On the Right Track

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The purpose of teaching mathematics is to point out mistakes and correct them! This seems to be a common understanding of mathematics education among students [1] We have even seen examples of pre-school children expressing the same view in role play about teaching mathematics.[2] One was playing the role of the teacher, the rest were "students" One "student" was supposed to work an exercise on the blackboard and wrote some serious-looking symbols in a long row Afterwards, the "teacher" erased a couple of those symbols and wrote some others, accusing the student of being mistaken. Thus, even before having any school experiences of their own, the children show an understanding of mistakes as being a central parameter in mathematics education

Marilyn Nickson has studied the culture of the mathematics classroom, and she concludes that "the pupil's role in the mathematics classroom can vary dramatically according to the view of mathematics projected by the teacher" [3] Similarly, we assume that student and teacher interpretations of mathematics education is a consequence of an implicit "philosophy" of mathematics From the studies of pre-school children we seem forced to assume that they possess such an implicit philosophy too. Thus, a particular philosophy of mathematics and of mathematics education must derive not only from the mathematics classroom itself.

In this article, however, we want to concentrate on the philosophy of mathematics that is disseminated in the classroom. the philosophy of mathematics may not be projected explicitly by the teacher, i e. it may not be put into words, but it can be studied through the classroom activities.[4] Our main questions are: In what way are students' implicit conceptions of mathematics and of mathematics education influenced by teaching even when the teacher does not explicate what mathematics is really about? How do teacher and students come to behave according to a certain philosophy? Our thesis is that the role of mistakes and corrections of mistakes in the classroom practice reveals a certain philosophy of mathematics and, further, that the way mistakes are handled produces such an implicit philosophy.

We shall refer to different situations in different classrooms.[5] We do not try to "prove" our thesis, we only want to illustrate and discuss it However, first an aside about the philosophy of mathematics

"Truth" in the philosophy of mathematics

One reason why the notion of "mistakes" seems so important in mathematics education can be related to the search for "truth" in mathematics A main task of a philosophy of mathematics has been to give an adequate explanation of "truth" *Absolutism* in epistemology is associated with the idea that the individual has the opportunity to acquire absolute truth This idea connects with the Euclidean ideal in epistemology. *Relativism,* though, maintains that truth is always located by someone in a certain context at a certain time and, therefore, truth cannot be expected to be located in absolute terms. With regard to mathematics, relativism has been put forward by the constructivist approach, especially as formulated by radical and social constructivism.[6]

Thus we come to two different notions of truth, one according to which it makes sense to talk about *the* truth of a proposition, another according to which truth depends on an agreement between people Steinar Kvale formulates it as follows: "Truth is something that is negotiated between people who share actions " [7] This definition makes truth an open concept. Truth depends on certain understandings and interpretations. We can be sympathetic to this definition by Kvale but we still have to ask ourselves what this relativism could mean when we look at a simple mathematical proposition like "12 times 13 equals 156".

"Mistakes" in a classroom philosophy of mathematics

If truth is something negotiated between people, it might be uncertain whether the outcomes of mathematical investigations are right or wrong. This leads to the question: Are mistakes really mistakes?

Normally the (theoretical) discussion of mistakes in the mathematics classroom has concentrated on the mistakes of the students We could as well look at teacher mistakes, teacher ways of interpreting their own mistakes, student ways of interpreting teacher mistakes, teacher ways of hiding mistakes, etc. The study of mistakes can take a variety of directions Nevertheless, we shall follow the mainstream and concentrate on student mistakes, and teacher ways of interpreting and correcting these.

Like the concept of "truth", the concept of "mistake" exists in two extreme versions—one absolutistic and one relativistic. The absolutistic interpretation has a sound basis in mathematics. To propose that 12 multiplied by 13

For the Learning of Mathematics 16, 1 (February 1996)
FLM Publishing Association, Vancouver, British Columbia, Canada

equals 157 seems a simple mistake. But the situation looks somewhat different if we come to the applications of mathematics If we measured one side of a playing ground and found it to be (about) 12 m and the other side to be (about) 13 m, its area may well be 157 m^2—the shape looks rectangular. Relativism may have a bearing when the application of mathematics is considered, although it also seems possible to make absolute mistakes when applications of mathematics are considered in mathematical textbooks.

