Symbol Sense: Informal Sense-making in Formal Mathematics

ABRAHAM ARCAVI

Prologue

Irineo Funes, the "memorious", was created from Borges's fantastic palette He had an extraordinarily retentive memory, he was able to remember ***everything****, be it the shapes of the southern clouds on the dawn of April 30th, 1882, or all the words in English, French, Portuguese and Latin Each one of his experiences was fully and accurately registered in his infinite memory On one occasion he thought of "reducing" each past day of his life to seventy thousand remembrances to be referred to by numbers. Two considerations dissuaded him: the task was interminable and futile. Towards the end of the story, the real tragedy of Funes is revealed He was incapable of general, "platonic" ideas. For example, it was hard for him to understand that the generic name dog included so many individuals of diverse sizes and shapes Moreover, he was also disturbed that the dog seen, and accurately memorized, at three-fourteen in the afternoon from a side view was the same dog he saw at three-fifteen from a front view. Precisely because Funes had such a monumental memory, he was unable to think, because, says Borges, "Pensar es olvidar diferencias, es generalizar, abstraer" [1989]—to think is to forget differences, to generalize, to abstract*

The argument

It is widely accepted that correct performance of arithmetical operations should not be the sole focus of arithmetic teaching and learning The knowledge of when to use an operation, and themes like "number sense" are nowadays receiving increasing attention. In general terms, "number sense" [NCTM, 1989; Sowder and Schappelle, 1989; Sowder, 1992] can be described as a "non-algorithmic" feel for numbers, a sound understanding of their nature and the nature of the operations, a need to examine reasonableness of results, a sense of the relative effects of operating with numbers, a feel for orders of magnitude, and the freedom to reinvent ways of operating with numbers differently from the mechanical repetition of what was taught and memorized

Is there a parallel situation with algebra? Does the mathematics education community no longer consider symbolic manipulations as the central issue in algebra instruction? The answer seems to be affirmative especially in the light of the emergence of symbolic manipulators and, in part, because many high school students make little sense of literal symbols, even after years of algebra instruction. Even those students who manage to handle the algebraic techniques successfully, often fail to see algebra as a tool for understanding, expressing, and communicating generalizations, for revealing structure, and for establishing connections and formulating mathematical arguments (proofs). Instruction does not always provide opportunities not only *not* to memorize, but also to "forget" rules and details and to be able to see through them in order to think, abstract, generalize, and plan solution strategies Therefore, it would seem reasonable to attempt a description of a parallel notion to that of "number sense" in arithmetic: the idea of "symbol sense". [1]

What is symbol sense? A first round

Compared to the attention which has been given to "number sense", there is very little in the literature on "symbol sense". One welcome exception is Fey [1990] He does not define symbol sense directly, but he lists "a reasonable set of goals for teaching" it, which include the "following basic themes":

- Ability to scan an algebraic expression to make rough estimates of the patterns that would emerge in numeric or graphic representation. . .
- Ability to make informed comparisons of orders of magnitude for functions with rules of the form $n, n^2, n^3, \ldots$, and n^k. . .
- Ability to scan a table of function values or a graph or to interpret verbally stated conditions, to identify the likely form of an algebraic rule that expresses the appropriate pattern. . .
- Ability to inspect algebraic operations and predict the form of the result or, as in arithmetic estimation, to inspect the result and judge the likelihood that it has been performed correctly. . .
- Ability to determine which of several equivalent forms might be most appropriate for answering particular questions . . . .

In this paper we attempt to extend the above situations both in number and content. Like Fey, we do not attempt to define "symbol sense"—the task is too complicated We think that "number sense" has been more widely discussed [e.g. Sowder and Schappelle, 1989], and yet a definition has proved to be extremely elusive. Therefore we will concentrate on describing and discussing behaviors which illustrate what we claim are examples of symbol sense

A methodological aside: Since we do not claim, either here or in the rest of this paper, to describe research on students' cognition and ways of learning, we can afford to be indulgent with the interpretations of the anecdotal data we provide. Thus we propose to dismiss the risks of misinterpreting (either by overcrediting or undercrediting) students' comments We bring the examples as mere illustrations of instances of what, in our view, symbol sense is

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Behavior #1: Making friends with symbols

We claim that having symbol sense should include the intuitive feel for when to call on symbols in the process of solving a problem, and conversely, when to abandon a symbolic treatment for better tools.

What are good friends for?

The following is a development of a lesson we have repeated several times. We start by introducing three-by-three magic squares, and then we present the first exercise: "Complete the empty cells to obtain a magic square with sum 9"

	3	
2		1

We follow this with

	2	
1		5

in which the sum is to be 6. This introduces negative numbers as legitimate entries. Then, depending on the class, we give one or two more simple magic squares, before the following, for which the requested sum is 8

	4	
2		2

Because all previous examples worked out so easily, this one, which does not, comes as a surprise. After checking their arithmetic more than once, or starting with a different cell, some students begin to suggest "mission impossible"; others continue doing the arithmetic guided by an implicit certainty that they must have made a mistake. Once all students are convinced that there is no solution, the natural question follows: "how come the others worked out and this one doesn't?" Clearly it must depend on the data, and students suggest many conjectures. For example, the last magic square did not work because the required sum is exactly the sum of the three given numbers. Discussions and conjectures, examples and counterexamples leave few students engaged.[2] In most cases, either it takes quite a long time before somebody timidly suggests the use of algebra, or we have to prompt its use directly (i.e. by asking "what tools do we have at our disposal in this situation to verify or refute the conjectures?"). On one occasion after algebra was mentioned, a skeptic even asked "How can letters help us here"?

We would claim that individuals who know how to perform algebraic manipulations, but do not consider the possible relevance of symbols to reveal the structure of a problem that has aroused their curiosity, have not fully developed their symbol sense. Having symbol sense includes the relevant invocation of algebra; or in other words, having symbols readily available as possible sense-making tools

A further indication of a lack of symbol sense was noticed when, in the process of solving the problem algebraically, some students were initially unable to recognize the solution even when they were in front of it. For example, on one occasion, in the process of completing the

2 $S-b-c$		1 $S-a-b$
3 $b+c-a$	$\underline{b}$	4
$\underline{a}$		$\underline{c}$

"general" magic square we reached the following stage: where a, b, and c are the given numbers, and S is the given sum. Completing the first cell (cell 1) involves the realization that its content should be expressed in terms of S, a and b. Some beginning algebra students may be tempted to introduce a new variable, because either they are not fully aware of what they are looking for or they do not have enough experience with how symbols display relationships. On this occasion, the cells indicated by 1, 2 and 3, were completed without trouble, and cell 4 was completed by the column sum to obtain $a + b - c$. At this point somebody noticed that the expression for the sum of the middle row is $3b$ and does not contain S. It took a while for the students in that class to realize that if we *want* the sum to be S, then $S = 3b$ expresses precisely the condition sought.

