Mathematical Abstraction as the Result of a Delicate Shift of Attention

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Students of mathematics often say that they find mathematics abstract, and give this as the reason for being stuck, for disliking mathematics lessons, or even for withdrawing from mathematics altogether Yet the power of mathematics, and the pleasure that mathematicians get from it, arise from precisely the abstract nature of mathematics. The aim of this article is to explore this unfortunate dichotomy of response to the idea of abstraction, to venture a technical use of the term which could be of help to mathematics teachers and students alike, and to provide a case study.

The first question is whether the word *abstract* is actually being used in the same way by frustrated students and by inspired professionals. The verb is usually pronounced differently to the adjective and noun, emphasis being placed on the prefix *ab* when it is used as a noun or adjective (as in the beginnings of papers, and when applied to an idea that seems unconnected with reality or which fails to inspire confidence—hence the pejorative meaning) and on the root stem *stract* when used to refer to a process akin to extracting Thus *extract means to draw out and to abstract* means to draw away. Eco, in discussing the meaning of *aesthetic* in the work of St Thomas Aquinas, writes that

> Aesthetic seeing involves grasping the *form* in the sensible. It therefore occurs prior to the act of abstraction, because in abstraction the *form* is divorced from the sensible. [Eco, 1988, page 193]

If emphasis is placed on *divorced,* we are reminded of the student's experience. It is easy to sympathise with the student's sense of abstract as *removed from* or *divorced from* reality (or perhaps, more accurately, from meaning, since our reality consists in that which we find meaningful). But perhaps this sense of being out of contact arises because there has been little or no participation in the process of abstraction, in the movement of drawing away. Perhaps all the students are aware of is the *having been drawn away* rather than the *drawing* itself.

If emphasis is placed on *form,* we are reminded of the expert's experience Tall [1988] speaks in a similar vein of

> abstraction (as) the isolation of specific attributes of a concept so that they can be considered separately from the other attributes

When forms become objects or components of thought, and when with familiarity they become mentally manipulable, becoming, as it were, concrete, mathematics finds its greatest power. But abstraction is not just higher mathematics for the few It is an integral part of speaking and thinking. C.S. Peirce, the noted creator of pragmaticism, having drawn attention to the abstractions which we call collections, namely pairs, dozens, sonnets, scores, etc, wrote:

> .... the great rolling billows of abstraction in the ocean of mathematical thought; but when we come to minute examination of it, we shall find, in every department, incessant ripples of the same form of thought ... [Peirce, 1982]

Far from being an abstruse activity of expert mathematicians, abstraction is a common experience. Why then does mathematical abstraction get the reaction it does?

My thesis is that the uses of the word *abstract* in mathematics by both novices and professionals refers to a common, root experience: an extremely brief moment which happens in the twinkling of an eye; a delicate shift of attention from seeing an expression *as* an expression of generality, to seeing the expression *as* an object or property. Thus, abstracting lies between the expression of generality and the manipulation of that expression while, for example, constructing a convincing argument. In that ever so delicate shift of attention occurs the drawing away of form from the sensible, the abstraction, referred to in the quotation from Aquinas.

When the shift occurs, it is hardly noticeable and, to a mathematician, it seems the most natural and obvious movement imaginable Consequently it fails to attract the expert's attention When the shift does not occur, it blocks progress and makes the student feel out of touch and excluded, a mere observer in a peculiar ritual Some students even master aspects of the form of the ritual without being able to explain why they do what they do. A few, through this process of habituation [Peirce, 1982; Mason and Davis, 1988] find themselves able to explain things to others, but many never completely lose their sense of alienation.