We shall look at different types of "mistakes" and how they are handled by teacher and students. We do this to locate an entrance to the students' "philosophy of mathematics". The way "mistakes" are pointed out in teaching is a forceful way to indirectly communicate a philosophy of mathematics This influences both the students' and the teacher's conceptions of the subject Even if it is the teacher who points out mistakes, the students' reaction to these corrections also adds to the teacher's interpretation of mathematics. Our thesis is that a philosophy of mathematics is carefully elaborated during classroom practice But as long as it is not verbalised, the philosophy becomes an unquestionable authority which may turn out to become a strait-jacket for mathematics education.

As "truth" is a key theme in the philosophy of mathematics, so are "mistakes" a key to grasping the implicit philosophy that prevails in the mathematics classroom We find that corrections of mistakes are the back door to the classroom philosophy of mathematics But although it certainly is a significant back door, nobody seems to be in control of what is brought in and out of this door

Classroom absolutism

We shall make a distinction between philosophical absolutism and *classroom absolutism* Philosophical absolutism in mathematics maintains that some absolute truth can be obtained by the individual. Classroom absolutism seems to focus on truth from the perspective of mistakes. This absolutism comes about when mistakes are treated as absolute: *This is wrong! You cannot query this!*

Our point is *not,* however, that no mistakes in the mathematics classroom should be said to be real mistakes If a child has multiplied 12 by 13 and got 165, we conceive this as a mistake. We do not want to maintain an absolute relativism.

We can conceive of different types of mistakes found in mathematics education [8] The mistake could concern the output of some algorithm: *This calculation is wrong!* The mistake could concern the algorithm used: *You should not add these numbers but do a subtraction!* The mistake could concern the sequence in which things are done: *When drawing a graph you first have to calculate some values of the function!* The mistake could have to do with the way the text is interpreted: *No, when the exercise is formulated like this, you first have to find the value of x!* Or it could have to do with the organisation of the teaching: *No, no, those exercises are for tomorrow!*

Although the contents of these mistakes are quite different, the corrections are expressed in absolute terms The basic axiom is that the aim of a correction is to correct a mistake. The phenomenon that all sorts of mistakes are treated as absolute, i.e as *real* mistakes, we refer to as classroom absolutism.

Correcting mistakes

Let us take a look at some examples from our observations to illustrate classroom absolutism. We want to make a distinction between corrections made in *public* in front of the whole class and corrections made in *private* in a dialogue between the teacher and the individual student. Public corrections are made in order to make everybody aware of the nature of the mistake At the same time public corrections can compromise the student who has made the mistake. Corrections made in private only involve the student who has made the mistake It is of no importance, though, for the existence of a classroom absolutism whether the corrections are made in public or in private Public corrections only make the absolutism more visible and observable in the classroom communication All of the examples below, however, illustrate private corrections

We also want to distinguish between the *form* of the correction and the *content* of the correction The form of the correction, understood as the teacher's way of expressing the correction, could be *explicit,* i.e. a direct mentioning of a mistake to be corrected, or *implicit,* i.e. a correction which can only be understood as a correction when you interpret the communication context

In our material we find many examples of explicit corrections:

(1) Teacher: This is wrong, you have to calculate it once again

(2) Teacher: There is a tiny mistake in both of them.

In (1) the teacher rejects the result and tells the student to try once again. Example (2) differs in form although it is certainly an explicit correction. The correction is modified by the word "tiny", which indicates that the student might be on the right track or that the teacher wants to encourage the student to continue without caring too much about the mistake. In neither of the examples, though, does the teacher argue how the student is mistaken, he just states it. Nor is there any advice or information about what the student is expected to do.

Implicit corrections can take several forms, e g :

(3) Teacher: You have to erase those numbers . you are not going to need them at all.

The teacher does not tell the student directly that he has made a mistake, but as one only erases things that are incorrect or insufficient, the student can easily understand the utterance as a correction of a mistake But still the teacher does not say anything about what kind of mistake the student has made or how it should be corrected.