Thus, we would like symbol sense to include the invocation of symbols where appropriate and the recognition of the meaning of a symbolic solution. Perhaps we would like to include a bit more than that as well. Even when symbols are used, and the solution they yield is recognized, it would be desirable that students appreciate the "power of symbols": only with the use of symbols can a conjecture of an argument be conclusively accepted or dismissed

Another simple and clear example of this is the following: Consider any rectangle. What would happen to its area if one of its dimensions were increased by 10% and the other decreased by 10%? Students' initial reactions include: "There is no change" (probably because of the "compensation"), "The change depends on which dimension is increased and which is decreased". Simple numerical calculations show that there is an apparent decrease in all cases, but it is only when we resort to symbols that the result becomes obvious and conclusive. If a and b are the original dimensions, then the area of the new rectangle will be either $1.1a \times 0.9b$ or $0.9a \times 1.1b$, namely $0.99ab$, in both cases. In a beautifully concise way the symbols express the whole scenario of the problem: Firstly, the area always decreases; secondly, it always decreases by 1%; and, thirdly, the result is independent of which dimension is increased and which is decreased.

We would like students to "see" the symbolic solution and to be convinced by it, even when it contradicts our initial intuitions about the problem. Experts to whom we talked regarded $0.99ab$ not only as the solution, but also as "carrying" its explanation. Thus, we claim that symbol sense should include, beyond the relevant invocation of symbols and their proper use, the appreciation of the elegance, the conciseness, the communicability and the power of symbols to display and prove relationships in a way that arithmetic cannot.

When friends are less friendly

If symbol sense requires us to invoke symbols when they are appropriate or indispensable, so symbol sense requires us to abandon symbols when we are likely "to drown" in the technical manipulations.

We consider two examples.

> "For what values of a does the pair of equations
> $$x^2 - y^2 = 0$$
> $$(x - a)^2 + y^2 = 1$$
> have either 0, 1, 2, 3, 4, 5, 6, 7, or 8 solutions?[3]

As we mentioned before, problems like this which are stated in algebraic terms, "invite" or induce, quite strongly, an algebraic solution. Therefore it is not surprising that many students plunge into symbol pushing without (or in spite of) realizing that the algebraic manipulation could be quite laborious and prone to error. The decision to discard the almost unavoidable initial temptation to proceed mostly symbolically, in favor of the search for another approach, requires a healthy blend of "control" with symbol sense. Briefly stated control is:

> "a category of behavior [which] deals with the way individuals use the information potentially at their disposal. If focuses on major decisions about what to do in a problem, decisions that in and of themselves may "make or break" an attempt to solve a problem. Behaviors of interest include making plans, selecting goals and subgoals, monitoring and assessing solutions as they evolve, *and revising or abandoning plans when the assessments indicate that such actions should be taken.*" [Schoenfeld, 1985 emphasis added]

In our example, the managerial decision to change a course of action involves more than that: it is also motivated and driven by a sense of aesthetics, elegance, efficiency, as well as an appreciation (or even belief) that mathematical work involves much more than the stoicism of embarking on "hairy" symbolic manipulations. The idea is to develop reactions of the sort: "this involves too much hard, technical and uninteresting work, there *must* be another approach"

The other approach may emerge from regarding the problem in a different way, or by changing the representation. In this case, a Cartesian graph and subsequent geometrical considerations suggest another way to look at this problem: the number of intersections between the two diagonals of the Cartesian plane ($x^2 - y^2 = 0$, namely $y = \pm x$) and a family of circles of radius 1 whose centers lie on the x-axis. From then on the solution is rather easy.

Similar considerations for abandoning the algebraic manipulations in favor of other representations can be made when solving, for example, $|x - 2| > |x - 6|$. The algebraic treatment of this inequality involves heavy use of logical connectives, lots of technical work and a high probability of making mistakes. Instead of pushing symbols, acting with symbol sense would imply "recovering" meanings: $|x - 2|$ is the distance of any number from 2, thus what the problem requires are the numbers whose distance from 2 is greater than their distance from 6. A simple number line diagram, or a mere verbal approach, can solve the problem. Another possible approach is to regard $|x - 2|$ and $|x - 6|$ as functions of x, in which case the solution to the problem is also immediate from the corresponding Cartesian graphs [Friedlander and Hadas, 1988].

In the above two examples, symbol sense leads to the premonitory feeling that persistence with a symbolic solution would cause hard work. Moreover, it would include the tendency to try other ways of representing the problem in the belief that more elegant and straightforward approaches may exist and should be considered. In other words, symbol sense includes the feeling for when to invoke symbols and also for when to abandon them.

Behavior #2: Manipulations and beyond; Reading through symbols

Solving simple algebraic equations does not call for much more than standard manipulation to yield the desired result, and the only meaningful "reading" required is to make sense of the answer in the form $x = \ldots$

In fact, from one point of view, this is one of the strengths of symbols—they enable us to detach from, and even "forget", their referents in order to produce results efficiently. Alfred North Whitehead [1911] pointed out the "enormous importance of a good notation" and the nature of symbolism in mathematics:

> "... by the aid of symbolism, we can make transitions in reasoning almost mechanically by the eye, which otherwise would call into play the higher faculties of the brain. It is a profoundly erroneous truism, repeated by all copy-books and by eminent people when they are making speeches, that we should cultivate the habit of thinking of what we are doing. The precise opposite is the case. Civilization advances by extending the number of important operations which we can perform without thinking about them. Operations of thought are like cavalry charges in a battle—they are strictly limited in number, they require fresh horses, and must only be made at decisive moments." [Whitehead, 1911]

However, we believe that Whitehead would also have agreed with Freudenthal [1983]:

"I have observed, not only with other people but also with myself . . that sources of insight can be clogged by automatisms. One finally masters an activity so perfectly that the question of how and why is not asked any more, cannot be asked any more, and is not even understood any more as a meaningful and relevant question."

