WRITING DOWN THE SOLUTION SEPARATE FROM SOLVING THE PROBLEM

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We believe that 'communication works for those who work at it' [1] – but how do we work on communication, especially the written communication, in school mathematics? Consider the following task: 'Investigate how many cycles of each type (bicycles and tricycles) there can be if the number of wheels is 21 in total and there must be at least one cycle of each type'. Think about how you would solve this task and then consider the documentation in Figure 1, from a 10-year-old boy who was asked to investigate this problem.

Figure 1 can be referred to as a mathematical *text* in line with literary theories on text being any set of signs that can be 'read'. The text uses pictures, symbols, sequencing, and other communicative resources to present all possible solutions to the problem. We know the text indeed was supposed to represent all possible solutions because we retrospectively interviewed the boy who produced the text, and we know that he was confident in having found all the solutions. He argued that his text shows this because there is a shift from presenting the tricycles rather than the bicycles first. He suggests that it is apparent that he realised the importance of the tricycles and from there it was just a matter of making sure that all the possible odd numbers were accounted for in his text. If, as a reader, you are well acquainted with this problem and its standard solution and if you have ample knowledge about the student and his abilities, it may be possible to conclude that the student's reasoning was in line with what we have described based on his text alone. We will however want to distinguish between assuming the student has reasoned in this way based on having particular and specific knowledge and observing this from the presented text. The latter is what we normally require in a mathematical proof. If you have all the prerequisites to understand the separate parts of a proof and its arguments, then you are supposed to understand the proof as a whole even if you have never seen it before and have no knowledge about the person who produced it.

In this article we aim to discuss how teaching, focused on developing students' ability to write this latter type of mathematical text, can be designed. We have reason to believe that neither research, nor curricular material offer guidance for teachers who aim to develop their students' communicational skills when writing in mathematics. We begin by describing three reasons for such a research focus.

Firstly, there are different conceptions of mathematics writing as either rule-based or discursive. In a rule-based system, writing is seen as fixed and stable, with rules and conventions that students need to learn to succeed in mathematics, whereas in a discursive model, writing is understood

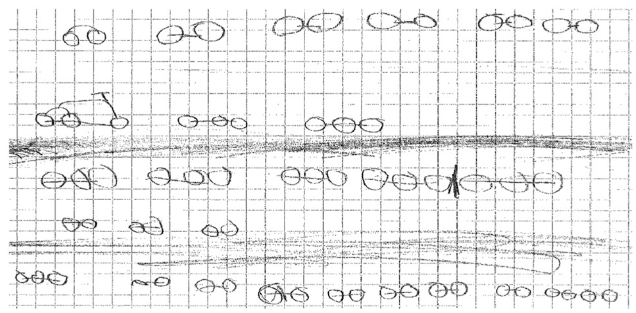

Figure 1. Documentation from a 10-year-old student.

as a social practice where meaning is shaped through the social structure of, for example, a mathematics classroom (Barwell, 2018). In classrooms however, these two conceptualisations are likely combined. Writing in the mathematics classroom consists of both mathematical notation, for which there are rules, and a more creative type of writing that uses mathematical notation together with non-mathematical language and other communicational resources, to construct a narrative that conveys a process (Steenrod, Halmos, Schiffer & Dieudonné, 1973). The latter is a type of writing where the rules are determined and negotiated in the social context of the classroom. Mathematics teaching that has focused on writing has mainly been concerned with the first type of writing, for which there are rules, while the second, more creative type of writing has been left to students to develop on their own.

Secondly, students' writing in mathematics is seldom separated from the problem-solving process itself, and thus not treated as a separate object of learning with a somewhat different set of skills. Mathematical writing is multimodal, and it is therefore likely that writing in mathematics requires distinctive skills. Unlike other types of writing, however, mathematical writing in school is seldom assessed or discussed based on its communicational merits, but rather on the mathematical knowledge it indicates, leaving teachers with few tools and limited terminology to discuss and give formative feedback to students on the quality of their writing. This lack of terminology makes it difficult to pinpoint what makes a good mathematical text *good*. In classrooms, this can lead to situations in which the discussions, instead of dealing with the texts as products, deal solely with the mathematical understanding that the texts indicate. In this way the aspects of quality in formal written mathematical communication are tangled with other aspects making it difficult for teachers to teach the skill and for students to discern ideas on what constitutes good writing in mathematics.