(4) Teacher: Mary, what was it, how much is 3/4 plus 3/4?
Mary: 6/8 (4 sec.)
Teacher: Don't you remember, I took these [pieces from a fraction game], this 3/4 and this 3/4, and that equals 6, and what are they still . . ? (5 sec.)
Mary: Mmm . . 3/4

In this example the correction is made implicitly with a certain questioning strategy by which the teacher tries to make Mary guess the answer. There is no negotiation in the sense of teacher and student trying to explain and understand their different perspectives.[10]

The content of the correction could be separated in three subcategories: the *object of the correction,* which means the phenomenon that is going to be corrected, the *sources of the correction,* i.e. the background which gives somebody the authority to define something as a mistake, and the *generality of the correction,* i.e. the scope of situations which the correction seems to concern.

Concerning the object of the correction, our empirical material shows that the teacher focuses either on the algorithmic procedure or on the result of the students' investigations.

(5) Teacher: Your numbers could be right, if you handle them right.

The teacher tells the student that he is wrong, by telling him that he "could be right, if ...". What he is wrong about is the algorithmic procedure: "if you handle them right". The mistake concerns a wrong algorithm or a wrong use of the algorithm.

(6) Teacher: The last two numbers are wrong, Marion, you must try to correct them.

In (6) the mistake pointed out by the teacher obviously concerns the result of the student's work.

Instead of the sources of corrections, we could also talk about the authority behind the correction. Something would be deemed wrong by a reference to some authority.

(7) Teacher: I would rather have you drawing a line than putting a cross up in the air (to show which numbers he has already used in the reduction)
Tim: This is just as easy.
Teacher: It isn't always just a question of easiness.

(8) Teacher: The first condition for calculating correcting is that you put it down correctly.

(9) Teacher: If they have made the exercise, they are the ones to decide whether it is right, aren't they?

In (7) the teacher wants a different marking of the numbers used by the student in a reduction exercise. The student argues that his method is quite as easy, which is indirectly rejected by the teacher. But he does not argue why drawing a line should be a better or more correct way of marking than putting a cross above the numbers.

In (8) the teacher's correction indirectly refers to the textbook: "that you have put it down correctly", i.e. that you have written exactly what is prescribed by the textbook. The argument is naturally that it is a mistake not to solve exactly the exercise spelled out in the textbook.

In (9) the student's result is compared to the answer book, which states a different result. The teacher argues that the answer must be true, because the authors of the exercise are the ones to decide whether the result is right or wrong. This might be right, but it might not necessarily help the student to an understanding of the problem and the way of solving it.

Concerning the generality of corrections, the correction is stated in a form which indicates that it applies to a whole set of situations.

(10) Teacher: You don't even have two decimal places, so ... this is totally far out.

The fact that there are various ways of doing things depending on the context is not included in the correction.

Normally classroom communication is characterised by an asymmetrical relationship between teacher and students. The teacher has the right to control what is going on. He controls what the students are allowed to say, when, and to whom. As Michael Stubbs puts it: "Anything the pupil says is sandwiched in anything the teacher says." [11] The teacher asks a question to which he knows the answer himself, the student answers, and the teacher evaluates the answer.

(11) Teacher: How much is 3/4 + 3/4?
Student: 1 1/2
Teacher: Very good

Not every sandwich is as simple as this one, but the student would often answer with one single word filling out the teacher's monologue.[12] Heinrich Bauersfeld analyses a certain kind of dialogue, which he calls the "funnel pattern".[13] The teacher's way of asking questions is narrowing the answering possibilities. The student cannot guess what the teacher is aiming at, and he does not reply at all or only by one-word sentences. In the end the teacher is most likely to answer the question himself. This form of communication also signals that every mathematical question has one right answer, which has to be stated.

Bureaucratic absolutism

All of the corrections we have put forward in the examples above illustrate a classroom absolutism. But how can this absolutism be characterised? We shall try to do this with particular reference to the object of the correction, the sources of the correction, and the generality of the correction.

The object of the corrections is the output from the students' mathematical activity, but the focus might as well have been placed on the ideas that the students have put into their activity. To make the output the object of criticism means that the students' reasons for coming up with some solutions are ignored.