Interrupting a mechanical symbolic procedure in order to inspect and to reconnect oneself to the underlying meanings, could be, to use Freudenthal's language, a good "unclogging" exercise.

Reading instead of manipulating

For example, while simplifying a linear equation in order to obtain a solution, a student arrived at $3x + 5 = 4x$ Instead of proceeding mechanically, namely "subtracting-$3x$-from-both-sides", she stopped and switched to a different mode: symbol reading She observed that in order to obtain $4x$ on the right from the $3x$ on the left, one would have to add an x, therefore the actual addend, 5 must be the value of x. Even though the standard method and hers are mathematically indistinguishable, psychologically there is a subtle but important difference. We suggest that interrupting an almost automatic routine in order to read and notice a symbolic relationship as in this case, is a small but healthy instance of symbol sense.

Reading and manipulating

The solution of simple algebraic equations, as usually posed in standard texts and as usually taught in the classrooms, automatically arouses an "instinct" for technical manipulation. Thus it requires a certain maturity to defer the "invitation" to start solving, for example, $(2x + 3)/(4x + 6) = 2$, and instead to try to "read" meaning into the symbols In this case, one might notice that, whatever x, since the numerator is half the denominator, this equation cannot have a solution. We claim that this *a priori* inspection of the symbols with the expectancy of gaining a feel for the problem and its meaning, is another instance of symbol sense.

One of our students went even further After noticing, as above, that there was no solution, he said: "OK, so this problem has no solution, but what if I "solve" it anyway?" Probably, this student, for whom to solve meant to apply the mechanical procedures leading to $x = \ldots$, seemed to be expressing his need to feel the way in which the algebra expresses the absence of solution Unfortunately, the algebra is not very forthcoming: technical manipulation yields without protest $x = -1\frac{1}{2}$ Our student was puzzled by the contradiction, and it took him a while to resolve it. When he substituted $x = -1\frac{1}{2}$ he realized that this is the one value one is *not* allowed to substitute, and thus confirmed that his "senses" were right. He did not say much; however, we suggest that he was learning something about "the language of symbols" and how tricky it can be to just "push them around" without sensible criteria for reasonableness

Reading as the goal for manipulations

There are situations in which reading through the symbols is essential For example: "What can you say about the numbers resulting from the differences between the third power of a whole number and the number itself $[n^3 - n]$? (This problem is borrowed from Fischbein and Kedem [1982]) Using a standard Pòlya heuristic of considering special cases, one may notice that the numbers obtained are always multiples of 6 Symbols are necessary to prove that this is always the case However, a mere algebraic manipulation, for example, the factorization $n^3 - n = n(n - 1)(n + 1)$, in itself does not help very much. Only by *reading* the meaning of the symbols, to realize that the right-hand side represents the product of three consecutive integers, and hence that at least one of them must be even and one of them must be a multiple of 3, will the argument be completed.

The above examples illustrate an aspect of symbol sense which consists of the search for symbol meaning, whether it is essential to the solution of the problem, or merely adds insight

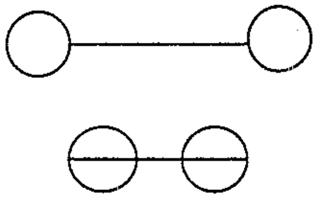
Reading for reasonableness

As an example of this aspect, we re-examine one of the classics of the mathematics education research literature: the students and professors problem.

"Write an equation using the variables S and P to represent the following statement: There are six times as many students as professors at this university." [See, for example, Clement, 1982]

The findings in Clement [1982] show that more than 30% of the 150 freshman engineering students who answered the test failed to solved this problem correctly. The typical wrong answer reported was $6S = P$. These findings are consistent with findings in other studies that investigated the ways students solve this problem.

We would suggest that, at least for some of the students, this error is not a manifestation of a deep misconception about the notion of variable, as it is not a deep misconception to believe that the two parallel segments in the following picture have different lengths.



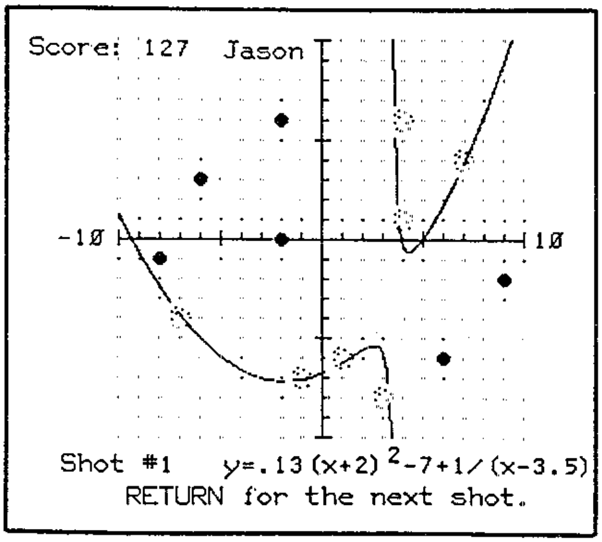
In the picture, the circles act as a distracting pictorial frame of reference which strongly biases our perception of the lengths of the two segments Similarly, as Clement himself points out, in the students and professors problem, there is a language distractor, i.e. the order of the key words (six times as many students) which may bias our sense towards a "word-by-word translation" as $6S = P$ Thus, in this case we do not claim that having symbol sense necessarily implies avoiding the error. Many students and even teachers may fall into the linguistic trap while building a symbolic model for problems of this kind and make the "reversal mistake". What we do claim is that, in this case, symbol sense would consist of developing the healthy habit of rereading and checking (e.g by simple substitution) for the reasonableness of the symbolic expression one constructs. Being aware and alert that one may be a victim of "symbolical illusions" may not avoid a misperception, but may reinforce the need for checking and thus overcoming it

Behavior #3: Engineering symbolic expressions

The computer game Green Globs [Dugdale & Kibbey, 1986] consists of a screen displaying a Cartesian grid with randomly placed "globs". The object of the game is to input a function in algebraic form whose graph will "hit" the largest possible number of "globs" in the grid. The player has to imagine a desired graph, whose corresponding algebraic form has to be engineered, since this is the only way in which the computer will draw the graph. Dugdale [1993] reports an example of an interesting student behavior with this game. In order to increase the number of globs hit, the student modified a simple quadratic function as follows

$$y = 13(x+2)^2 - 7 + \frac{1}{x-3.5}$$

The graph obtained [Dugdale, 1993] shows a copy of the screen



By adding the rational term $\frac{1}{x-3.5}$ to the function $y = 13(x+2)^2 - 7$ whose graph was first envisioned as a parabola, the student showed considerable symbol sense. He was able to craft an algebraic expression for a graph he wanted to be a parabola almost everywhere, but behaves like a vertical line in a desired vicinity. Thus he added a rational term which does not affect significantly the graph except when x is close to 3.5, but in the vicinity of its discontinuity, the graphs "breaks" and "jumps" to pick up three extra globs.

