ON THE PITFALLS OF ABDUCTION: COMPLICITIES AND COMPLEXITIES IN PATTERNING ACTIVITY [1]

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I came to Charles Saunders Peirce's notion of *abduction* when I was confronted with four epistemo-ontological issues in relation to patterning activity. First, Hoffmann's (1999) extrapolation of underdeterminist thinking in induction could perhaps explain the linear forms of algebraic generalizations that various cohorts of learners have developed for certain sequences of figural and numerical cues such as the patterns in Figures 1 and 3 (Becker & Rivera, 2005, 2006; Rivera, 2007; Rivera & Becker, 2004, 2006, 2007). Such pattern sequences oftentimes appear as a set of three or four consecutive particular cues or instances that students then use in constructing a generalization.

Second, a reviewer of a recent manuscript that I co-authored was particularly troubled by the rather ill-defined way in which patterning tasks with only three instances could actually suffice in establishing a generalization:

> [Those] tasks … are not well-defined, mathematically speaking. The patterns are not defined in such a way that it is clear what figure will come next. For example, if these patterns are repeating patterns, the fourth figure in a pattern may be a duplicate of the first figure, the fifth a duplicate of the second figure, etc.… [Those] tasks … focus students on the recursive relationship that exists by providing consecutive figures or pictures … [which thus should] lead the authors to reconsider the generalizability of their findings based on the use of such narrowly defined tasks.

A concern I had with such a critique was mathematical in nature – that is, what are those necessary and sufficient assumptions that need to be clearly articulated so that a pattern sequence has a well-defined closed form? Should every pattern correspond to a unique algebraic general formula? And, how do we negotiate between the reviewer's critique and the mathematical fact that there exists a relationship between the geometric conditions necessary to determine a locus of points and the algebraic conditions necessary to determine the corresponding equation of the locus? For example, in the two passages in Figure 2 taken from two different analytic geometry texts, the authors state the mathematical significance of the minimum number of points required to establish a polynomial equation in closed form.

Third, recent seminal semiotic investigations in the learning of mathematics (*cf.* Dörfler, 2003; Duval, 2006; Otte, 1997; Radford, 2006) have surfaced the complicated relationship between form and content that makes the task of comprehension to be a primary issue in how learners construct and negotiate with others various cycles of referent-representation-personal meanings (*i.e.*, Peirce's

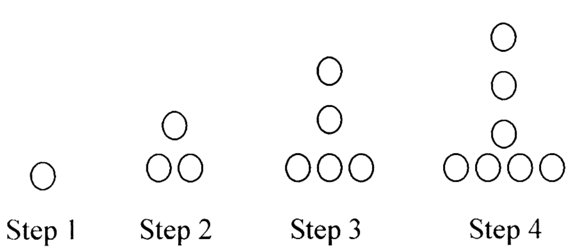


Figure 1. Circles pattern

47. The Equation of the Circle Derived from Three Conditions. — The correspondence between the geometrical conditions necessary to determine a circle and the algebraic conditions necessary to determine its equation are analogous to those discussed for the straight line. The number of independent constants in the general linear equation

$$Ax + By + C = 0$$

is two, for if we divide each term by A, we have the form

$$x + B'y + C'' = 0.$$

Geometrically, a straight line is determined by two points, or in general by two geometric conditions. Both forms (13) and (14) of the circle equation involve three arbitrary constants. Geometrically a circle is determined by three points not on a straight line. A circle may be determined in other ways than by the condition of passing through three points, but the determining condition is in any case threefold.

(a)

183. The equation of a conic through given points. The general equation of a conic may be written

$$Ax^2 + 2\,Hxy + By^2 + 2\,Gx + 2\,Fy + C = 0, \quad . \quad (1)$$

and contains five parameters, the five ratios between the coefficients A, H, B, G, F, C. Since five equations, or conditions, will determine those parameters, in general five points will determine a conic. That is, in general, *a conic may be made to pass through five, and only five, given points.*

(b)

Figure 2. *(a)* Analytic Geometry *(Wilson, 1937, pp. 69–70); (b)* An Elementary Course in Analytic Geometry *(Tanner & Allen, 1898, p. 307)*

Toothpicks are used to build shapes to form a pattern. The table below shows the number of toothpicks used to build a particular shape.