The teacher, the textbook, and the answer book make up a united authority, which hides the nature of the sources of the correction. It becomes unnecessary for the teacher to specify the authority that is put behind different types of corrections. This means that the students are not met with argumentation but with reference to a seemingly uniform and consistent authority, even though the sources of corrections might be very different. Some rest upon mathematical features, some rest upon practical matters of organising the educational process, etc. All mistakes are treated as absolute; they are pointed out by the teacher, with no explanation or argumentation as to what should be done differently or why.

Finally, the generality of the corrections seems never to be questioned. This is caused by the fact that the correction is not contextualized but stated in general terms, not referring to the context of the problem solving process.

A client, applying to a bureaucracy, has to behave in a certain way. The bureaucracy might have different reasons for refusing the application: The client may not really need the favour; the application appears too late; some information is missing; there is no money left, etc. These reasons for refusing the application are quite different. But when the client faces the bureaucracy, the denial of the application turns out to be of the same "logical form" whatever reason the bureaucracy might have: the application is refused. Good reasons or bad reasons, moral reasons, administrative reasons, other reasons—all appear in the same way. Things either fit into the schemes of the bureaucracy or not.

Students meet the same phenomena in a mathematics classroom. Therefore we want to characterise classroom absolutism as a *bureaucratic absolutism* that states what is right to do and what is wrong in absolute terms, and not by explicating the reasons for this distinction. Further, bureaucratic absolutism is characterised by the difficulty of getting in contact with the "real" authority: *We cannot do anything about this, it is outside our reach.* Things are as they are according to some standards: It is not possible for the person behind the desk to alter things. The client can argue, but it is not possible to change things all the same. Similarly, the absolutistic mathematics teacher is not supposed to change the fact that students have to deal with a certain kind of exercises, and that the formulae they have to use are those put on the top of the page. Further, the teacher does not question the fact that these exercises have to be solved. Bureaucratic and classroom absolutism prescribe: *We cannot do anything about this now. That is how it is.* Bureaucratic absolutism faces students in many mathematics classrooms.

We also find that even if a teacher shows great sympathy with alternative forms of teaching he has difficulty in practising his own ideas because bureaucratic absolutism has taken hold of him. It is personalised in his educational practice, it is built into his basic structures of communication. This puts the teacher in a paradoxical situation. On the one hand he feels that he has to follow a textbook to lead the students into the best possible situation to cope with the described questions.

This absolutism determines a philosophy of mathematics. In other words, the classroom communication is produced by, as well as reproduces, a philosophy of mathematics both for the students and for the teacher. The communication establishes an idea about mathematics and simultaneously produces a verification of this philosophy.[14]

Here we have got to the centre of the production of an "ideology of certainty".[15] This ideology encompasses the idea that when mathematical calculations are involved in some sort of problem solving, the result must be trustworthy. Mathematical competence is exactly established when mistakes are ruled out! The "ideology of certainty" is a reflection of the patterns of communication which we have characterised as bureaucratic absolutism. In this way a working "philosophy of mathematics" becomes rooted in a communicative practice.[16]

Being involved in a certain discourse implies being entangled in a certain "myth". A language is not a simple tool but part of a structure which anticipates interpretations. In this sense a language exercises "symbolic power". Therefore the question is: What features are embodied in the discourse of bureaucratic absolutism? Bureaucratic language seems able to divide statements about "reality" into two halves: the true ones and the false ones. We consider the true-false dichotomy as a basic symbolic aspect of the language of bureaucratic absolutism.

This "myth" need not be intended by the teacher or the authors of the textbook. Nevertheless, it may be developed, and it may bring about an unintended absolutistic philosophy of mathematics. In that sense the form the correction takes becomes a backdoor entrance to a conception of mathematics.

Negotiation of perspectives

How is it possible for the students and the teacher to communicate using a different paradigm than bureaucratic absolutism? Is the only alternative that relativism has to prevail? Our point is that relativism is one alternative to bureaucratic absolutism, but it might be possible to identify a different alternative—certainly with reference to the same empirical situations.