We regard this reasoning as showing a higher cognitive level of symbol sense than that illustrated in Behavior #2. There, we suggested that, given the symbols, symbol sense should include "reading" meaning from them. We now propose that symbol sense also includes: firstly, an appreciation that an ad hoc symbolic expression can be created for a desired purpose, and that we can engineer it; secondly, and more specifically, the realization that an expression, with certain characteristics (in this case a rational term) is what is needed; finally, symbol sense should include the ability to engineer that expression successfully.

Behavior #4: Equivalent expressions for "non-equivalent" meanings

We start this section with two examples. When dealing with the formula for calculating the arithmetic mean between two numbers, a student observed that a simple symbol manipulation transforms $(a + b)/2$ into $a/2 + b/2$. But, she did not stop at this, she came up with a new conceptualization of the mean of the two numbers: "It is a number made up by the half of one of the numbers and half of the other". This reconceptualization emerged from regarding equivalent symbolic expressions not as mere formal results, but also as possible source of new meanings.

The following is another example of a similar behavior "Take an odd number, square it and then subtract 1. What can be said about the resulting numbers?" The problem can be represented as follows: $(2n - 1)^2 - 1$. Then we may proceed to obtain the equivalent form $4n^2 - 4n$ in order to reach a general conclusion. At first sight, the conclusion is that the resulting number is a multiple of 4. However, by rearranging the symbols we first obtain $4n^2 - 4n = 4n(n - 1)$, and if we now read into the symbols, we may notice that the result is always a multiple of 8 (since n and $n - 1$ are consecutive integers and one must be even). Further rearrangement of the symbols reveal even more. If we write $4n(n - 1)$ as $8[n(n - 1)/2]$, it is not only more evident that the resulting numbers are multiples of 8, but also that they are very special multiples of 8: those in which the other factor is a triangular number

These two examples, share a common "story". Both gained richer meanings emerging from equivalent expressions derived by symbol manipulations. We suggest that the feel for, and the confidence in, symbols which guide the search for new aspects of the original meanings constitute another facet of symbol sense.

Behavior #5: The choice of symbols

When we translate a situation into symbols, one of the first steps is to choose what to represent and how. That choice, as we see in the following examples, can have crucial effects on the solution process, and on the results.

First consider the last example in the previous section. If the given odd number is represented by n instead of $2n - 1$, the expression obtained is $n^2 - 1$. Although the choice of variable is certainly legitimate, the resulting findings are likely to be less informative. Thus if we factor $n^2 - 1 = (n - 1)(n + 1)$, we will be able then to read from this expression that the resulting number is always the product of two consecutive even numbers. One of these numbers must be a multiple of 4, thus we can conclude that the result is a multiple of 8. However, which multiples of 8 are obtained is not clear. The choice of $2n - 1$ to represent an odd number, as opposed to n, shows that more of the given information can be built into the choice of symbol, and thus the final outcome can display more aspects of the "structure" of the situation.

Other simple examples of alternative choices of variables are: electing to represent the sum of two negative numbers as $a + b$ or as $-a - b$ (depending on whether we select a and b to represent negatives or natural numbers), or electing to represent a rational number as a or as p/q (where p and q are natural numbers). Symbol sense helps to make the appropriate choice by taking into account the goal of the problem.

The choice of symbols may not only obscure part of the situation, as in the first example above, but it may also impede the solution altogether. Consider the following problem.

John went to the bank to cash a check for a sum under \$100. The cashier confused the cents with the dollars. (for example, if the check was for \$19.45, he paid John \$45.19). John got the money and after spending \$3.50, he realized he had exactly twice the amount written on the check. How much was the check for? [SNARK, 1977].

If we elect to represent the amount of the check as x, any progress is extremely unlikely. On the other hand, if we represent each of the four digits by a variable, things will soon get very complicated. The optimal choice would seem to represent the dollars by one variable and the cents by another, each variable standing for a two digit number.

We do not wish to imply that an unfortunate choice of variable at the beginning of a solution process necessarily indicates lack of symbol sense, but what we would like to expect is awareness of at least three issues. Firstly, the freedom to represent the problem as one wishes. Even when the symbols represent the same "kind" of number, there may be different ways to choose them. For example, three consecutive integers, can be represented either as n, $n + 1$, $n + 2$ or as $n - 1$, n, $n + 1$ or as $n - 2$, $n - 1$, n. The choice may well simplify the calculation, if not the final result. Secondly, the realization that the initial choice of symbols is in no way binding, and that the problem may be re-represented if one wants to, or if the original choice is seen as unproductive. Thirdly, in some cases, as in our first example, we would also like to include in symbol sense some premonitory feeling for an optimal choice of symbols.

Behavior #6: Flexible manipulation skills

Even in those contexts in which we fully agree with Whitehead that one should forget meaning, and be able to perform manipulations mechanically, there still should be a guiding "technical" or "formal" symbol sense in control of the work. The correct manipulation of symbols consist of much more than a dry abidance by the rules: there are many aspects of sensing the symbols, *qua* symbols, even when we reasonably forget or ignore their referents. For example: the realization of a potential circularity in symbol manipulation, the "gestalt" view of some symbolic expressions, and manipulations directed towards formal "targets".

Circularity

By circularity we mean the process of symbolic manipulation which results in an obvious or tautological identity, which is uninformative and unproductive. The ability to anticipate such circularity is a manifestation of symbol sense, and when the circularity is not anticipated, symbol sense would prevent paralysis, i.e. it would trigger a natural response to search for other approaches. See, for example, the quote from Wenger [1987] below.