Thirdly, in school mathematics, the kind of writing that students use as a cognitive tool to record, organise, and visualise a problem during their problem-solving process tends to merge with writing used to communicate with others, such as reporting or explaining the solution. Morgan (1998) has suggested that these two types of writing represent fundamentally different processes of communication and should therefore be treated differently. Written communication, as already Plato pointed out, can be used to communicate with others across time and space, meaning an author must consider possible audiences outside of the immediate context. The process of using pen and paper as tools to minimise the cognitive load when solving a mathematical problem, however, is a personal process for which there is no audience. This process is instead closely linked to the thinking process, something many teachers are interested in. Teachers want to analyse and assess students' thinking process because it contains evidence of their grasp of concepts, reasoning, and procedures, but because students' documentation of their thinking process is not primarily meant for someone else, they will typically use idiosyncratic representations and fail to produce clear, precise, and efficient communication. They are busy thinking about the problem rather than busy thinking about how and what to communicate.

Designing a model

In the absence of research-based frameworks that describe quality in the type of formal written mathematical communication we are interested in, the project that we will describe adopts a bottom-up perspective. We aim to develop a model for teaching writing that is dependent on students' and teachers' ideas on different quality aspects in mathematical texts that report on problem solving. We choose to view the rule-based and discursive conceptualisations of writing in mathematics as complimentary rather than dichotomous, in line with Stylianides' (2007) ideas on proof in primary school mathematics. Stylianides decomposes the conventional requirements of a mathematical proof and transforms them to function in a school context that includes young students and their school mathematical discourse. In this way he shows that it is reasonable for third graders to produce proofs that are understandable to their peers but also have a clear connection the general requirements for mathematical proofs. On a similar note, we argue that mathematical writing in a school context has elements that are socially negotiated but also elements that are guided by rules, conventions and norms that are universal. This view of mathematical writing can be compared to Brousseau's (1997) concept of institutionalization, which has the role of tying together local representations of knowledge, developed by individual students or groups, to knowledge represented in established ways, on an institutional level.

We are using the phrase 'communication works for those who work at it' here to introduce the idea that the mathematics classroom should be a place where we systematically work on improving communication, in our case, students' written communication that is part of reporting on problem solving activities. We design teaching that is built on the separation of the two processes of (i) solving the problem and (ii) communicating a written report on how the problem can be solved, to someone else. To make this separation clear we need to change the way we talk about the two processes. We reserve the term *solving* for the first process, the one where the students are solving the problem, either individually or in groups. During this part of the design the students explore, investigate, and discuss the problem and come up with ideas on how to solve it. The mathematical writing produced during this phase is personal, processual, and explorative and various arguments can be proposed and discarded. The second part of the design is about formal communication. In this phase we no longer talk about the solving, but rather the solution, the communication of a line of arguments that, in their totality, can convey to a reader that the presented answer is indeed the answer to the specified problem. We will call such a mathematical text *a formal written mathematical communication,* to indicate its status as a product and not the communicational process.

Following this separation between solving a problem and producing a formal written mathematical communication the first part of our teaching design is a problem-solving lesson. Students are presented with and work on a mathematical problem. This part of the design can take different forms depending on what students are used to. They can work individually, in pairs, or in groups when exploring the problem and trying out different strategies. There can be discussions and cooperation that are organised in different ways during this phase. It is important that as many students as possible take part in explorations and discussions. During this phase, the writing that takes place is the type of writing that assists students in their explorations, calculations, and discussions. We will not here go into detail on other important aspects of problem solving in school mathematics since this is not our mission. When the exploration phase is over some type of whole-class discussion should take place. This discussion can be organised in different ways in line with ideas from mathematics education research (see, for example, Stein, Engle, Smith & Hughes, 2008). During the whole-class discussion, the students will, ideally, present and discuss different ideas on how to solve the problem and they will present and be subjected to valid mathematical arguments for their different strategies and calculations. Before the whole-class discussion is concluded, it is the role of the teacher to ensure that all students have grasped the problem and at least one way to solve it.

Following the discussion of solutions, we ask all students, individually or in groups, to produce a formal written mathematical communication that reports on how the problem can be solved. This part of the design differs from the way writing is normally handled in school mathematics, since students are not expected to describe their thinking but rather describe and justify one of the ways the problem can be solved. This means that individual students do not have to report on the solution they themselves used in solving the problem, they can choose to use the ideas of a peer or any solution that came up during the whole-class discussion.