Bureaucratic absolutism is not a feature of every situation that we have observed. The teacher sometimes changes the pattern of communication. The essential point is, however, that it is not possible to do this simply by some sort of educational decision. The possibility of taking this step depends on a change in the educational situation. The organisation of the classroom influences the communicative practice, hence the way mistakes are corrected, and finally the students' conception of what mathematics is about. In this and the following section we shall mention two steps in challenging bureaucratic absolutism.

To understand the meaning of a sentence it is not sufficient to understand the words of the sentence. The meaning of a sentence is established by the use of the sentence in a certain situation. While you are saying something you are also doing something. To talk is to act. Meaning is constituted in the context of the use of language. Therefore, meaning cannot be discussed without reference to a specific situation and to the perspectives of the persons who communicate.

A *perspective* is normally not presented by an explanation. It is part of the background for communication. Normally, nobody finds it necessary to communicate a perspective explicitly. In fact it is not obvious how one should do this. Where could one begin? A perspective belongs in a strict way to the background of communication, but it is against this background that statements find their meanings. A perspective is a source of meanings. Without a perspective no communicative act will take place.

The perspective is decisive of the things one chooses to see, to hear, and to understand in a conversation, and it manifests itself through our use of language, in the things we choose to talk about and not to talk about, and in the way we understand each other. The purpose of a conversation can be to explain one's perspective, to understand the perspective of the other person and perhaps to agree upon a common perspective or upon the fact that you have different perspectives on which you want to insist. For example, the students and the teacher can share the perspective that the whole educational task is to arrive at mastery of some techniques and to be able to pass an examination. Teacher and students can also have different perspectives, for instance, the students may focus on the result of an exercise while the teacher wants them to explain the algorithm.

It is a common situation that one perspective dominates another. When the teacher points out mistakes he maintains a perspective which the students are supposed to accept to the extent that they try to avoid new mistakes. To ask for a correction of mistakes is a normal way of maintaining this perspective. Corrections mould a perspective. The teacher is in a position of power because he states the mistakes. The correction of mistakes, then, maintains the existence of an authorised set of meanings which it seems impossible to discuss.

A *shared perspective* can be established, even in cases where nothing much has been said. Nevertheless, the perspective becomes, so to say, the factory for the production of meaning. The converse may also be the case: Even when everything seems to be stated explicitly, it may be difficult for the parties to communicate if the communicators cannot understand or accept each other's perspectives, or if no perspective is shared. In this case, the wheels of the factories run uselessly.

How is it possible to come to share a perspective? This is a complicated task, because words and concepts used in normal communication address content matter questions, not the concepts themselves. To negotiate perspectives seems to presuppose that we shall be able to talk about how the concepts we use acquire their meanings. In that sense a discussion of perspectives calls for a platform beyond language. A disagreement may concern a state of affairs, but it may also concern how we interpret things. A disagreement because of non-shared perspectives may be interpreted as a disagreement about facts. In this way a disagreement at a meta-level becomes translated into the level of facts.

In order to be able to negotiate, the teacher and students must both be aware of the teaching purpose.[17] The teacher has to give explicit information about the matter taught in order to give the students the opportunity to take part in the process. If the students do not have sufficient information about the subject being taught, they have to guess the teacher's thoughts and imagine what the lesson is all about in order to be part of the classroom communication. Thus, instead of using their effort talking about mathematics, the communication is concentrated on questions like: "What is the teacher thinking of?" and "How do I become a part of this lesson?" Not telling the students about the teaching purpose is therefore an obstacle to the negotiation of perspectives. The students have no chance to ask mathematically relevant questions or to express their understandings of the matter taught.

A first step in trying to overcome bureaucratic absolutism is the teacher and the students coming to discuss and negotiate their perspectives.

Negotiation of meaning[18]

"Meaning" in mathematics education could be interpreted in two ways. We could talk about the meaning of a mathematical concept: What does a "fraction" in fact mean? We could also talk about the meaning of a mathematical task: Why do we have to learn about fractions? When we talk about the negotiations of meaning we think in terms of both the meaning of concepts and the meaning of tasks.