My approach to the thesis makes use of the theory of shifts of attention [Mason and Davis, 1988] and proceeds via the Discipline of Noticing [Mason, 1987]. I present a few episodes from a mathematical case study in which you the reader are asked to participate. The case study is based on part of what happened during a weekend of mathematical problem solving devoted to the theme of axioms and abstracting, with participants varying from mathematically naive undergraduates to sophisticated graduates

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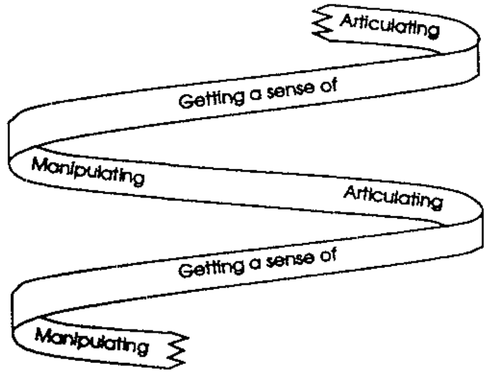
and tutors of the Open University.

By engaging in the mathematical tasks, it is my prediction that you will experience aspects of abstracting which I have found significant. By means of these examples, you will (through the process of generalising from my examples) construe what I mean by various terms, such as *shift of attention, characterising,* and *abstracting,* and because the descriptions are related to mathematical experience, you will be more likely to notice their applicability in mathematical moments in the future. This in turn will enable you, should you so wish, to take alternative action, or at least to be more sharply aware of aspects which might otherwise have been overlooked. As St. Augustine wrote

> In the halls of memory we bear the images of things once perceived, as memorials which we can contemplate mentally, and can speak of with a good conscience and without lying. But these memorials belong to us privately. If anyone hears me speak of them, provided he has seen them himself, he does not learn from my words, but recognises the truth of what I say by the images he has in his own memory. But if he has not had these sensations, obviously he believes my words rather than learns from them [St. Augustine, 389]

Even with the influence of a modern translation, this observation of St. Augustine neatly expresses the purpose of my offering a case study, and the effect I intend it to have. The validity of my thesis lies in the extent to which you recognise the aspects which I stress, and the extent to which you find such awareness helpful in the future. This methodology is itself a process of abstraction, moving as it does from experience, to expressing experience, to taking such expressions as descriptions of properties of many experiences, to manipulating labels of those experiences in subsequent descriptions.

In Floyd *et al* [1982], a helix was used to describe the experience of moving from *manipulating* objects (physical, pictorial, symbolic, mental) to *getting-a-sense-of* some feature or property of those objects, to *articulating* that property as an expression of generality, to finding that expression becoming a confidence-inspiring entity which can be manipulated and used to seek out further properties. I suggest that the process of abstracting in mathematics lies in the momentary movement from articulating to manipulating



Articulation of a seeing of generality, first in words and pictures, and then in increasingly tight and economically succinct expressions, using symbols and perhaps diagrams, is a pinnacle of achievement, often achieved only after great struggle. It turns into a mere foothill as it becomes a staging post for further work with the expression as a manipulable object. The helix image not only helps to locate abstracting in a flowing process, but also reinforces the notion that what is abstract for one person can become concrete and confidence-inspiring for others [Mason, 1980]

The language of *process* and *object* is a little too glib, for in the stressing which the words imply, there is a tendency to ignore the experience of a *sense-of,* which accompanies or is associated with the expression, and which does not disappear when the expression becomes an object of attention. On the contrary. The expression acts as a signal to recall salient associations. Just as there is a huge difference between drawing your own figure in order to stabilise your mental imagery and to extend your thinking power, so there is a huge difference between expressing your own generality and doing someone else's algebra. Algebraic expressions provide a generic example of how meaning can remain connected to symbols: it is important, for algebraic thinking to develop effectively, to maintain a dual awareness of expressions, as entities or objects, and as statements about how a calculation is to be performed [Mason, 1982]. In the language of Tall and Vinner [1981], the *concept image* is extended and made more powerful by the manipulability of the expression, not narrowed and refined

Case study

> *A sequence*: Express to yourself in action (by doing it) and in words (by talking to yourself or a colleague) a rule for continuing the following array [Honsberger 1970 page 87]:

			10	
		5	11	
	2	6	12	
1	3	7	13	. .
	4	8	14	
		9	15	
			16	

> Now attend to the central line, and generate more terms. Find a way of generating even more terms without filling in the other numbers. Express your rule in general terms, and as a formula. Pause now and try it.