We believe this aspect of the design represents a move away from a situation in which the mathematical writing of students is used exclusively to assess their thinking. There are other ways to assess students' mathematical thinking and we are striving to create a specific teaching situation focusing on how mathematical texts can report on problem solving, including justifications and arguments. One important aspect of this is to make certain that a thorough discussion on students' use of various strategies for solving the problem has already been dealt with. In our design, this first phase typically represents one mathematics lesson, and is concluded with the teacher collecting the students' formal written mathematical communications.

The second part of our design, which mostly coincides with a second lesson separated in time from the first, aims to discuss the formal written mathematical communications that the students produced in the previous lesson. In this part we focus exclusively on the students' communications as products that can be assessed based on their communicational merits. The teachers that we collaborated with prepared this second lesson by reading the students' texts and selecting a few to discuss with the class. In trying out the design we, as researchers, discussed the students' texts together with the teachers to identify interesting differences between them in terms of communicational quality. The general aim with the sessions, where students' formal written mathematical communication is discussed, is for students to identify different qualities in such texts, something that will improve their ability to produce better written communication. We are, in our present work, aiming to design an introduction to our teaching model that will allow new teachers to implement our ideas without researcher support, but this effort to scale up is a matter for another article.

A classroom example

To give an idea of what a discussion on the quality of students' texts looks like we will describe a lesson one of the teachers conducted. The focus was this problem:

> In a class there are 12 girls and 8 boys. In the last month 50% of the girls and 25% of the boys have borrowed and read a book from the library. What percentage of the class have borrowed and read a book from the library?

The students had worked with the problem in a previous lesson, and they had all produced a formal written mathematical communication describing how the problem could be solved. The teacher had selected five of the texts and, to avoid a discussion on who wrote what, she had reproduced each one in her own handwriting while taking care not to change elements such as sequencing, placing and use of space.

The five texts were posted on the wall and the students, aged 11-12, walked around in groups of three to carefully read and discuss each one. When the students were satisfied, they were asked to vote for the text they considered the best. The voting was followed by a whole-class discussion and a second voting where the students individually were offered an opportunity to change their mind based on the discussion. This was the third time the teacher followed the study design, and the students had become familiar with criteria that the teacher had formulated. These were:

> Someone who does not know the problem should be able to read the text and understand the solution
>
> There should be arguments to justify the calculations
>
> Information needs to come in an order that helps you understand the entire process
>
> It should be efficient; it should not contain unnecessary information

The criteria were based on a framework that we will discuss below (see Table 2). The teacher's role in the discussion was to encourage the students to justify why they considered one text superior to another. Below we present outlines of the whole-class discussion on each of the formal written mathematical texts. The texts have been re-written and translated by us to faithfully represent the ones that were discussed in class.

When discussing the text written by Student 1 (see Figure 2), the students believed that there is enough information for a person who does not know the problem to understand what is to be calculated. Some students were happy with the arguments presented while others were unhappy with the way they are presented as part of a narrative that comes after the calculations, it is not very efficient, and the order is messed up. The students were also not satisfied with the conclusion. They believed that there could have been more information on what was meant by 40%.

The text written by Student 2 (see Figure 3) was the most popular one before the discussion. The students believed it

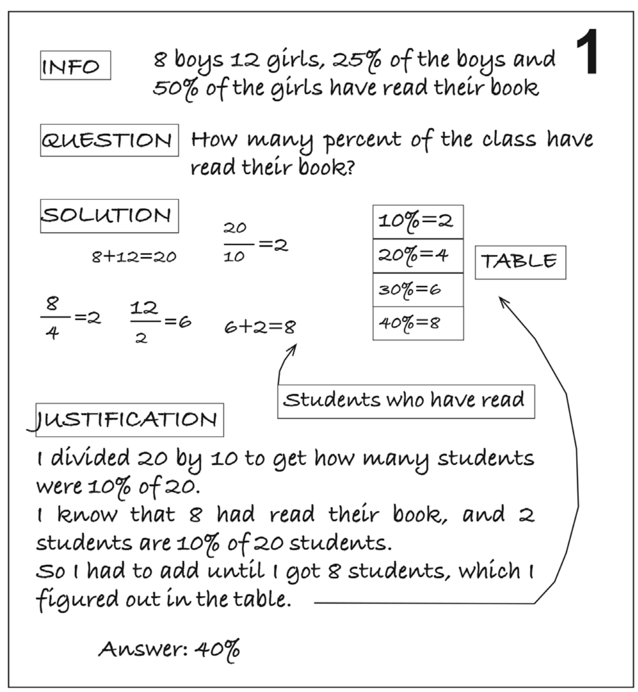

Figure 2. A formal written mathematical communication from Student 1.