We cannot think of mathematical meaning as transmitted from teacher to student. Neither can we think of mathematical meaning as being constructed individually by each student. Mathematical meaning rather emerges between the participants in the interaction of the teaching and learning process. This moves the educational focus from the mathematical knowledge to the discussion about the mathematical knowledge developing between teacher and students.[19]

Of course negotiation of meaning is impossible if one party is allowed to control the other. In order to talk about negotiation, the two parties must accept each other as equals, or at least they have to respect each others' perspectives.[20] Otherwise there is nothing to negotiate. Through dialogue both teacher and students have the opportunity to control their understandings of what the other says. They can make use of meta-communication—e.g. they can communicate about the ongoing communication in order to make sure that they understand each other. For example the teacher can ask the student: "Would you explain once again what you just said?" Negotiation means actively participating in the process. The students are learning by doing as well as by talking about mathematics. Thus, to explain one's perspective, method, problem solving, etc., in a dialogue can be described by the term *learning by talking* which we use as a metaphor for the point that knowledge develops not primarily in the student's mind but through dialogue.[21]

Therefore we shall emphasize negotiation of meaning as the second essential step towards challenging bureaucratic absolutism.

A difficulty

Although formulated as simple pieces of advice, negotiation of perspectives and meaning are not simple activities. These activities will be met by obstruction from "the logic of schooling". And, as indicated, one of the obstacles to be considered is the students' already-established implicit philosophy which says that nothing has to be discussed. It is not a simple task for a teacher to create a "climate of negotiation".

To change bureaucratic absolutism does not simply presuppose a change in the teacher's attitude. This absolutism is not only rooted in an attitude but in the whole structure of schooling.

We also witness the possibility of inconsistency between what the teacher thinks he is doing and what he in fact communicates in the classroom. The style of communication may be more conservative than the teacher's own conception of mathematics education. The teacher may maintain the importance of communication with the students in order to understand how they construct knowledge while his actual style of communication fits the absolutistic paradigm. And one obvious reason for this discrepancy between what is said and what is done is that the absolutistic paradigm fits many students' perspective on mathematics education.

It is important to realize that the students are brought up within a certain school discourse, which influences their expectations of the teacher's role compared to their own, which again influences the way they think things can be put forward and talked about in the classroom. For example, students often expect the teacher to take the lead, to decide and control what is going to happen, and to present the knowledge which he wants them to gain. These students would not insist on their own perspective because they expect that they are going to be evaluated and corrected by the teacher. This means that the students do not need to take full responsibility for their answers—the teacher will always provide the right algorithm or the right result. Such preunderstanding of school discourse naturally prevents all negotiation of meaning. Consequently it is necessary to change the way of communicating in a more general sense of the term in order to be able to negotiate meaning.[22]

The students' good reasons

To take the perspective of the students seriously, whatever it might be, is an alternative to that of bureaucratic absolutism in mathematics education, which has made the corrections of mistakes the highway to mathematical competence.[23] Looking at a student's mistakes is the same as evaluating his perspective about being wrong—often before the sources of the mistake are examined and explicated. But to point out mistakes is not the only way of searching for a mathematical truth. One might look at truth from a different point of view—namely, that student proposals are attempts to find a truth, and that he might have some *good reasons* for thinking and acting the way he does. To take the student's perspective seriously means to make his perspective and his good reasons visible in the classroom and use them in negotiation.

The students might not be mistaken if they do not understand a certain problem, or if they have solved an exercises in a different way, than the teacher expects them to do. They could also have a different perspective, which they might have good reasons for having. They might even be able to argue why they have got this and not that perspective. Perhaps negotiation will show that the student's perspective lacks consistency, or that it does not work within a certain framework, but this is less important. The important thing is that the perspectives of the students are often rejected by the teacher before being examined. The student's perspective could be utilized as a starting point for the learning process.

Let us again take a look at a dialogue, which in short form reveals bureaucratic absolutism:

(12)	Teacher:	Are you on the right track, Poul? I think your numbers are too big . they do not look quite . how can this become zero?
	Poul:	It can, if you say 5 minus 5.
	Teacher:	But it says 5 multiplied by 5 . I erase it then, because none of it is right.
	Poul:	Oh, no!
	Teacher:	Yes, then you must start crying [said fondly, ironically].
	Poul:	Good wasted efforts

Poul learns that mistakes should be erased immediately, because you cannot learn anything from them. But there might be a good point in examining Poul's idea of subtracting the numbers: "It can, if you say 5 minus 5." By asking him about his good reasons for subtraction, it would be possible to make him reflect upon his proposal, whether it is reasonable or not. The teacher does not give him this opportunity by referring to the textbook or by asking him to erase his result. Poul knows that he has to start all over again—he must try to work within the authoritative perspective of the textbook and the teacher.