Comment: The action of filling in more numbers, whether performed physically or just mentally, serves to clarify and crystallise a sense of what is going on, and prepares the way for a verbal statement of a general rule. For example, the presence of the square numbers in certain salient positions is usually noticed. Furthermore, the instruction to attend to the central line already focuses attention on certain aspects of the array, and away from others. The instructions are intended to prompt a move towards expressing a general rule, through the experience of particular cases. Young students in their early exposure to algebra can often express a general rule but find it difficult to see that expression as a manipulable object. The theory of shifts suggests that they need continued exposure to such acts of expressing so that they begin to find it relatively easy, and thereby find some of their attention free to view formulae as objects. Formulae are abstract, in the pejorative sense, if they are drawn away from or unconnected to meaning, or in other words, if they fail to be confidence-inspiring as potentially manipulable objects.

> *Observation* Notice that if you start at the 3 and count along a further 3 terms, a further 3, a further 3 still, . . . , those terms are all divisible by 3 Similarly if you start at 7 and count along from there 7, 14, 21, . . . terms, the terms you reach are all divisible by 7 Will this always happen for this sequence? Pause now and try it

Comment: The first task is to get a sense of the *this* in the statement that might "always happen," and often there is more than one way to make sense of it The observer has stressed certain features of the sequence, has edited out or ignored others, and expects the reader to be stressing in the same way. But the reader construes this stressing by responding to instructions, and then reconstructing the generality which may or may not be what the observer has already experienced.

To answer the question convincingly, after having tried numerous examples and becoming convinced that it does work, it is useful to express the *it* that always happens in a manipulable form, and then to manipulate it as part of an algebraic argument. It is not at all easy in this case, even for sophisticated students of mathematics, because it involves a term as part of a subscript of a term. In other words, an expression such as t_n must have become confidently manipulable as an entity. The movement from a formula for t_n, to that formula seen as an entity, is precisely the delicate shift of attention which I associate with abstracting The shift from looking-at t_n to working-with t_n is ever so slight, yet it constitutes a fundamental, and for some people, difficult shift.

During a different seminar in which the same sequence was offered and participants invited to express the same divisibility property, one person observed that:

> *Observation*: If d divides t_n, then d divides t_{n+d}.

There then followed a discussion as to whether this was actually the same or stronger than version involving t_n dividing t_{n+kt_n} For some it was obvious that they said the same thing, others saw them as completely different expressions. By specialising, we were able to convince ourselves that the first implied the second but not vice versa. It was then noted that in order to discuss whether one statement implies another, there had to be another abstraction shift: seeing the expressions (which until then had been attempts to say what someone was seeing in the given sequence) as objects of attention while looking at logical connections between them. Peirce was right; there are incessant ripples at every level

> *Extending*: What is special about this particular sequence 1, 3, 7, . . ?

Comment: The question draws attention to an observation which might have been made during the proof of either of the previous observations, that the form of argument does not make use of the particular formula for t_n, but only of the fact that it is a polynomial. Such an observation is most likely to come from a sophisticated mathematical thinker, because it requires a splitting of attention, one part involved in calculation while another part remains aloof but observant. This is the monitor on the shoulder [Mason, Burton and Stacey, 1984], the executive [Schoenfeld, 1986], which for most students has to be brought to life, encouraged and strengthened in order to get it operative It is the second bird described so succinctly in the stanza from the Rig Veda [see for example O'Flaherty, 1982]:

> Two birds clasp the self-same tree;
> One eats of the sweet fruit, the other looks on without eating.

Even if this awareness-monitor is not present (since it requires an attention split which is characteristic of abstracting), the extending question is readily available to the less sophisticated student since it is standard behaviour [for example Polya, 1970; Mason *et al*, 1984; Mason, 1988] to try to place any result in a more general context. But this sort of mathematical behaviour requires training [Mason and Davis, 1987] precisely because it involves an abstracting shift

> *Extending rephrased*: What sequences satisfy the condition that t_n divides t_{n+kt_n}?