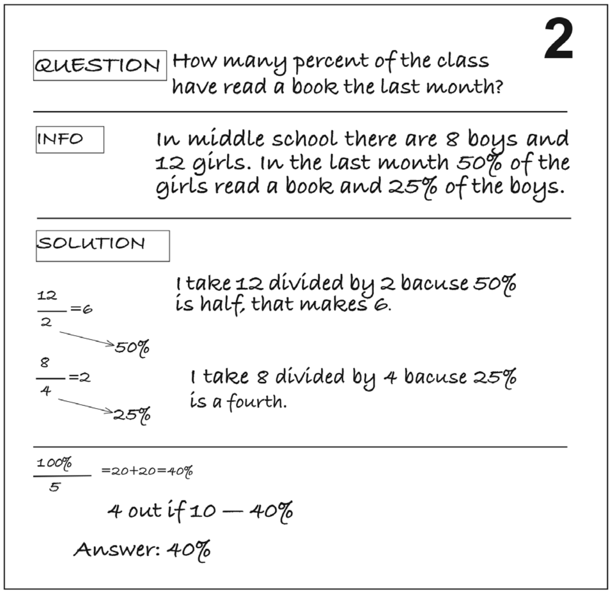

Figure 3. A formal written mathematical communication from Student 2.

was clear what the problem was, and why the first two calculations were performed. The last calculation, however, was not justified at all, and the students agreed that the text did not explain how it arrived at the answer 40%. About half of the students believed that the text was efficient, despite the initial mass of text, but they agreed that there should have been a more comprehensive conclusion.

The third text (see Figure 4) created a lot of discussion, and the opinions were diverse. Students who did not like it argued that there was too little information to understand the problem. Some students were happy with the calculations and viewed the arrows as very efficient tools for communication, while others argued that all the long arrows made it more difficult to follow the process.

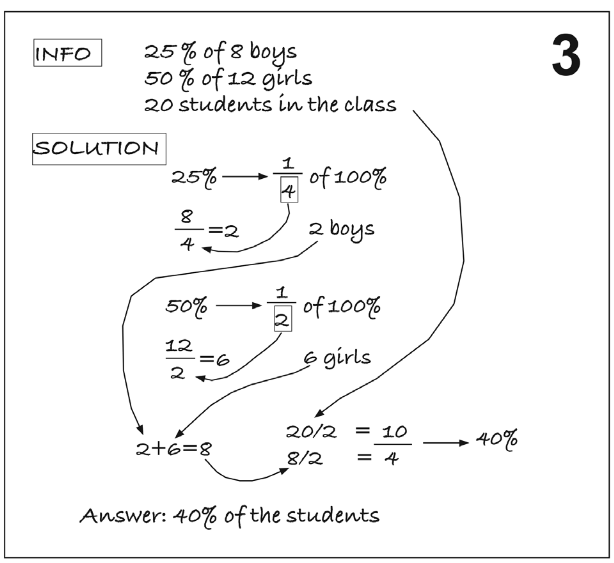

Figure 4. A formal written mathematical communication from Student 3.

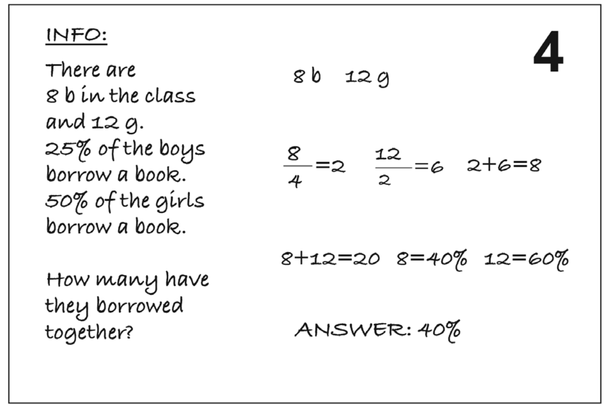

Figure 5. A formal written mathematical communication from Student 4.

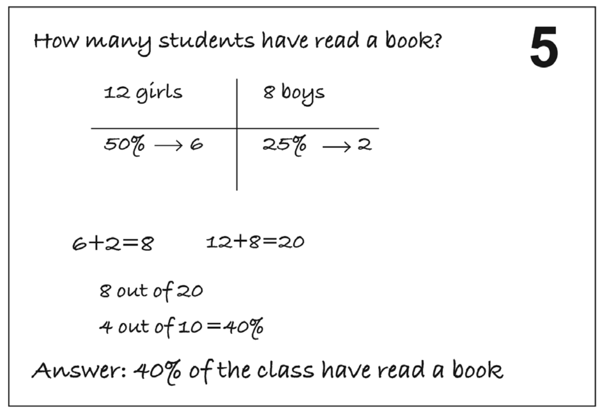

Figure 6. A formal written mathematical communication from Student 5.