Shape number	1	2	3	4	5	6	7	8	20	60	N
Number of toothpicks	3	7	11	15	19	23					

Figure 3. Toothpick pattern

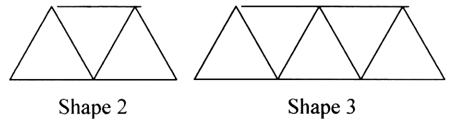


Figure 4. Elisa's figural interpretation of the toothpick pattern in Figure 3

interpretant). For example, when Elisa, an 11-year-old sixth grader, was asked to obtain a generalization for the sequence of numbers in Figure 3 prior to a teaching experiment on generalization, she initially used toothpicks and formed one triangle for Shape 1, followed by two and three adjacent triangles, respectively, for Shapes 2 and 3 (Fig. 4). Thus, from both psychological and epigenetic perspectives, we still do not know to what extent and how early learners have access to the possibility that patterns such as those in Figures 1 and 3 could also be interpreted as anything besides a linear function. Given that the patterning tasks I have used in all published studies were oriented by the assumption that students were learning about the formal process of generalizing for the first time, I thought that figural cues with a simple linear structure would need three - and generously, four - initial pictorial conditions following conventional practices. Little did I know that the patterning activities that I implemented were evaluated as being "narrowly defined" and somehow complicit in the propagation of what De Bock, Verschaffel, and Janssens (2002) have, rather perniciously, referred to as the "illusion of linearity."

A fourth motive for exploring abduction deals with more than a decade's worth of studies that surface difficulties of children and young adults in establishing general formulas involving patterns of mathematical objects (see ZDM, 2008). Certainly, there is more to generalizing from particular cases (or samples) to the entire domain (or population) than simply implementing a process of "applying a given argument in a broader context" (Harel & Tall, 1989) or "deriving or inducing from particulars, identifying commonalities, and expanding domains of validity to include large classes of cases" (Dreyfus, 1991, p. 35). That is, a complex of factors (cognitive, cultural, extra-cultural such as linguistic, etc.) influences the construction of a "generality" that, according to Dörfler (2008), is a way or practice of using and interpreting "signs, like graphs or letters, [that] are not general by themselves" (p. 1). What is induction, really? If it is, indeed, a type of reflective action that infers a general law after examining a number of specific cases, then why does it seem to be so much easier said than done? For example, results of at least five years of performance assessments on generalization given to more than 60,000 middle and beginning high school students in the Bay area in Northern California reveal a stable ceiling value of 20% success rate in the construction of a general formula (Rivera, 2007; Rivera & Becker, 2005); global results appear to be within the same percentage rate.

Reading Sharpe's (1970) account of Peirce's perspective of scientific practice, I became convinced of the complicit role of abduction in induction - which, I thought, has not been fully unpacked in reported psychological accounts of generalizing in mathematical contexts, especially and, in particular, in patterning activity. Sharpe (1970) nubs Peirce's abduction-induction process in the following simple but elegant manner: "We enter inductive reasoning with a preliminary hypothesis [*i.e.*, abduction] about what character the instances have in common, the hypothesis being that they have the character that has been predesignated" (p. 22). In the third section of this essay, I propose and empirically demonstrate a pattern generalization scheme at the level of elementary algebraic thinking in which abduction and induction are both seen as necessary and prior before any final statement of generalization is inferred. Such a scheme assumes some kind of a cognitive unity between argumentation and an inferred justification of a general formula for a pattern with argumentation activity as reflecting a nonlinear, synergistic relationship between abduction and induction.

At this stage, the most significant task is explain what abduction is all about and how it is similar and different from other inferential acts that we employ in doing mathematics and, in particular, in developing mathematical reason. Two more examples from Elisa's class of sixth graders provide further starting points. In the next section, I explore abduction in greater detail beginning with a general discussion of Peirce's thoughts and then quilting relevant studies in mathematics education that illustrate key aspects of abduction. The interpretive structure in the next section is my own making that should provide readers with a sense of how I characterize abduction and explain the arguments presented in this essay. When James, near the closure of a teaching experiment on generalization, was asked to establish a formula for the pattern in Figure 1, he developed the form $C = 2n - 1$, where C represents the total number of circles in step n with the unstated assumption that the pattern was increasing. He then explained his formula when he first noticed the constant addition of two circles ("keep adding two") and then saw that each step involved "doubling a row and minusing a chip" (Becker & Rivera, 2006, p. 99). All learners do not *see* - or *objectify* in Radford's (2006) sense - the same way as James which perhaps represents the desired intentional version. That is to say, institutional forms may not likely align with the situated character of some learners' generated forms. In Figure 5, Jenna perceived the next four steps in Figure 1 in a different way. While Jenna did not have the algebraic means at her level to express a general formula more formally beyond a general description, it was nevertheless clear to her how the pattern grew which then determined for her the generalization she established (Fig. 6). Keeping in mind Sharpe's (1970) thoughts in the quote above, James and Jenna approached the generalizing task in Figure 1 using two different "explanatory hypotheses" or abductions.