Comment. Note the momentary abstraction move which this question signals The condition which involved a struggle to express when first encountered in the sequence, has become a property, akin to an axiom The search is now on for other examples which satisfy this property. Possible ideas that people might try include other polynomials (how else might you generate a sequence?) Fibonacci and other similar sequences, geometric progressions, hyper-geometric progressions, etc. Not all of these are satisfactory however.

The sequence which was the goal of study in the beginning of the investigation comes to be seen as an object, a particular instance of a class of similar objects, with similarity lying in the "divisibility criterion." Earlier I described abstraction as a shift of attention, from an expression of generality as the apex of a struggle to articulate, to the expression as object or property. But simple reification is not quite adequate as a description. The divisibility property begins as foreground, as the object of attention, the goal, and then, when expressed, recedes into the background and becomes a description, those specifications of a property. The foreground becomes the sequences which satisfy the property. Meaning is enriched not impoverished by the divorce of form from sensibility, because the sensibility lies in the background, ready to be called upon when required. So in abstraction there is an element of reification, and an element of change in stressing ignoring, of alteration of foreground and background, of summarising rich experience in a distilled essence of form.

During the event on which this article is based, someone observed that 3 not only divides 21 (three terms on), but the quotient is 7, which is the term following 3. The person tried one other example in the head and then announced a conjecture.

> *Conjecture:* Not only does t_n divide subsequent terms t_n apart, but the quotient is also one of the terms of the sequence. Check this for yourself, and formulate a more precise conjecture.

Comment: There is a taste of abstraction in the shift from a pattern appearing in the numbers, to the pattern being expressed and experienced as a conjecture. Most students, upon detecting a pattern, dwell in the pattern and its expression. They may even invest self-respect in the validity of their conjecture (manifested as a repeated return to the conjecture in the face of other conjectures or even of counter-examples), without experiencing their pattern *as* a conjecture. The teacher has a role here, in drawing attention to the status of the expression and thereby inviting a helpful shift of attention, releasing students from the confines of dwelling *in* the pattern.

In the particular event on which this case study is based, participants were attracted to this conjecture, so we worked on expressing it and then looking for other sequences that satisfy the same property. Since my aim was to focus attention on the shift from "sequence with a property" to "property satisfied by what sequences" as a generic example of the shift abstraction in mathematics, I was content to follow their inclination.

As expected, we kept being drawn back to the expression $t_n . t_{n+1} = t_{n+t_n}$ as people found that what they thought they understood proved to be slippery. The formula was written in blue on an OHP, with a blue box around it, and frequently someone would interject with a question such as "what are we assuming?" Someone else would then say "the blue box formula." Many participants found the movement from verifying the formula on the sequence 1, 3, 7, 13, ..., to looking for other sequences satisfying the same condition an unstable shift.

People proposed various polynomials, and we tried a few simple ones such as $t_n = n$ and $t_n = n^3$, and concluded that the property seemed only to apply to quadratics. We then tried to found out which quadratics, but the necessary algebra was too complicated for public calculation on an OHP. Later it was confirmed that the nontrivial quadratic $t_n = an^2 + bn + c$ satisfies

$$t_n \; t_{n+1} = t_{n+t_n}$$

if and only if $a = c = 1$. Here we have a clear example of the mathematical force to characterise, seeking another condition or property which identifies those objects which satisfy the given property. In this case, the argument requires considerable algebraic facility, including the technique of equating coeffecients of n^k in two different polynomial expressions. This in itself requires an abstraction shift, from the equality of the values of two polynomials in *n*, to seeing the equality as a statement which must hold for all *n*. It requires a confidence with the coefficients *a*, *b* and *c* as the objects under consideration, whereas, until recently, it was the *n* which was being investigated, a shift of foreground and background mingled with allowing what were previously place-holders (*a*, *b* and *c*) to become variables.

This is where the case study ends. It had been a long evening (this work being but part two of the session, and some 80 people attending closely to what they were thinking and to what others were saying), and I felt that attention had been sufficiently sharply focused on the issue at hand—the process of abstracting.