The text written by Student 4 (see Figure 5) was the least popular one, and it did not receive any votes. The students argued that the question from the problem should have been included for a reader to understand what was being calculated. The presented question is different from the original question. They also pointed out that there were no justifications for the calculations and that the option to put the mass of text to the left and the calculations to the right did not result in efficient communication.

When the text from Student 5 (see Figure 6) was discussed, several students felt that it was easy to understand the problem, although they still wished for more information. They agreed that it was efficient, but they also pointed out that justifications for the calculations were missing. The students all agreed that the conclusion was well presented since it was clear what had been calculated.

The students asked if they could vote for more than one text and if they were allowed to refrain from voting if they did not accept any of the texts as good examples of written mathematical communication. The results are shown in Table 1.

Text	1	2	3	4	5	more
Number of votes first round	3	7	2	0	2	9
Number of votes second round	0	3	0	0	5	13

Table 1. The results of the voting.

Different qualities in formal mathematical texts

In designing the second part of the two lessons an initial problem for us was that there is very little research on what can be considered quality or progression in students' formal written mathematical communication if you look exclusively at the communicational merit of their texts. We have searched for frameworks that suggest what should be included in students' texts that report on problem solving. Elements suggested in the few frameworks we have found include recounts of the steps of the problem-solving process, explanations, accounts of reasoning, and referrals to context (Hughes Markelz & Cozad, 2019; King, Raposo & Gimenez, 2016; Kosko & Zimmerman, 2019). Mathematical language, or the use of mathematical terms, can also be viewed as elements that are either present or not in the text (King, Raposo & Gimenez, 2016; Kline & Ishii, 2008) but we prefer to view mathematical language as a form of representation rather than an element that carries a particular meaning. After reviewing this literature, we concluded that the separation between students' thinking in the problem-solving process and a formal written mathematical communication of a solution was difficult to discern. We also searched for frameworks that describe different levels of quality in students' writing, but we found that very few describe any type of progression in students' production of mathematical texts. Kosko and Zimmerman (2019) describe six different levels of quality in students' arguments to justify their calculations in a problem-solving context, but we believe arguments make up only one of several elements in formal written mathematical communication.

From the, albeit few, descriptions, or guidelines on what should be included in mathematical texts that report on problem solving we initiated the development of a framework to use in our design. Based on the research presented above and our own previous research on students' mathematical writing, we suggested that students' formal written mathematical communication should contain 1) a description of the problem and the premises for solving it, including facts and assumptions, 2) the calculations needed, 3) arguments that justify the calculations, and 4) a conclusion or an answer to the question posed in the problem. Two of these elements may need elaboration.

Describing the problem and its premises and facts, is something we believe that students do only if they are expected or told to since problem solving often takes place in a context where the participants are familiar with the problem. We chose to include this element because we believe it is important that students learn to produce a formal written mathematical communication that stands on its own and is not dependent on contextual information. We also believe it benefits students to differentiate between necessary and unnecessary information and to clarify assumptions they make in order to solve the problem. This can be done in different ways and does not mean they necessarily need to repeat the entire problem. Sometimes an efficient and clear way of communicating the assumptions made and what the problem is about is to add a different representation such as a figure, a table, or a graph. Providing arguments to justify calculations is also something that is dependent on the classroom norms. In Swedish schools, students are rarely expected to provide arguments to justify their calculations but rather to provide a coherent description of their problem-solving process, *i.e.,* 'what they thought'. In King, Raposo and Gimenez's (2016) study this element is called 'reasoning'. We have chosen not to use this term as we believe that reasoning is not sufficiently defined and does not present the opportunity to focus exclusively on the calculations. Justifying calculations is not about describing procedures but about making clear why a particular calculation is made and where the numbers used come from.