Another example from the same class shows that the students are aware of this kind of correction procedure:

(13)	Teacher:	Are you on the right track, John?
	John:	I certainly hope so
	Teacher:	I hope so, too
	John:	I don't feel like erasing it all again.
	Teacher:	No, it also gets to look so bad

If the student is on the wrong track, he is supposed to erase his results, and he knows that. This is a silent agreement between teacher and student.

We have also seen examples of the students maintaining the view that they are wrong, and the teachers trying to make them look at the calculations they have made before erasing them:

(14)	Teacher:	Jannet!
	Jannet:	Yes, I'll erase it
	Teacher:	No, please don't erase it. It is good to see what you might be wrong about. Please don't erase it
	Jannet:	It isn't good to see at all
	Teacher:	Please don't erase it
	Jannet:	It isn't good to see.
	Teacher:	Jannet, please don't erase it
	Jannet:	It is completely wrong
	Teacher:	Not necessarily
	Jannet:	Yes it is.
	Teacher:	Now, look.

In this dialogue, Jannet holds the view that she has made a mistake, which she wants to erase. The teacher does not evaluate at once, because he wants to look at Jannet's calculations in order to find out if she is wrong, which is "not necessarily" so, and anyway it is worth considering what she "might be wrong about". In other words the teacher is willing to negotiate meaning and examine Jannet's good reasons for having solved the problem in a possibly wrong way.

This is the essential opening. The students' good reasons can change the purpose of the communication. "Mistakes" has dominated the classroom philosophy of mathematics. We do not suggest the introduction of "truth" as the ultimate perspective, but we suggest the students' good reasons as the cornerstone to such a classroom philosophy.

These reasons can become a departure point for a different teaching-learning process. It does not make sense to correct reasons in any bureaucratic form. The object of corrections should not be the output of any calculations, but the reasons for these calculations can be negotiated, in order eventually to challenge the students' good reasons. Good reasons can be expressed through argumentation. Therefore the most important thing is no longer whether the answers are right or wrong. The main purpose is to make students able to understand, reflect upon, and talk about mathematical problems and procedures.

The source of corrections has to be drawn towards the light. If negotiations of perspectives and meanings are to take place, a secret source of authority cannot set the standards. Dogmatism cannot prevail. Instead the students' good reasons must be confronted with other good reasons. It is also obvious that corrections cannot be stated in any general form. Good reasons are exactly that: a challenge to any supposed generality.

Our thesis was that the role of mistakes and corrections of mistakes reveals a working philosophy of mathematics which can be characterised as an "ideology of certainty". This philosophy results from a form of communication which we have characterised as bureaucratic absolutism. To challenge this absolutism, we must concentrate on the students' good reasons instead of concentrating on mistakes. These reasons could be a starting point for a different conception of mathematics. But, as emphasised, this is not any simple change.

Notes

1 Alrø & Lindenskov [1994]

2 This example was presented in 1992 by Trude Fosse at a seminar at Bergen University in Norway

3 Nickson [1992] p. 110.

4 In our classroom observations we did not find any explicit discussion of the nature of mathematics

5 The empirical examples in this article are part of a corpus of observation material made at three schools in a Danish city. Helle Alrø followed one 5th and two 6th grade mathematics classes during four weeks. Three of the four weeks, i.e. twelve lessons, were videotaped. One female and two male teachers participated. After each course the teacher was interviewed on the basis of the observations. The teachers involved are all experienced mathematics teachers. Some of the teaching may be called "classical", dominated by the textbook and the teacher's effort to explain what was outlined by the text. Others included free enterprises, like problem solving and problem posing processes. However, these differences are not in focus in what follows.