There is more to be learned from the exploration, however. In preparing for the session, I had searched and searched for some structural property characterising sequences for which t_n divides t_{n+kt_n}. Starting from the initial sequence a variety of different sequences with the same property were found, but there were so many that it was not at all obvious what to do next.

One approach is to look at the structure of the set of sequences, by seeing the sequences as objects, and looking for operations on the sequences by which new objects can be constructed. (Note the etymological links between *structure* and *to construct*.) This is typically mathematical: drawing back (some would say abstracting) from the examples, and considering the entire set of such sequences, together with operations on that set which preserve the divisibility property. This led to a number of fascinating questions, but yielded little in the way of a characterisation, partly because of a lack of closure under term-wise addition and multiplication. Somehow more structure was needed; an extra constraint had to be found.

Returning to the original sequence, a further observation emerged, illustrating again the mathematical process of abstraction. Mathematical abstraction is intimately connected with the mathematician's sense of structure, which again seems to come down to that momentary shift from an expression of generality, to the expression as object to be manipulated. To illustrate this, consider the following development of the case study.

> *Observation:* In the original sequence, terms two places apart (e.g. 1 and 7, 3 and 13, ...) have their difference divisible by two; terms three places apart (e.g. 1 and 13, 3 and 21) have their difference divisible by three. Does this always hold?

Comment: Again attention is drawn to finding out what the *this* actually is, by specialising, experimenting with particular terms in the sequence, and trying to re-interpret for yourself what general pattern was observed. It is a stronger statement even than the one reported earlier involving *d* dividing t_n implying that *d* divides t_{n+d}. The rapidity of movement to a formula for t_n, or even to a general polynomial, is a measure of the confidence developed in the formula as an expression of generality *and* as a manipulable object. I conjecture that this is precisely where sophisticated mathematician-teachers, unaware of the momentary abstraction in themselves, miss the need to attend to the abstracting movement in their students.

> *Extending.* What sorts of sequences have this particular divisibility property?

Yet again there has been a shift, from the particular sequence as context in which a property is being checked, to considering all sequences satisfying that property. This is again typical of the mathematician's move to characterise, which is so intimately bound up with abstraction. The very notion of characterising requires the abstraction move to which I am trying to draw attention.

Not all polynomials satisfy the property (for example, triangular numbers). By looking for operations on such sequences which preserve the property (such as finite differences, deletion of initial terms, addition term by term, multiplication term by term), conjectures and a sense of structure begin to emerge. Asking whether operations such as finite difference and first-term-deletion can be inverted leads to insight about the divisibility property and what it implies.

Note however the further acts of abstraction in switching from a few examples of sequences satisfying a property, to contemplating the set of sequences whose members satisfy a property, to verifying that a pair of unknown sequences both satisfying a property can be combined or operated upon to give yet another property-preserving sequence. To be able to treat a sequence s_n as an object with only one thing known about it, namely that it satisfies a divisibility property, requires considerable letting go of particularity and detail. It requires a true *drawing away* from specificness. The experienced mathematician does this without thinking, and sometimes leaves the novice gasping. Mathematics teachers would like their students also to make the delicate shift of abstraction without thinking.

The case study is at an end, but what purpose has it served? Has your attention been caught by the mathematical ideas, and if so, have you also observed the various abstraction shifts which I claim are likely to be present? This case study is no different from apparatus or from any mathematical example offered to students as an aid to experiencing a generality. The very particularness of the example or case study tends to attract attention to the particularity. The same is true of many computer programs. The presence of a thing which is intended to be generic, to act as a window through which the general is to be experienced often does quite the opposite [Mason and Pimm, 1984]. From my perspective, the case study illustrates or highlights certain features of mathematical thinking to which I wished to draw attention. The same is true of mathematical examples. To the teacher, they are examples of some general idea, technique, principle or theorem. To students, they simply are. They are not examples until they achieve examplehood.