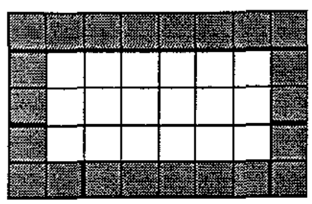
"Gestalt"

Having a "gestalt" view is sensing the symbols not only as a concatenation of letters, but as arranged in a certain form, as, for example, in Wenger [1987]:

> "If you can see your way past the morass of symbols and observe that equation #1 ($v\sqrt{u} = 1 + 2v\sqrt{1+u}$ which is required to be solved for v)is linear in v, the problem is essentially solved: an equation of the form $av = b + cv$, has a solution of the form $v = b/(a - c)$, if $a \neq c$, no matter how complicated the expressions a, b, and c may be. Yet students consistently have great difficulty with such problems. They will often perform legal transformations of the equations, but with the result that the equations become harder to deal with; they may go "round in circles" and after three of four manipulations recreate an equation that they had already derived .... Note that in these examples the students sometimes perform the manipulations correctly ... "

The first part of the above quote is an example of what we have called "gestalt"; the remarks in the second half relate to circularity. In the same paragraph Wenger also says that students "often appear to choose their next move almost randomly, rather than with a specific purpose in mind". This remark leads us to the next subsection.

Formal targets

> "A rectangle (the term includes a square) is drawn on graph paper as shown and its border cells are shaded.
>
> 
>
> In this case the number of shaded cells in the border is not equal to the number of unshaded cells in the interior. Is it possible to draw a rectangle such that the border—one cell wide—contains the same number of cells as the interior?" [Problem 87 from Longley-Cook, 1965, as it appears in Gardner, 1983]

The following is a solution produced by a mathematician. He started by equating $2a + 2(b - 2)$, the total number of cells in the boundary, to $ab - [2a + 2(b - 2)]$, the number of cells in the interior. After reducing terms, he obtained the equation $ab - 4a - 4b - 8 = 0$ (1). This expression prompted him to search for a possible factorization, from which it might be easy to read all possible integer values of a and b. His first trial yielded $a(b - 4) - 4(b - 2) = 0$, but the factorization cannot be completed. However, he noticed that if he transformed $(b - 2)$ into $(b - 4)$ the factorization can be completed. He did so by adding 16 to both sides of (1) yielding $(b - 4)(a - 4) = 8$, from which he obtained the only two different rectangles which solve the problem, a 6×8 and a 5×12.

This solution illustrates symbol sense in two stages: first the solver envisioned a symbolic target and its form which is easy to handle and interpret. Then, he purposefully chose the formal manipulations needed to obtain the selected target.

Behavior #7: Symbols in retrospect

In one class (at the undergraduate level) we observed a very different solution to the above rectangle problem. After many students spent some time "pushing" the symbols without making much progress, someone came up with the solution we brought above. At that point, a mathematician who was auditing that class, decided to present her own solution which (surprisingly?) did not make use of symbols at all. According to her own testimony, her inspiration came from her expertise with *origami*, the Japanese art of paper folding. She mentally "folded" one border row onto the immediate adjacent interior row, and thus the number of cells in the interior and the border matched, except for two extra border cells. By repeating with the other border row, two further unmatched cells were added. Then she folded the two border columns (without the corner cells already folded in) getting four additional unmatched cells to make a total of 8 (two extra cells from each corner). Now, in order for the number of cells in the interior to match completely the number of cells in the border, all she needed was to match these 8 cells. For this purpose she needed 8 interior (unmatched) cells which form a rectangle. And there are only two possibilities: a 1×8 and a 2×4. By "unfolding" the *origami* in each case, she obtained the two original rectangles the problem is asking for. The verbal description may be a bit laboured but it becomes very clear when one does the actual paper folding. This is a very ingenious and beautiful solution in itself: completely general, without any symbols, and visually appealing.

One of the students in that class noticed what for him was an interesting and surprising feature of this alternative solution: even though it does not use any symbols, it is closely related to the algebraic solution previously presented in that class (and described in behavior #6). This student remarked that when we solve an algebraic equation which models a problem, we detach ourselves from the meaning of the symbols and their referents. Intermediate steps, are usually not regarded as having meaning with reference to the situation, they are rather mechanical (and thus efficient) manipulations towards the solution. However, in this case, the symbolic and the *orgami* solution have a significant point of contact. In the algebraic solution the 8 seemed to emerge as a mere artifact of the symbolic manipulation in the process of obtaining the formal target: the factorized equation $(b - 4)(a - 4) = 8$. However, *a posteriori*, the *origami* solution gave the 8, and the whole equation, meaning.

When one solves a problem by building a mathematical model of a situation, symbols and meanings are usually connected at the beginning (when the model is established) and at the end (to interpret the result in terms of the situation modeled). At intermediate steps one usually makes progress by performing formal derivations whose meanings, in terms of the situation, are disregarded. We suggest that symbols "spoke loudly" for this student, because he was able to recognize meaning in them at intermediate steps, where it is neither needed, nor usual, to do so. We claim that this behavior is a nice instance of symbol sense.

Behavior #8: Symbols in context

Consider, for example, the case of the general linear relationship $y = mx + b$. Even though x, y (the "variables") and m, b (the "parameters") represent numbers, the kinds of mathematical objects one obtains by substituting in them are very different. In terms of the Cartesian plane, choosing numerical values for x and y, fixes a point from the set of all points, and numerical values for m and b fix a line from the set of all lines. Thus $y = b$ can be interpreted in two different ways depending on the context. If it is the result of substituting $x = 0$ in $y = mx + b$) then what we have found is the corresponding y-coordinate (or a family of y-coordinates), i.e. the point where a line of the form $y = mx + b$ intersects the y-axis. But if, on the other hand, $y = b$ is obtained as the result of substituting $m = 0$ in $y = mx + b$, what we have found is the family of lines of slope zero.[4]

In this section we suggest that a desirable component of symbol sense consists of the *in situ* and operative recognition of the different (and yet similar) roles which symbols can play in high school algebra. That recognition entails sorting out the multiplicity of meanings symbols may have depending on the context, and the ability to handle different mathematical objects and processes involved [Sfard, 1992; Moschkovich, Schoenfeld and Arcavi, 1993].

We bring two examples of 9th grade students who came for help with their homework. In both cases the students were confused by the contextual meaning of the symbols. In the first case, the student was able to resolve her difficulties by resorting to her common sense applied to symbols. In the second case, the situation was less fortunate.