Arriving at the question of what can be considered quality or progression in a mathematical text, we again faced a near-complete lack of research to lean on that is relevant for exclusively analysing the communicational merits of formal written mathematical communication from primary to secondary school mathematics. This time we turned to research outside of mathematics education, more specifically research on communication as a competence, where we picked up the aspects *efficiency* and *appropriateness* (Rickheit, Strohner & Vorwerg, 2008). Efficient communication can be described as communication that achieves its goal with as little effort as possible. Mathematical notation can be seen as the ultimate example of efficiency as its symbols are the result of increased standardisation and narrowing of linguistic options throughout the last two centuries. Appropriateness is about choosing the linguistic or semiotic options best suited to the context and the expected reader or recipient of the communication, *i.e.,* to 'speak the customer's language'. Efficiency was a term we could use without further elaboration, but in the context of the classroom, appropriateness was transformed to clarity, answering the question *is it understandable to the students*. We also added the aspect of correctness in using mathematical notation, resulting in the framework described in Table 2.

Discussion

The teaching model presented above addresses the three problems surrounding teaching mathematical writing in classrooms that we presented above. The model offers teachers opportunities to have whole-class discussions, in which students' writing is the object, by providing a framework for assessing mathematical writing using an appropriate terminology.

The model addresses the two conceptualizations of writing in mathematics as rule-based or discursive and combines them in a way that allows written communication to be the object of learning. By employing a framework that attends to the 'what' as well as the 'how' of mathematical writing teachers and students can explore possible differences in

Elements / Quality	Problem description	Calculations	Arguments, justifications	Conclusion
Efficiency				
Clarity				
Notation				

Table 2. A framework for discussing students' formal written mathematical communication.

quality and suggest choices that will increase the communicability of the formal written mathematical communication. Discussions take place in the social practice of the classroom and students can have different ideas on what is considered efficient and clear. These discussions, however, are based on an ambition to be understood outside of the immediate context. Because of this, they must institutionalize meaning, and the most obvious way to do that is to use formal elements of mathematical writing, for example mathematical symbols. We argue that this combination of rule-based formality and socially negotiated norms makes our model useful and efficient in mathematics education.

One of the most important parts of our model is our way of addressing the problems that arise from the fusion of problem solving and writing. When presenting our teaching model, we are often met with curiosity but also with scepticism regarding whether it is possible to separate the process of solving a mathematical problem from the process of communicating its solution. We believe that with the lesson design we presented above it is possible to create this separation, to treat students' writing as the object of learning separate from a discussion on different ways to solve the problem. What we have described above is part of a research project where we are working with several teachers who are using the design. Even though the students' ages range from 10 to 17, we have seen similarities in the outcome of the two-lesson cycle. As in the examples from the whole-class discussions presented above, the students display a distinct and detail-oriented critical stance towards the texts exclusively as communicative products. This stance has developed after only a few cycles in the around twenty classes where we are so far piloting this intervention, strengthening the belief in the principle that the separation of the problem solving, and the communication, is not only possible but also efficient and useful. We believe however, that the mathematical side of the solution process (the actual problem solving) and the communicative side of the process (the creation and design of a formal written mathematical communication) are intimately related in the sense that students' problem-solving abilities will benefit from developing communication competence and vice versa.

With our model we also address the problem of treating writing for oneself and writing for others as two fundamentally different processes. When we ask students to consider an audience in their writing, we are creating a different mindset, moving away from a focus on their thinking process. One example is that we are asking students to justify their calculations and provide arguments for them. This hopefully produces texts in which the mathematical reasoning of the student is easy to follow, and with that one can argue that the fundamental role of the communication is fulfilled. But in this project, the role of communicating is not primarily to help the teacher understand what students are thinking. Instead, the communication should explain to a reader why the described reasoning solves the task. By shifting focus from the individual student's thinking to arguments and justifications that are general, we are decontextualising the students' methods for solving the problem, thus institutionalising them. The context for formal written mathematical communication ceases to be student to teacher. Following Stylianides' argument (2007) on how to view proof and proving in primary school, the context should be communication between peers. As with proofs, it is a reasonable requirement that a peer, not familiar with the problem or the solution, should be able to make sense of why certain calculations are relevant and how they lead to a solution. In maintaining a focus on the mathematical text, we believe that our model offers opportunities for students to enhance their ability to produce and formulate arguments that justify calculations with an audience in mind. This in turn offers the students better opportunities to make sense of the underlying mathematical ideas. Not only because their texts are formulated in ways that are socially negotiated to be clear and precise, but also because the students must contemplate the problem twice, or in two different ways. First in relation to solution strategies where their use of writing is personal, idiosyncratic, and connected to the process, and second, in relation to producing formal written mathematical communication that others can understand, resulting in a product.

We believe our project shows that communication works, for those who work at it.

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Note

[1] This phrase is popularly ascribed to John Powell, but we have been unable to trace its source.

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