Most mathematics educators have a story about abstraction, and many have been attracted into mathematics education (in addition to their teaching) by the desire to help students to cope with mathematical abstractions. Each new device, from Cuisenaire rods and geoboards to Dienes blocks and dotty paper, to calculators and computer programs, is seen by succeeding generations as offering assistance. Thus for example, Kaput and Pattison-Gordon [1987] reported on extensive work with students using computer programs designed to

> ramp students from their concrete, situation-bound thinking about the conceptual field of multiplicative structures to more abstract and flexible thinking ... An important, perhaps defining, characteristic of these (computer) environments is their systematic linking of concrete representations, beginning with iconic representations, to the more abstract representations on which much of advanced mathematics is built, e.g. coordinate graphs and algebraic functions. This systematic linking of computer representations is the foundation of our strategy for building rich and flexible cognitive representations. It is an attempt to build meaning and understanding gradually in this conceptual field in a way that will support long term competence—computational and conceptual competence.

One of the many features of computer programs, apart from their dynamic and interactive qualities, is that they present objects. Thus they can be used to support the reification which is so fundamental to mathematical thinking, though whether they then prove in some circumstances to focus attention so strongly on screen objects that it becomes difficult to recognise the underlying form, is another question which needs a great deal of research.

Ever since HMI have written about school inspections in the U.K., stress has been placed on the need for experience with apparatus [McIntosh, 1981]. Through Dienes [1964] it has become usual to speak of mathematics as being embodied in apparatus, and this has come to imply that mere contact with the apparatus will cause students to experience mathematical ideas. Hart [1989] has led an attack on this myth, asking for direct evidence that apparatus actually helps. Unfortunately, embodiment is in the eye of the beholder. It is not the apparatus but how it is perceived that matters. Mathematicians, wishing to help students, sometimes look for apparatus, for physical objects (or physical situations, diagrams, images, or supposedly already familiar mathematical contexts) in which they can see manifestations of their mathematical awareness, their abstraction. They then offer their *representation* to students, perhaps forgetting that students may not be seeing things from the same perspective. Students stress and ignore according to their experience, and not according to the experience of the expert. Thus it behoves the expert to seek ways to draw attention to what is being stressed, to help students to see through the particular apparatus to a generality being exemplified, and then, having gained confidence in articulating that generality, to make the abstraction shift in which the generality becomes object

In examining the issue of abstraction experienced by undergraduates being taught linear algebra, Harel [1987], observes that embodiment is the reverse process to abstraction (and hence embodiment-abstraction is analogous to specialising-generalising). Looking at textbooks, he found that all but one author offered examples of vector spaces (at various levels of generality/abstraction), and then launched into abstract definition with little or no reference to the examples except as later "checks." It is as if authors fail to recognise the fine detail of what students actually need. Only Pedoe [1969] defined the same technical term three times, in connection with three examples at three different levels of generality and abstraction. Once abstraction is recognised as a shift of attention, it is possible to be of direct assistance to students, by constructing activities which call upon their powers of generalising to express in their own terms what is the same about a number of, for them, situations, for the author, examples, and then explicitly draw attention to the movement in which the expressions become manipulable entities. Advice can be offered about how to prepare for this move, and mathematical challenges can be offered which attract attention away from the generalising, and so help students to subordinate the generalisation aspect of the expressions through integrating it into their thinking [Gattegno, 1987] and so participate in or experience the abstracting.

The notion of embodiment is encircled by the teacher's dilemma of not knowing what it is that the students are attending to. Frequently teachers are not aware of what they are attending to either, in the sense that they are not aware *that* they are stressing and ignoring, nor that their students may not be stressing the same features. Being aware of the fact of stressing and ignoring, in the midst of a teaching moment, enables a teacher to focus attention on, to stress explicitly, the act of stressing.

