitutes for me a sufficient reason for its discussion.

Imagine that prospective teachers of literature in a certain country are made to believe through their professional preparation that all fairy tales, fables, fantastic stories are useless. When told a fable, one where animals speak to each other, they cannot comprehend and enjoy it in a normal way, as in fact children do, but exhaust their imagination in figuring out how could it happen in real life: perhaps the animals were especially trained to speak? Perhaps they underwent some operation? Perhaps they were disguised people? etc.

I certainly do not mean that all children need be or are interested in cumbersome or far-fetched plots. I do not propose to use arbitrary or chaotic problems. On the contrary, my point is that word problems are and should be *mathematical* problems, presented in a form accessible for children and their quality depends first of all on the quality of their intrinsic mathematical structure and also on their elegance and accessibility. This means that they should not be overburdened with arbitrary or irrelevant details

A good word problem should be as aesthetically appealing as a piece of art. Take Aesop's fable 'The Crow and the Fox'. On the one hand, it uses images known to every child; on the other, it has been purged of all irrelevant details. But, from this strange viewpoint, it is as if Aesop's fable is useful only for those who have the chance in some already-pre-perceived future to perch on a tree branch with a piece of cheese in their mouths.

We are dealing here with some of the most fundamental laws of culture: human culture never describes reality one-to-one. It condenses, simplifies, idealizes. Geographical maps are not, cannot and should not be equal to those landscapes which they represent. Creations of the human mind are subject to the drastic law of economy: redundancy should be avoided, every detail must serve a purpose.

A good word problem shares all the same attributes with which General Chuck Yeager (the first person to fly faster than sound) cryptically praised an aeroplane's engine system: "simple, few parts, easy to maintain, very strong". (cited in Bentley, 1989, p. 6) Many so-called 'real-world' problems are cumbersome and loose. The real world is full of waste, redundancy, confusion and boredom, all of which should be excluded from the mathematics classroom.

References

Bentley, J. (1989) *Programming Pearls*, Reading, MA, Addison-Wesley

Boote, D. (1998) 'Physics word problems as exemplars for enculturation', *For the Learning of Mathematics* **18**(2), 28-33

Gardner, M. (1959) The *Scientific American Book of Mathematical Puzzles and Diversions*, New York, NY, Simon and Schuster

Gerofsky, S. (1996) 'A linguistic and narrative view of word problems in mathematics education', *For the Learning of Mathematics* **16**(2), 36-45

Perelman, Ya. I. (1976) *Recreational Algebra*, Moscow, Nauka

Smith, M. (1994) *Humble Pi*, Amherst, NY, Prometheus Books

Thomas, R., Gerofsky, S. (1997) 'An exchange about word problems', *For the Learning of Mathematics* **17**(2), 21-23

Thorndike, E. L. (1926) *The Psychology of Algebra*, New York, NY, The Macmillan Company

Tulviste, P. (1991) *The Cultural-Historical Development of Verbal Thinking* (Hal, M. J., trans.), Commack, NY, Nova Science Publishers

Usiskin, Z. (1995) 'What should not be in the algebra and geometry curricula of average college-bound students?', *Mathematics Teacher* **88**(2), 156-164