Suffice it to say, Lee's (1996) assumption about patterning as eventually taking the route of *algebraic usefulness* - that

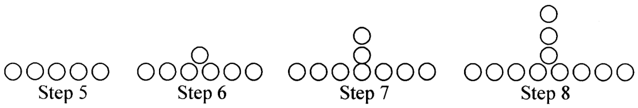


Figure 5. Jenna's extension of the circles pattern in Figure 1

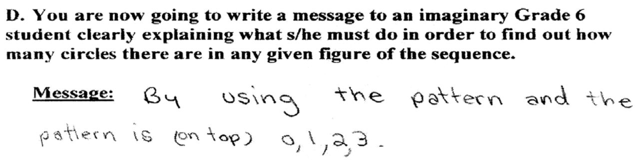


Figure 6. Jenna's general description of the pattern in Figure 1

is, an expressly articulated desire to produce generalizations that always leads to a formula – is always, already a complicated task because learners tend to impose several different abductions. Further, taking into account their differing generalizing abilities, which have been empirically shown to take several forms (Radford, 2006; Rivera & Becker, 2006), it seems the more relevant pedagogical task at this time involves assisting learners to develop plausible algebraically useful abductions in patterning. "Plausibility," Peirce writes, "is the degree to which a theory ought to recommend itself to our belief independently of any kind of evidence other than our instinct urging us to regard it favorably" [CP 8.223]. However, taking the cue from Jenna and James, with any claimed plausible representation of a pattern of a few named instances that is demonstrated to be algebraically useful, one will always have to wrestle with the issue of underdeterminism in which the instances could be captured in several different ways unless additional conditions that have not been drawn from the instances themselves are imposed. Following Peirce, one of these conditions involves situating induction in a follow-up confirmatory role. Where and how one draws the line between abduction and induction is pursued in the next section. Further, how all these conditions factor together are pursued in the third section in which I submit a pattern generalization scheme that I demonstrate with an example from a clinical interview with Shawna, Elisa's classmate, in relation to her work on the pattern in Figure 1. In the proposed scheme, I claim that complete abduction is necessary before any final statement of generalization is accepted in relation to a pattern sequence.

The treacherous zone of abduction [2]

Peirce developed abduction in the late nineteenth century in relation to induction and deduction. Quite recently in mathematics education research, abduction has figured prominently in a few investigations such as algebra (Radford, 2008; Reid, 2003; Rivera & Becker, 2007), the history of early algebra (Heeffer, 2006), arithmetic (Sáenz-Ludlow, 1997), calculus (Ferrando, 2000), geometry (Arzarello, Micheletti, Olivero, & Robutti, 1998; Pedemonte, 2008), and problem solving (Cifarelli, 1997, 1999; Cifarelli & Sáenz-Ludlow, 1996).

De Souza's (2005) diagrams of the deductive and abductive cycles in Figure 7 show the difference between,

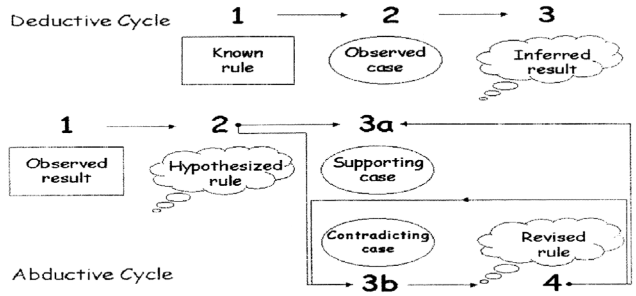


Figure 7. Deductive cycle and abductive cycle (De Souza, 2005, p. 43; reprinted with permission)

including the processes involved in, the two inferential types. What is not clearly articulated in the diagrams is the important role played by the available knowledge base that determines which inferential act to use in a particular situation. For Peirce, the validity and "truth-preserving" (Josephson & Josephson, 1994) nature of deduction depends on having a complete knowledge base – that is, we simply cannot deduce a true conclusion from an incomplete set of true premises. The monotonic nature of all deductive inferences (*i.e.*, the canonical form "If p, then q; p. Therefore q.") already precludes the need to assume additional hypotheses – that is, the stated premises must contain all the necessary facts in order to state a valid conclusion. Thus, a deductive inference primarily *predicts in a methodical way* an inevitable necessary and valid conclusion or result because all the required premises are known.

With abduction and induction, also called ampliative inferences [CP 2.709], conclusions or rules are drawn from an incomplete knowledge base; hence, they are deductively invalid. Such inferences harbor generous and possibly fallible conclusions because they are not necessarily drawn from the premises (Deutscher, 2002). Cifarelli and Sáenz-Ludlow (1996) point out their rather perfidious nature, that they are basically "plausible hypotheses on probation" (p. 161). In the above abductive cycle, there is the "absent presence" of *creativity* that fuels the construction of possible hypothesized rules or abductions. Especially in the case of mathematics, Ferrando (2000) insists on the necessity of creativity that not only assists in establishing a mathematical proof but pervades in all phases of mathematical thinking and practice.