Traditionally, mathematics has been presented to students in a straightforward, rehearsal mode, in which the expert rehearses publicly the honed presentation of theorems, definitions, proofs and examples. Those who begin to make the move to abstraction spontaneously go on to study mathematics further, while the others remain mystified, though sometimes intrigued. Exposition is currently blamed for most of the failures of mathematics teaching, but it is not the mode or style which is responsible. Rather, it is the location of the teacher's attention, and the extent of the teachers' mathematical being which determines what happens within any mode of teacher-student-content interaction [Mason, 1979]. Despite current emphasis on exploration and investigation, many students may still not experience the shift of abstraction unless they receive explicit assistance. By being explicit, by focussing attention on the brief but important moments of abstraction movement, more students could be helped to appreciate the power and pleasure of mathematics, and to get more from their investigating. One of the roles of a teacher is to help students to draw back from detail, to shift into a more reflective contemplation of what they are doing, to keep track of the large picture while students pursue details. This is what Vygotsky signalled with his *zone of proximal development* and the role of the teacher in providing an extended awareness [Bruner, 1986].

Dubinsky and Lewin [1986] found difficulty in prompting students to abstract. Using the language of Piaget, they claim that

> the most powerful, and cognitively, the most interesting form of equilibration is that in which particular cognitive structures re-equilibrate to a disturbance by undergoing a greater or lesser degree of re-construction, a process known as "reflective abstraction." [p. 61]

After discussing two mathematical topics which pose major hurdles to undergraduates, they conclude that one of the most serious obstacles to overcome is the propensity of students

> to persist in applying an incorrect understanding of a concept even after a teacher has pointed out the error and tried to re-teach the concept. ... students will accept the criticism and instruction overtly, but revert to incorrect usage as soon as possible, somewhat amused by the teacher's apparent pedantry. [p. 88]

Dubinsky and Lewin suggest that either the student has no overall sense of the target concept, or has an integrated but incorrect concept, and that disequilibration can only take place when the student realises that thinking can be subject to disconfirmation and alteration. Thus, Dubinsky and Lewin seem to be invoking a metacognitive shift in awareness in order to bring about a cognitive shift in perspective. Such metacognitive shifts are not all that difficult to bring about, being best supported by undertaking to establish a mathematical, conjecturing atmosphere in classes, where what is said is assumed to be said *in order that it may be subsequently modified* [Polya, 1970; Mason *et al*, 1984; Mason, 1988].

But there are other explanations for the robustness of current ways of thinking, and other ways supporting a cognitive shift of attention. It is not just student wilfulness that makes teaching the frustrating task implied in the previous quotation. The didactic tension, or the didactic transposition as Brousseau [1984] calls it, is always present:

> The more explicit I am about the behaviour I wish my students to display, the more likely it is that they will display that behaviour without recourse to the understanding which the behaviour is meant to indicate; that is, the more they will take the form for the substance.

Students are interested, as are we all, in minimising the energy they need to invest in order to get through events. Thus they look for cues, in what the teacher says and does, for what they should do. The result is often a superficial production of behaviour, without roots adequate to enable that behaviour to be modified and adapted in slightly different circumstances (as when the questions are posed slightly differently on a test). Behaviour is trainable, but training alone is not enough to secure flexibility and competence [Mason, 1989].

A temporary shift can take place through students producing the behaviour being demanded, but a robust shift of attention, or rather a flexibility to be able to attend in more than one way, is what I believe Dubinsky and Lewin mean by the term reflective abstraction. By working on the locus and structure of their own attention, and through that, on students' attention, teachers can contribute to a more rounded and balanced experience for students than a narrow focus on training behaviour. There are strategies for harnessing emotion and educating awareness as well as training behaviour [see for example Griffin, 1989] When computer programs are constructed to illustrate or manifest mathematical ideas, we have an example of stressing made manifest But the program by itself is not sufficient for most students They also need help in construing what they see and do, weaving it into a story in their own words [Mason, 1985]

The helix depicted earlier was introduced precisely in order to assist teachers to notice opportunities to help students in shifts of attention such as abstracting. It is intended to provide an image, linking the labels *manipulating, getting-a-sense-of, articulating* with personal experience, and so awakening a teacher to the value in noting such transitions explicitly and allowing students time to convert hard-won expressions of generality into confidently manipulable objects. There is no doubt that such frameworks help [Mason and Davis, 1987], and indeed may in the end be the single most effective tool for helping teachers to develop their teaching and their sense of what it means to be mathematical, and thereby helping students to experience the delicate shifts of attention which mathematicians call abstraction

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