In the first case the problem was: "Find the coordinates of the center of the circle through (a, b), (-a, b) and (0, 0)". One may claim that in this problem a first sensible step would be to "see" the three points behind the symbols, their location, their symmetry, etc., and thus temporarily abandon symbols in favor of a graphical approach which will help to find part of the solution: the center of the circle should lie on the y-axis. Our student did not proceed graphically, she stayed with symbols. She initially had some difficulty in writing down the relevant equations because she was used to the conventional use of the letters a and b (In the general equation of a circle in the form $(x - a)^2 + (y - b)^2 = r^2$, with which she was familiar, a and b are the coordinates of the center, whereas here they are given as coordinates of points *on* the circle.) However she overcame the difficulty and stated the following three equations, with (m, n) as the coordinates for the center she was looking for:

$$(a - m)^2 + (b - n)^2 = r^2$$
$$(-a - m)^2 + (b - n)^2 = r^2$$
$$m^2 + n^2 = r^2$$

Then she equated the first and the second equation, cancelled terms and obtained $ma = 0$ She knew that the equation implies that either $m = 0$ or $a = 0$, but she did not know which one was the result she was looking for. The effort devoted to the symbol manipulation seemed to have made her forget the meaning of the symbols, or perhaps her purpose in solving the equation In order to resolve the impasse, she decided to check the implications of both possibilities. After a while she noticed that if $a = 0$ there would be no circle (the three given points then lie on the y-axis), and then she also remembered that a was given in the problem as a general point, and thus she was not in a position to choose its value. Thus she concluded correctly that $m = 0$ is the result sought, and that it means that the center of the circle has to be somewhere on the y-axis

This story seems to suggest that symbol sense does not necessarily imply an immediate recognition of the roles of symbols: in some situations this recognition can be very difficult for students But symbol sense can manifest itself in the resourcefulness and common sense which may help students recognize mistakes and start to clear up confusions. As in the case of a symbolical illusion (described in behavior #2), symbol sense should also include the resourcefulness to extricate oneself from confusion by resorting to whatever tools one has available to regain symbol meaning.

The following is an example of a less fortunate instance in which a student was not able to make much sense of the different roles symbols can play. The problem from his class text [Resnick, 1991] was: "In the following expressions, find, if possible, a number which when substituted for d, a linear function will be obtained" Twelve expressions followed this general instruction, and the last one was $y = (x^2 - 4)/(d - 2)$ Provided one has a clear distinction between the different roles of symbols, the problem seems straightforward: no matter which number one substitutes for d (and certainly not 2), this expression will never become a linear function. We were surprised by what this student did. Displaying a "gestalt" view (as described in behavior #6) he decomposed $x^2 - 4$ into $(x - 2)(x + 2)$, because he noticed that $x - 2$ and $d - 2$ can be cancelled by making $d = x$ ($d \neq 2$) to obtain a linear function, $x + 2$ In this case, his otherwise healthy technical symbol sense which enabled him to envision the cancellation, interfered with the distinction between different roles symbols can play He was not disturbed by the legitimacy of his move, nor did he get direct feedback from the problem itself to alert him that something was wrong; quite the contrary, he seemed happy to achieve the goal requested. Thus he did not question the meaning (or meaninglessness) of his action Our reaction at the time was not very fortunate either and we did not succeed in calling his attention to what he had done It may be that the subtleties involved in the "illegal" step $d = x$ were beyond the scope of what he was able to cope with at the moment.

What is symbol sense? A second round

If we inspect the common features of the above behaviors, we may now be in a better position to attempt to characterize symbol sense. In general, we would say that symbol sense is a complex and multifaceted "feel" for symbols. Paraphrasing one of the definitions provided by the Oxford Encyclopedic English Dictionary for the word "sense", symbol sense would be a quick or accurate appreciation, understanding, or instinct regarding symbols.

Following the stories above, symbol sense would include:

—An understanding of and an aesthetic feel for the power of symbols: understanding how and when symbols can and should be used in order to display relationships, generalizations, and proofs which otherwise are hidden and invisible.

—A feeling for when to abandon symbols in favor of other approaches in order to make progress with a problem, or in order to find an easier or more elegant solution or representation

—An ability to manipulate and to "read" symbolic expressions as two complimentary aspects of solving algebraic problems On the one hand, the detachment of meaning necessary for manipulation coupled with a global "gestalt" view of symbolic expressions makes symbol-handling relatively quick and efficient. On the other hand, the reading of the symbolic expressions towards meaning can add layers of connections and reasonableness to the results.

—The awareness that one can successfully engineer symbolic relationships which express the verbal or graphical information needed to make progress in a problem, and the ability to engineer those expressions.

—The ability to select a possible symbolic representation of a problem, and, if necessary, to have the courage, first, to recognize and heed one's dissatisfaction with that choice, and second, to be resourceful in searching for a better one as replacement.

—The realization of the constant need to check symbol meanings while solving a problem, and to compare and contrast those meanings with one's own intuitions or with the expected outcome of that problem

—Sensing the different roles symbols can play in different contexts

At this point we would like to elaborate on two important limitations of the above characterization of symbol sense: firstly, the above "catalogue" is far from being exhaustive, and secondly, there is much more to symbol sense than a catalogue, regardless of how complete it might be.

About the incompleteness of the catalogue

We collected instances from our own experience by informally watching students and teachers doing algebra, and we believe that the collection can be helpful as a first step towards describing symbol sense. We (teachers, mathematics educators, and researchers) can use it not only as it is, but mainly as a way to raise our awareness of similar behaviors related to the several aspects of symbol sense. Thus it can serve as a springboard for further observations. We maintain that the work towards a satisfactory "definition" of symbol sense should blend theoretical and philosophical ideas with detailed empirical observations of novice and expert behaviors. There is no doubt that careful observation of problem solving behaviors will enhance and extend our descriptions

A catalogue in itself is not enough

We do not wish to convey the impression that defining symbol sense is a matter of merely expanding a catalogue. Even though it might be very tempting to have a comprehensive check list to evaluate the absence or presence of symbol sense, that would be too simplistic. Symbol sense involves complexities and subtleties of which we may not be fully aware. We suggest, as a beginning, pointing to some of those complexities. For example:

—Different aspects of symbol sense may interact with each other in many ways, not necessarily positively. For example, a global (and otherwise healthy) view of the symbols in $y = (x^2 - 4)/(d - 2)$ tempted a student to simplify and to conclude that linearity will occur when $x = d$

—There are instances in which students who do not seem to conform to the "expected" or "correct" behavior, are in fact capable of overcoming their difficulties with symbols and they may do so by resorting to whatever tools they have available. Thus the student who might have been regarded as not displaying symbol sense in sticking to symbols to find the coordinates of the center of a circle through (0, 0), (a, b) and $(-a, b)$ (as opposed to using graphical considerations), did in fact show considerable symbol sense in dealing her way with $ma = 0$.