Abductive claims are in no simple terms the same thing as conjectures. What I consider to be partial abductions amount to mere conjectures that is a necessary action in Phase 2 of the abductive cycle in Figure 7. Thus, surprising facts and conceptual leaps from single instances to plausible hypotheses are all manifestations of partial abductions. However, complete abductions go through all four phases of the abductive cycle. Further, within and beyond the abductive cycle itself, induction is implemented in order to test a revised rule repeatedly. This explains why, for Peirce, induction is stronger than abduction. Induction generates a conclusion that has already been drawn from repetition which could then be evaluated as being false, true, or true to some degree unlike abduction which fuels discovery on the basis of one known case or a few instances. Thus, induction, defined in the general, classical way as the shift from the particular to the general, necessitates, first and foremost, the stipulation of an abductive claim that is then subjected to repeated testing, which becomes the essence of induction. In other words, while abduction involves discovering a new hypothesis, induction establishes the strength of the hypothesis through an experimental confirmatory process that produces tendencies in support of the hypothesis and are, thus, not novel facts. Hence, there is closure in induction with a statement of a probable obvious generalization unlike abduction that never seems to do so since it works primarily in discovering or suggesting new events (Abe, 2003) or novel actions (Cifarelli, 1999). For Flach (1996), an abductive claim is always confronted through inductive testing, and increased inductive success means increased confidence in the abduced claim. Hoffmann's (1999) point astutely captures the original Peircean sense in which abduction forecloses any illusion of a perspective-free induction: "induction is not what can be generalized from a sample of data, but only a quantitative determination of what is already given by abduction" (p. 272). Hence, the foregoing sense in which generalization was conceptualized articulates the mutual, complicit relationship between abduction and induction.

Considering the "truth-producing" (Josephson & Josephson, 1994) power of abduction, how do we choose, evaluate, select, and justify the most sensible explanatory hypothesis out of several ones that are offered? Clearly, a sensible action is to allow induction and deduction to take place. For example, Pedemonte (2008) allowed her students initially to use abduction to develop conclusions about certain mathematical contexts such as the famous angle-sum property of triangles in Euclidean geometry. Next, they employed induction to confirm their generalization, which they then proved deductively in the final phase. Thus, developing an abductive claim that then undergoes inductive and deductive justification can certainly increase the validity of the claim. Boero, Garuti, and Mariotti (1996) describe such a process as cognitive unity. That is, learners first enter the argumentation phase that allows them to produce abductions. Then they transition into the formal phase in which they provide a structure for their arguments leading to a proof in the form of a valid logical chain. Pedemonte (2008) reinscribes the cognitive unity thesis within the abduction-induction-deduction perspective and with abduction situated as being prior and necessary to deduction. Further, induction arises in situations when students repeatedly test particular cases and eventually produce a generic case that has relevance on the anticipated deductive argument.

Josephson and Josephson (1994, p. 5) have proposed the following version of what I consider to be a complete abduction with the addition of Premise 3, which requires that H be *the inference that yields the best explanation.*

D is a collection of data
(facts, observations, givens). (1)

H explains D (would, if true, explain D). (2)

No other hypothesis can explain
D as well as H does. (3)

———————————————

Therefore, H is probably true. (4)

Thus, compared with the monotonic form of a deductive inference, Arzarello *et al.* (1998) point out the reverse deductive nature of a complete abductive inference as shown above. In the context of this essay, generalizing patterns on the basis of partial abductions is not sufficient; it has to evolve into a complete abduction following the above four steps, which provide further criteria and support to the abductive cycle in Figure 7. Josephson (1996) further suggests the following points in evaluating the strength of an abductive conclusion:

- how good H is by itself, independently of considering the alternatives;
- how decisively H surpasses the alternatives, and;
- how thorough the search was for alternative explanations. (p. 3)

Josephson also claims that if a generalization is used to explain a perceived commonality among a given class of instances, "it does not explain the instances themselves" (p. 2). For the warrant in an explanation lies in its capacity to "give causes," and a generalization cannot and does not provide an explanation that causes the instances. For example, while the general abduced form $C = 2n - 1$ explains the relationship between elements in the class $\{1, 3, 5, 7, \ldots\}$ in Figure 1, however, it does not cause them. Thus, the nature of an abduced generalization is determined by the observed "frequency of [a] characteristic" in both the small and extended samples (p. 3). In other words, what needs to be explained or be given a "causal story" deals with how a generalization could sufficiently represent the nature of the frequencies involved in a class. In the case of patterns especially, an abduced generalization may be explained in terms of how well it fits and is driven by the largest number of observed cues in the relevant domain. For example, unlike Jenna's partial abduction (Figs. 5 and 6), the complete abductions of James and Dung (Fig. 8) are expressed articulations of their explanations of the general formula $C = 2n - 1$ only after they have established an observation about the frequencies that were presented to them, including the ones that they calculated. Their explanations fit all the frequencies that they perceived to be behaving in a particular way that they then inductively projected (*i.e.*, observations → All A's are B's → The next A will be a B) onto all the unknown instances beyond their perceptual field.