6 According to (radical) constructivism, knowledge cannot be transmitted, it can only be developed by the individual. Further, knowledge cannot copy the role of "reality". In fact, it becomes difficult to combine (radical) constructivism with any sort of ontology maintaining the existence of "something" which mathematics is about. Therefore knowledge, in the terminology of radical constructivism, takes a pragmatic turn. Knowledge must be seen in relation to the intentions and tasks of the individual. And knowledge becomes "useful schemes for interpreting behaviour". This also makes sense in relation to John Dewey's interpretation of learning which says that "truth" is not the primary concept in an epistemology; "inquiry" gets that position instead.

7 This formulation was used by Steiner Kvale in a lecture in Ebeltoft in 1992. In Kvale [1989] we find, with a reference to M. Salner, the following formulation: "Truth is constituted through a dialogue; valid knowledge claims emerge as conflicting interpretations are *'communicated and negotiated among people who share decisions and actions'* "

8 In what follows we talk about "mistakes' in the broadest way, including "real" mistakes as well as other sorts of (mis)conceptions

9 Here we refer to the situations that the quoted remarks belong to.

10 See the following sections 'Negotiation of perspectives' and "Negotiation of meaning"

11 Stubbs [1976] p. 99. The aspect of control shows itself in the so-called I-R-F structure (Initiation—Response—Feedback) of the classroom communication

12 Cf. Pimm's "cloze' procedure, in: Pimm [1987] p. 52ff

13 Bauersfeld [1988] p. 36.

14 We do not agree with the assumption, though, that a philosophy exists beforehand—that the teacher has got a certain philosophy that he consciously wants to transmit to his students.

15 Borba [1991] has drawn our attention to this notion.

16 This ideology concerns many aspects outside mathematics, including applications of mathematics as well as of formal methods in general. The ideology of certainty comes to support a *real* bureaucratic absolutism, expressed in the way *real* applications of mathematics and formal methods are exercised

17 Cf. Alrø & Skovsmose [1993].

18 This notion is also used by the German interactionists. Cf. Bauersfeld [1992], and Voigt [1994].

19 As mentioned before, a perspective is often implicitly shared. So it is possible to come to some sort of agreement, even though nothing is discussed. To share a perspective is, therefore, not a sufficient condition for students and teacher to challenge the conception of mathematics produced by bureaucratic absolutism. Much mathematics teaching is characterised by the harmony between the perspective of teacher and students, even in cases where the teaching by many standards cannot be characterised as "good" or "acceptable". This means that harmony between perspectives cannot simply be stated as a feature of "good" teaching

20 With reference to Erving Goffman's terms of "framing" and "keying", Götz Krummheuer [1983] talks about *Arbeitsinterim* as representing the asymmetrical teaching discourse, and *Modulation* as representing the process where the teacher and students try to attune their frames in order to come to a shared understanding in spite of their different backgrounds

21 Stieg Mellin-Olsen [1993] has also talked about dialogue as a tool for cognitive processes. The need for dialogue turns up when contrasting or complementary forms of knowledge meet: for example, when two theories contradict each other. Mellin-Olsen believes that contradictions stimulate thinking and learning processes. Therefore there is no need for the teacher to establish consensus or compromise between contradictions. They can exist side by side and inspire new ways of thinking. When the teacher presents two different theories to the student in an external dialogue this will cause an internal dialogue in the student, who then goes through a recognition and learning process by discussing with himself. This suggestion is interesting, because it allows one to look at different perspectives and keep them side by side without having to reject one of them or try to reach a compromise

Continued on page 22.

collaborative mathematical inquiry. It also suggests penpal letters can be a valuable addition to existing instructional practices in the mathematics classroom.

Acknowledgments

We would like to thank David Pimm for "listening mathematically" and contributing to the development of this project We would also like to thank the principal, parents, and students of Kerridsale Elementary and UBC preservice teachers for their support and participation

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Continued from page 8

22 We have to remember that mathematics education need not be the only source of ideas about what mathematics is about. Some strong ideological forces may be produced from outside school We must remember the phenomena described by Trude Fosse We have not in this article come up with any explanation of the phenomenon that even children in kindergarten seem to accept bureaucratic absolutism as a form of communication as far as mathematics is concerned

23 To take the student's perspective seriously also refers to the notion of "critical mathematics education" See Skovsmose [1994]

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