—A catalogue for symbol sense is inevitably incomplete because students will have little sense for a representation (in our case symbols) in isolation, if they are not able to carry the meanings of those symbols flexibly over to other representations. And, conversely, moving back and forth among different representations will result in enhancing the understanding of each particular one. Symbol sense will grow and change by feeding on and interacting with other "senses", like number sense, visual thinking, function sense, and graphical sense

These and other potential facets of symbol sense need to be taken into account in addition to the enhancement of a list.

Regarding the inner nature of symbol sense as a feel, it may be illustrative to make an analogy to the "physiological" meaning of the word *sense*, as brought by The Oxford Encyclopedic English Dictionary: "any of the special bodily faculties by which sensation is aroused". We can adapt this to symbol sense as: "any of the special mathematical faculties by which meaning is aroused". It would be a desired goal for mathematics education to nurture symbol sense so it becomes an indivisible part of our mathematical took kit, similar to the way in which our physiological senses are an integral part of our biological being. The analogy suggests that symbol sense should become part of ourselves, ready to be brought into action almost at the level of a reflex. But also, just as when our senses fail us we tend to develop substitutes, our symbol sense should tend to develop ways to overcome "failures" (for example, the awareness to overcome by any means situations involving "symbolical illusions").

Finally, we suggest placing the question "What is symbol sense?" in the larger context of mathematics thinking and learning. Symbol sense is the algebraic component of a broader theme: sense-making in mathematics. Sense-making in, and with, mathematics seems to be the goal at large for most, if not all, of mathematics education, as it is reflected in the spirit of the NCTM Standards. It implies the meaningfulness of all the activities one engages in, the empowerment it provides to understand and manipulate situations, and the usefulness of the mathematical tools to make progress in and beyond mathematics.

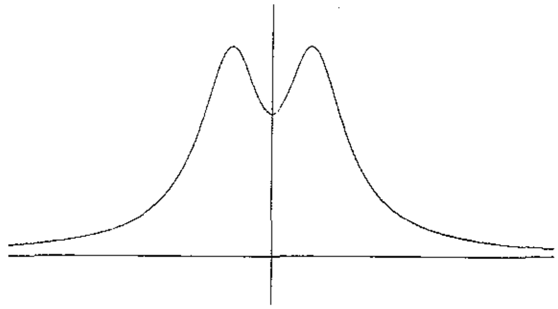
Instructional implications

It may be presumptuous to describe/prescribe full-fledged instructional implications of a not yet fully-developed idea such as symbol sense. Moreover, it can be argued that even if a more satisfactory characterization of symbol sense is achieved, we still need some answers to many open questions in order to discuss the instructional implications. Some of the open questions are not trivial at all: How do people acquire symbol sense? What is the underlying knowledge required? What is the role of the technical manipulation of symbols? Do drill and practice precede, are they concurrent with, or do they impede the development of symbol sense? Is symbol sense an expert-like posture or can it be expected from novices as well, and to what extent? And so on.

Nevertheless, and in spite of the unknowns, we do think that we are at a point at which a discussion of instructional implications can and should be initiated. In the following we suggest implications which can be derived from our descriptions.

1) We begin by proposing that symbol sense is at the heart of what it means to be competent in algebra and the teaching of algebra should be geared towards it. Whereas it may be easy to understand and accept (even if often forgotten) that being competent in arithmetic includes much more than performing flawlessly the computational algorithms, when it comes to algebra one is more likely to encounter the belief that knowing the formal manipulation rules is a central goal of algebra instruction. Therefore we suggest that a (probably obvious but nevertheless crucial) first implication for teaching, is that symbolic manipulations should be taught in rich contexts which provide opportunities to learn when and how to use these manipulations.

2) We follow by referring to the ways in which we can harness technology in the service of the development of tasks and problems which, in the hands of a skillful teacher, have the potential to foster symbol sense. These tasks should rely on the computational power which leaves free mental resources for developing and enriching meanings and connections. Consider, for example, the following problem, which is not usually found in traditional textbooks "Aided by a graphing calculator or a graphing tool, find one algebraic expression for the function in the following graph "[5]



This task is not easy, and many students (and teachers) rebel against it because there is no readily available routine with which to start. By attempting to craft a symbolic expression for the function, many questions arise. Can the function be a polynomial? Why not? is it a rational function? If so, what are the degrees of the numerator and denominator? Given that the function is defined all over the real domain, what does it mean in terms of its denominator? How does the symmetry of the graph affect the choice of functions? Once the general shape of the curve is envisioned, which parameters should be changed in order to make it more like the given graph? And so on In our experience, this project keeps expert teachers busy for about an hour and a half, during which the graphing technologies are used to check the expressions conjectured and then to adjust them. Different kinds of knowledge, including informal knowledge, are engaged in surprising ways. We invite readers to work for a while on this problem and to notice how many facets of their own symbol sense it stimulates. We suggest that more tasks of this nature should be developed and tried.

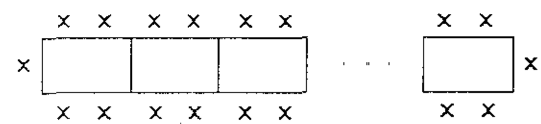
3) We do not wish to imply from the above that what we need to do is to dismiss traditional curricula completely and to concentrate on the creation of novel tasks or problems which make use of technologies. Instead, we claim that, whereas new problems and tasks are needed, they in themselves will not "embody" symbol sense. No matter how interesting or innovative a task may appear, it will be the *activity* the students are led to engage in that will determine if it supports the construction of symbol sense. And conversely, a seemingly traditional or even dull task may be a potential source of insightful discussions. For example, consider one of the above problems which is certainly traditional. "Find the coordinates of the center of the circumference through (a, b), (-a, b) and (0, 0)" However, instead of letting students jump immediately to pose and manipulate equations, a teacher can direct the activity towards collectively embarking on some sense-making activities, for example, drawing a Cartesian sketch Such activity should promote discussion during which students may reveal implicit aspects of the problem which will not be apparent if all they do is manipulate equations in order "to find a solution". for example, during the discussion they may find that:

—From symmetry it looks as if the center should lie somewhere along the y-axis, so the x-coordinate of the center is 0 (This hypothesis can be confirmed on the basis of some geometrical knowledge related to perpendicular bisectors, if the class has that knowledge)

—The coordinates of the center depend on both a and b Students can be invited to play with different values to gain an intuition about the nature of this dependence. It can be noticed, for example, that whereas the sign of a is irrelevant, the sign of b will determine the sign of the y-coordinate of the center. Moreover, attempts to draw the circle indicate that for small absolute values of b, the absolute value of the y-coordinate of the center (and the radius) will be large, and vice versa.