While Peirce situates abduction as being prior to induction – a view shared by Pedemonte (2008) and Boero *et al.* (1996) – Arzarello *et al.* (1998) consider abduction, induction, and deduction to be working dynamically in "coexistence" with one another. In their model, learners

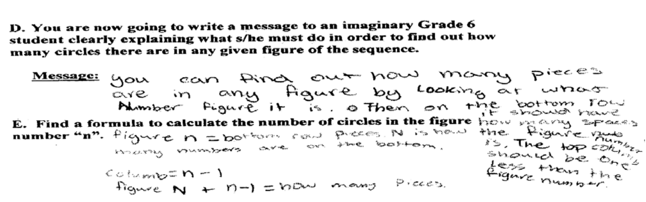


Figure 8. *Dung's explanation of his generalization $C = 2n - 1$ for the pattern in Figure 1*

transition from conjecturing to proving in geometry as soon as they change their attitudes and modes of control in various phases of exploring and selecting relationships about a target object that they both construct and perceive. Further, while they situate abduction in the "most delicate cognitive point" (p. 31) in their model, they also claim that it in fact permeates the "whole conjecturing and proving processes" (p. 30), since it is the driving action that enables students to produce conjectures that lead to conditional statements and to the proof(s) and then again when further exploration and proof become necessary. This heady, dynamic, and almost playful characterization in which Arzarello *et al.* (1998) endow abduction captures the original Peircean intent when they describe abduced inferences as "put[ting] on the table all the ingredients of the conditional statements" (p. 30), with some ingredients being more necessary and significant than others.

In the case of patterning activity, Radford (2008) positions abduction at the beginning phase of generalizing action in algebra. He extrapolates, on the basis of his investigations with groups of middle school children, the following scheme relevant to the algebraic generalization of patterns: First, learners notice a local commonality *C* among the known instances in a sequence *S*. Abduction then takes its course when they generalize *C* so that it applies to both known and unknown terms in *S* in which case *C* transforms into a warrant that enables them to test it and finally deduce an algebraic expression for *S*.

Dead on, (complete) abduction is not simply about entertaining an explanatory hypothesis – that is, a partial abductive claim – as it is also about needing to have it "entertained interrogatively" [CP 6.524] using induction. My own concern about putting abduction to work, especially in pattern generalization, deals with how to assist learners to choose wisely in the abductive space that has opened up a farrago of plausible explanatory hypotheses that may or may not lead to complete abduction. For Peirce, the key is "intelligent guessing" [CP 6.530] with the "power of guessing right," since the "human mind is akin to the truth in the sense that in a finite number of guesses it will light upon the correct hypothesis" [CP 7.220]. However, we also require that all complete abductive claims involving patterns to be as much as possible algebraically useful.

A generalization scheme for patterns

Figure 9 illustrates a dynamic pattern generalization scheme in which abduction and induction are both situated at the kernel of the generalizing process, thus, enabling a candidate regularity *R* to transform into a viable general form *F* after it is empirically verified in several extensions of a given pattern. The additional requirement of *F* needing to be evaluated as being the best inference opens the scheme to the possibility of a spiral movement before a final generalization is deductively inferred onto the pattern (*i.e.*, following the deductive cycle in Fig. 7). I have drawn the scheme from results of clinical interviews that I and a colleague conducted with twenty-nine sixth grade students whom we asked to establish and justify a generalization for patterns (see, for *e.g.*, Figs. 1 and 3). In the following interview episode, I demonstrate the scheme using Shawna's work in relation to the pattern in Figure 1.

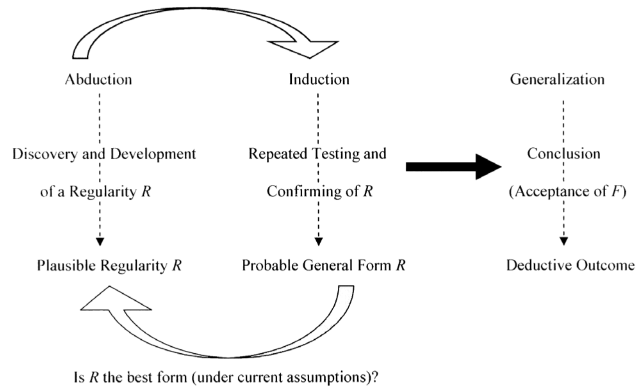


Figure 9. A pattern generalization scheme

When Shawna was first asked to obtain the total number of circles in step 10, her initial answer was 5. Unsure of her response, she obtained totals in individual cases and then inferred that each case "adds by 2." The additive inference marked the beginning of her abduction, a partial abduction in phase 2 of Figure 7. To further support her rule, she drew step 5.

JRB: How did you know how many to put going ahm vertically and how many to put horizontally?

Shawna: [*Referring to step 4.*] Because it adds one on here [top of the *column*] and one on here [*right end ofthe row*].

Shawna then continued to draw steps 6 and 7 (Fig. 10) in a way that preserved the structure of what she saw on the questionnaire with the drawn steps. Both pattern steps 6 and 7 indicate the use of induction in which Shawna confirmed her abduced hypothesis in two more cases. When then asked if there was an "easier" way to obtain the total for each step, she interpreted it as constructing figures not by hand but by using the colored chips on the table. She built steps 8, 9, and 10 with the chips and concluded 19 circles by "adding two more each time … right there [*top of the column*] and there [*right end of the row*]." In dealing with the far generalization task of finding the total count for step 100, she suggested adding "more than 2 instead."

Shawna: Instead of adding 2 to the figures, add 4.

JRB: Add 4 and how is that gonna help you figure out step 100?

Shawna: It would be more easier because ahm instead of adding 2 it just does that by 2. So with 4, it's gonna be more faster.

Figure 10. Shawna's illustrations of steps 5 through 7 for the circles pattern in Figure 1

Shawna's suggestion of adding 4 was driven by efficiency. Using the colored chips, she began the difficult process of obtaining the total count for step 100. Sensing Shawna's uncertainty about building steps by repeatedly adding 4 and not knowing how to account for the correct step number, the interviewer then tried to scaffold her to see an invariant structure among the pattern cues on the basis of what she had already begun to notice earlier (lines 4 to 6).

13 JRB: Do you have any idea how many you
14 would have [*referring to the row*] on
15 step 100?

16 Shawna: Maybe about 50?

17 JRB: Are you sure about that? So tell me how
18 you would build step 100? Are you gonna
19 make all the ones between [*step 8 &*
20 *step 100*]?

21 Shawna: Uhum.

22 JRB: That'll gonna take a pretty long time,
23 huh? Is there an easier way to do that?
24 That's why you want to do 2 by 4s, right?

25 Shawna: Maybe add by 5?

At this stage in her thinking, Shawna has already offered an abduction ("add 2") and two counting suggestions ("add 4;" "add by 5") that were both driven not by the pursuit of an explanatory structure but by the immediate need to be efficient. In fact, what prevented her from fully considering her two counting suggestions had to deal with her inability to organize her data so that she could easily monitor how many circles would go with which step number depending on whether it was constantly adding by fours or by fives. So, once again, the interviewer tried to help her notice a structure in pattern step 7 indicative of a method that Mason, Graham, and Johnston-Wilder (2005) refer to as specializing:

26 JRB: Let's look at step 7. How many would
27 you have on this side [row]?

28 Shawna: Seven.

29 JRB: And how many would you have going up?

30 Shawna: Six.

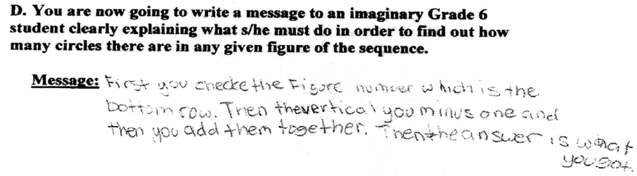
JRB and Shawna once again investigated steps 6 and 8 which provided additional supporting cases in order for Shawna to obtain complete abduction through specializing. JRB then asked her to consider pattern step 100.

31 Shawna: The bottom would be 100 and the
32 vertical would be 99.

33 JRB: And how many would there be altogether?

34 Shawna: 199.

Shawna's message (Fig. 11) to an imaginary student, in which she described how to obtain the total number of circles in any given step, is reflective of a final generalization


D. You are now going to write a message to an imaginary Grade 6 student clearly explaining what s/he must do in order to find out how many circles there are in any given figure of the sequence.

Message: First you checke the Figure number which is the bottom row. Then thevertical you minus one and then you add them together. Thenthe ansuer is what you got.

Figure 11. Shawna's generalization for Figure 1

that involves the addition of n circles horizontally and $n - 1$ circles vertically.

Two points are worth noting. First, Cifarelli and Sáenz-Ludlow (1996) have already noted the slippery and transitory nature of abduction. However, in terms of reflective action, it should not be difficult to delineate between stating a plausible hypothesis (abduction) and confirming it (induction). Shawna's articulation of an initial abduction, which involves adding by twos, was followed by a verification process that involved constructing pattern steps 6 and 7. Her recursive rule is already a complete abduction in itself and could be characterized as an arithmetical generalization (following Radford, 2006) on the basis of noticing both numerical and operational schemes. Shawna's counting suggestions of adding by fours and then by fives are nonproductive guesses, albeit necessary for her as she was attempting to develop a more significant explanatory hypothesis. It is simply not possible to tame the sources of abduction – that is, it could come to the knower as a surprising fact, a flash, an insight [CP 5.181], a conceptual leap (Deutscher, 2002), a novelty, or an anomaly (Hoffmann, 1999) and is driven by factors that are never predetermined or easily predicted such as, in Shawna's case, efficiency and the joint activity with the interviewer. Thus, conjectural activity is the domain of partial abductions that do not necessitate a follow-up induction process (*i.e.*, phases 1 and 2 of the abductive cycle in Fig. 7). Complete abduction, however, necessitates a combined abduction-induction process. Finally, generalization necessitates transposing the complete abduction into a deductive hypothesis within the framework of the deductive cycle shown in Figure 7.