These informal sense-making findings are very informative of the nature of the answer to this problem. Having this informal knowledge as an anchor, the outcome of the algebraic manipulation, namely $y_c = (a^2 + b^2)/2b$, becomes more meaningful, and students have an *a priori* sense of its reasonableness, against which it can be checked. Thus, a task, which could have been mostly technical can become an environment in which aspects of symbol sense are nurtured, because of the activity in which students engage.

4) Algebraic symbolism should be introduced from the very beginning in situations in which students can appreciate how empowering symbols can be in expressing generalizations and justifications of arithmetical phenomena [Friedlander et al., 1989; Hershkowitz and Arcavi, 1990]. By displaying structure, algebraic symbols are not introduced as formal and meaningless entities with which to juggle, but as powerful ways to solve and understand problems, and to communicate about them. The following problem [adapted from Gamoran, 1990], is one example of a whole class of such situations, which can be presented very early in algebra classrooms.

> "In the following arrangement of n tables, X indicates a seat for a single person, and "... " indicates a variable number of tables.
>
> 
>
> How many people can be seated?"

One possible way of solving this problem is to regard it as a perimeter problem; thus the number of seats will be $2(2n) + 2 \times 1$, namely $4n + 2$. Another way is to count the number of seats at the tables which can seat only 4, namely $n - 2$ and then add 10 more seats for the two extremes, giving $4(n - 2) + 10$.

One can pose the reverse question: Which way did one count if, for example, the resulting expression was $5n - (n - 2)$? Thus, besides the stating of the symbolical model for the problem, we can also capitalize on students' different solutions and go in the reverse direction, from the symbolic equation back to a reconstruction (from the non-reduced expression) of the way the counting was done.

This kind of activity can be very revealing about the purposefulness of manipulations and about the invariance of the final expression, regardless of the way one performs the counting. In sum, in tasks of this nature, manipulations are at the service of structure and meanings

5) One habit teachers can establish is the "postmortem" analysis of problem solutions, which can be helpful in monitoring the use of automatisms. In such discussions it is likely that alternative approaches are encountered. The inspection of these alternatives is usually helpful in establishing connections between symbolic and other approaches. It was during such a discussion that a student, who after seeing the symbolic and the *origami* solution to the problem of the border cells of the rectangle, reimposed meaning on to each formal step in an algebraic manipulation in order to connect the two different solutions. We suggest that one instructional implication for nurturing the development of symbol sense is to provide opportunities to make these connections or to see others making them.

6) Classroom dialogues and practices should legitimize and stimulate "What if" questions in general, and especially regarding the role of symbols and their rules. For example: "$y = mx + b$ is the general expression for a linear function. If you substitute values for m and b you will obtain a specific linear function. Can you make any sense of the result of substituting for x and y to obtain, say $2 = m3 + b$?" Questions like these, if allowed, and often asked, would help students to regard symbols as entities which can be the object of their constant reinspection, and not just governed by rules arbitrarily imposed on them from above.

Finally, rather than extending further a *prêt à porter* collection of instructional implications, we would like to conclude with an invitation: if we create and work on problems of the kind described above (both in classrooms and in teacher courses) promoting appropriate practices of thought and discussion, we will sharpen our sense of what symbol sense is and should be. Moreover we will be able to collect more implications for instruction which arise form observations of our own environments

Epilogue

"I once had dealings with a man who carried on his business from his apartment. He had a large clientele. He lived in the fourth floor of the apartment house, and access to his apartment was gained from a rear door on the ground floor adjacent to a parking lot. This door was normally locked. Now, the manner of going up to his apartment was this. After calling him up just before your appointment, you parked your car in the lot and then honked your horn. The man would then come out onto his back porch and let down a key to the ground floor on a long string. You would open the door with the key, find your way up to his apartment and give him back the key.

This procedure struck me as highly eccentric and mysterious. One day I asked him to explain what was going on.

"Why don't you simply have an electric lock installed so you could push a button in your apartment and it would open the bottom door?"

"There is an electric lock already on the door."

"Then for heaven's sake, why don't you use it?"

"I'll tell you. About ten years ago, my wife and I were divorced. It was a nasty business, and for months and months after the divorce she would come around here causing me trouble. One day I simply decided that when someone pushed the button downstairs, I would ignore it, but of course I hade to make provision for my clients. So there you are; it's not really mysterious at all."

"I take it she still comes around and tries to make trouble for you?"

"No. She died about five years ago!"

" " "

"Formalism, in the sense of which I still use the term, is the condition wherein action has become separated from integrative meaning and takes place mindlessly along some preset direction."

" • "

"Over the ages, mathematicians have struggled to restore thought and meaning to mathematics instruction, to provide alternatives to the formal and ritualistic mode of learning in most mathematics classrooms, but in spite of new theories, new applications, new courses, new instruments, the battle is never won. The fight against formalized, unthinking action is perpetual."

" • "

"I do not mean to suggest that formalism is a material substance that resides in the atmosphere and can infect us, I use the term formalism only in order to call attention to a natural tendency of form and function to get out of balance or to part ways. We cannot prevent their separation and probably should not try to do so but we should be aware of the process, so that we may institute countermeasures when it gets out of hand"

[All the preceding quotes in this section are from Davis and Hersh, 1986]

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Notes

[1] Throughout this paper, by symbols we mean literal symbols as used in high school algebra

[2] The dynamics of these dialogues is usually interesting and rich, but this is not the subject of this paper

[3] This problem is borrowed from Alan Schoenfeld's Mathematical Problem Solving Class

[4] The careful use of set notation would require in the first case $\{(x, y)/x = 0, y = b\}$, whereas in the second case it would be $\{(x, y)/y = b\}$.

[5] Another version of this problem includes specific numbers on the axes

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May not Music be described as the Mathematic of Sense, Mathematic as the Music of Reason? the soul of each the same?

J. J. Sylvester