Second, while it is the case that abduction and induction are two conceptually distinct inferential acts, in patterning activity, I view both as mutually determining each other and that, depending on the learner, can spiral a number of times before a final generalization is sufficiently constructed and justified. Only complete abductive sequences (*i.e.*, R_1, R_1, …, R_i) could lead to a sufficient generalization (F). My use of the term *spiral* is a metaphorical gesture in the sense that I visualize the emergence of increasingly sophisticated complete abductions in relation to a pattern sequence and not a mere single event of continuous curvature that winds around a beginning partial abduction and then smoothly progressing towards the best generalization through succeeding abductions. Breaks can occur when newer and better abductions arise – abductions that may not be related to the previous ones.

Conclusion

I began this essay with four epistemo-ontological trepidations that are all webbed together in some way. First, how to deal with underdeterminism in induction is a fundamental age-old philosophical conundrum brought about by the fact that there is no way of comprehending what we can never fully perceive except through the patterns that we project onto, say, a collection of similar objects. Second, how to characterize a well-defined pattern sequence with a well-defined (closed) formula or generalization that removes any doubt of underdeterminism has now become a mathematical issue. Third, how to assist learners to comprehend patterns mathematically is a psychological concern, especially because there are implications in the instruction and learning of generalization that is, without a doubt, a desideratum in all mathematical activity – or, in Mason's (1996) words, the heartbeat of mathematics. Fourth, what does it mean when one performs induction in patterning activity? Are generalizing and inducing one and the same thing, and should they be? My basic response to all four issues is abduction, a concept which Peirce has initially relegated the inferential function of "choosing an explanatory hypothesis." I refer the readers to excellent philosophy books on induction, for my primary concern in this paper is generalization in patterning activity.

In thinking about the inevitable condition of underdeterminism in patterning activity, including the reviewer's explicit articulation of his or her concern in the introduction, I am well aware that it is possible to obtain several different generalizations for an always, already incomplete pattern sequence of cues. Philosophers such as Reichenbach (1938) have dealt with such skepticism by submitting the simplicity hypothesis that they justify on the basis that it is an accepted scientific practice. In fact, a simpler generalization is more capable of being contradicted with new evidence than a complex one (Barker, 1957). Also, it seems to be the case that any generalization is in the eye of the beholder – that is, it is relative to a learner's current state of mathematical competence. For example, in the pattern in Figure 1, second graders may only see an oscillating sequence, while sixth graders can perceive a nontrivial increasing linear sequence, and precalculus students may even be able to produce a nonlinear polynomial generalization. It goes without saying that abduction is a subjective process. The fact that a sixth grade 11-year-old child (completely) abduces a linear generalization for, say, the pattern in Figure 1 is no reason to conclude that he or she harbors the illusion that it is the only valid generalization, for the simplicity of the linear model might be what the learner finds cognitively appealing. Somewhat related to this idea of a simple model is the Gestaltist law of *prägnanz* – the rule of conciseness – which both acknowledges and foregrounds our natural cognitive disposition towards the parsimony of forms, of trying to see and analyze patterns of whatever form by imposing a structure that is simple, symmetric, and has order in some way. Especially with complex figural patterns, some students oftentimes break down aspects of what they perceive to be structures of the patterns into simpler components or units. Going back to the basic idea that is pursued in this essay, I observe that students' difficulty with patterning is not because they cannot notice something invariant; in fact, they see several plausible structures and, thus, selecting the best one – that is, (complete) abduction – is the most pressing concern for them.

Shawna's case provides an interesting demonstration of a spiral progression in generalizing activity. Jenna's attempt at a generalization, unfortunately, failed to spiral further, which might have led to a closed formula. Her partial abduction (Fig. 6) could be interpreted as a conjecture about how she comprehends or sees growth in the pattern in Figure 1. Indeed, while she was able to confirm her hypothesized rule on four near generalization cases (Fig. 5), she was unable to either support the rule or provide a contradicting case that might have led to a complete abduction. In the scheme shown in Figure 9, I define generalization as the result of a complete abduction that involves a combined abduction-induction process. Hence, the issue of constructing generalizations, especially in patterning activity, is not only a problem of induction but of abduction as well. Jenna's thinking about the pattern in Figure 1 is typical of some children and adults who, armed with much patience, would produce extensions (*e.g.*, case by case and organized in a table) while remaining oblivious to a generalization. Also, the issue of abduction in patterning activity is complicated by the fact that the repertoire and content of available strategies in establishing an algebraically useful generalization actually depend on the level of learners' current domain-specific knowledge. For example, in the case of the pattern in Figure 1, Shawna needed some scaffolding from the interviewer before she began to notice a global invariant structure through specializing, unlike Dung whose work is shown in Figure 8. Like Dung, who did not need any assistance whatsoever, James easily perceived the doubling of a row and the minusing of a chip that led him to his closed formula. Such analogical ways of reasoning are useful abductive strategies that many students, in varying stages of learning through more experience, do not possess.

Cifarelli (1997) has surfaced the heuristic dimension of abduction in problem solving. He claims that abduction fits squarely within learners' "ongoing sense-making processes" (p. 20); it is a structuring resource that enables them to "organize, re-organize, and transform their mathematical actions" (Cifarelli, 1999, p. 219) in the course of developing a solution to a problem-solving situation. Cifarelli (1999) characterizes the problem-solving space as marked by "idiosyncratic activity such as the generation of novel hypotheses, intuitions, and conjectures" that appears to students as "surprising facts" (p. 217). His analysis of undergraduate majors' thinking on word problem tasks surfaces the creative power of abduction, including its capacity to drive, mediate in, and structure the emergence of a problem-solving situation among them, engendering them to develop interpretations that oftentimes lead to more "questions and uncertainties" (Cifarelli, 1999, p. 218; *cf.* Cifarelli & Sáenz-Ludlow, 1996). Cifarelli's (1999) empirical demonstration of abduction is reflective of Arzarello *et al.*'s (1998) position in its pervading role throughout cognitive activity. Because learners tend to generate inferences and more facts, abductive actions predispose them to simultaneously problem pose and solve.

For Hoffmann (1999), however, abduction involves both inferential and heuristic dimensions. I think that if abduction is simply interpreted as a heuristic strategy for generating novel hypotheses, then it undermines its distinct conceptual character from, say, acts of conjecturing. Cifarelli's (1999) investigations on students who employed abduction as a heuristic in developing "surprising" novel facts to solve closed problems do not push abduction far enough. Pedemonte's (2008) extrapolation of abduction in relation to geometry learning with mediation by the Cabri, a dynamic inductive-driven software due to its built-in dragging function, and Radford's (2008) architecture of algebraic pattern generalization paradigmatically demonstrate how inducing a fact depends on the abductive conditions that constrain the probable fact which then becomes a warrant in the fold following a deductive inference. The result of students' activities in the studies of Pedemonte and Radford reflects, in Hoffmann's (1999) terms, the coming together of the inferential and heuristic in abduction whose strength has undergone inductive confirmation. Radford (2008) contrasts generalizations resulting from complete abduction from those which have been established by "naïve induction," such as when students derive a generalization by trial and error, in the sense that a general form is produced as a consequence of accidental uniformity among a limited set of particular cues.

Harel (2001) explored the roles played by *result pattern generalization* and *process pattern generalization* in inductive reasoning, with the latter seen as being more significant than the former in the construction of a mathematical induction proof. The two pattern generalization types seem to conceptually shadow what Dörfler (1991) has referred to as empirical and theoretical generalizations, respectively. While result pattern generalizing focuses on regularities among the particular instances, process pattern generalizing dwells on stable relationships among the local processes used. However, the complexity of abduction that occurs underneath the enactment of a necessary smooth transition from the first result or process to the next and then still to the next leading to a generalization is oftentimes the root of many a student's dilemma, an empirical fact that seems afloat but is not significantly addressed in Harel's (2001) generalization types, or Dörfler's (1991) for that matter. I use a similar argument in the case of Simon (1996) who proposed transformational reasoning as a way of explaining the series of physical and/or mental enactments which learners manifest in a problem-solving situation in order to sense or gain experience about how some rule, operation, or system works in the situation. Perhaps transformational reasoning is "beyond inductive and deductive reasoning"; however, a more fundamental dilemma involves ascertaining factors that drive such enactments, since learners simply cannot perform an operation on an object without first assuming an abductive claim.

Thus, all reasoning types or generalization processes need to foreground the fundamental act of selecting and choosing a hypothesis. Peirce, and Hegel and Kant before him, postulated the necessity of abduction, a "cognition by experience" (Redding, 2003) in ampliative activity such as generalizing and problem solving. He, in fact, developed abduction when he "absorbed" and "assimilated" the logic of mathematics within the logic of science and, thus, saw the need to account for an inference with a nature that could not be categorized deductively or inductively (Fisch, 1986, p. 392). Fisch (1986) interprets Peirce's justification of abduction, and rightly so, in the following manner: "even the pure mathematician goes through the same three stages of inquiry as the scientist; the difference is that his experiments are performed upon diagrams of his own construction" (p. 392).

Notes

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[2] Characterizing Peirce's thinking about abduction necessitates making a distinction between the early and late Peirce. Flach (1996) categorizes the two perspectives as syllogistic and inferential, respectively. In this essay, I focus on the implications of the inferential domain in relation to pattern construction and generalization.

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