ENGAGEMENT WITH MATHEMATICS AS CATALYST FOR BACKWARD TRANSFER

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"We know that $y = x + 1$ moves up, so why does $y = (x + 1)^2$ move left? Why is there such an inconsistency?" This was a question from a prospective teacher, as Rachel's [1] class discussed transformations of quadratic functions and considered different ways of explaining the 'counterintuitive move' to the left of the canonic parabola $y = x^2$ indicated by the '+1'. At this moment, Rachel had a realisation, which was new to her: While usually the graph of $y = x + 1$ is considered as a vertical translation of $y = x$, one unit upwards, the transformation also can be seen as a horizontal translation, one unit to the left. Becoming aware of this relationship created a sense of consistency for Rachel.

We look at Rachel's story as an example of an experience that occasionally happens to people engaged with mathematics. However, following Goldenberg and Mason (2008) in seeing any example as an example of something, we ask, "Rachel's story, what is this an example of?" In the following two sections we introduce two theoretical constructs—backward transfer and thickening understanding—and exemplify them with Rachel's story. We then describe a study inspired by Rachel's story in which interweaving the two constructs contributes to enriched understanding of expert learning.

On backward transfer

We consider Rachel's realisation as an example of 'backward transfer'. Hohensee (2014) defined *backward transfer* as "the influence on prior knowledge by the acquisition and subsequent generalisation of new knowledge" (p. 13). In accord with this view, Hohensee, Gartland, Willoughby and Melville (2021) conceptualised backward transfer as a type of learning that influences students' prior ways of reasoning. We expand on this view, replacing "student" with any individual engaged in a mathematical activity, interpreted broadly. That is, backward transfer is an experience that influences individuals' prior knowledge and prior way of reasoning related to a mathematical idea. Hohensee, Gartland, Willoughby and Melville (2021) differentiated between backward transfer that enhances prior learning and that which undermines it, and our focus is on the former. Backward transfer enhances reasoning when prior ways of reasoning become more closely connected to the "structural core" (Hohensee, 2014, p. 138) of the concept.

The idea of 'backward' in Hohensee's research related more advanced mathematics for learners, such as quadratic functions, to simpler and previously learned topics, like linear functions. Rachel's story also connects backward these two concepts—quadratic and linear functions—as it pertains to their graphs. Rachel's new reasoning, that influences her prior knowledge, acknowledges that a translation of a linear function's graph can be conceptualised in different ways, acknowledging consistency with translations of parabolas.

On thickening

Building on research in anthropology, Liljedahl, Sinclair and Zazkis (2006) use the "adjective 'thick' to describe a learner's layered, rich, contextual, and often affective understanding of a mathematical concept. Likewise, [they] use the adjective 'thin' to describe a learner's understanding of a mathematical concept that lacks richness and connectivity" (p. 254). As above, we expand this description to account for any person engaged in a mathematical activity, rather than restrict to students. Noting that 'thickness' is relative, a mathematical activity can thicken one's understanding by adding layers and context. Liljedahl, Sinclair and Zazkis noted two dimensions of extension: an increase in the number of conceptual layers related to a concept and an increase in the thickness of each conceptual layer.

Returning to Rachel's story, we suggest that her understanding of linear functions and their graphs has thickened by acknowledging multiple interpretations of how the graph of $y = x$ is transformed into the graph of $y = x + b$. Her previous 'thinner' understanding was limited to vertical translation, with 'b' as y-intercept. The thicker layer of transforming the graph included the possibility of horizontal translation. Additional thickness was achieved by connecting the layers of linear and quadratic graphs and achieving consistency with the counterintuitive direction of the horizontal translation in the case of quadratic and other polynomial functions.

Inspired by Rachel's story: the study

We wondered about the existence of experiences, similar to Rachel's, of people engaged in mathematical activity; that is, the experience of acquiring new understanding as a result of backward transfer. This led to the following research question: In what ways does engagement in mathematical activity enhance an individual's understanding of previously learned mathematics? We interpret the notion of mathematical activity broadly, to include studying mathematics, mathematical problem solving, and activity related to teaching mathematics, such as preparing for instruction or addressing students' queries.

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We shared Rachel's story with individuals who held at least a master's degree in mathematics, and asked them to share stories of similar experiences, if any occurred, in their mathematical careers. In what follows, we relate six of these stories, chosen to illustrate a variety of experiences across mathematical concepts and kinds of engagement with mathematics.

In presenting the participants' stories we follow Mason (2002) in distinguishing between accounting-of and accounting-for. The term *accounting-of* provides a brief description of the key elements of the story, suspending as much as possible evaluation or reflection. This serves as data for *accounting-for*, which provides explanation, interpretation, and theory-based analysis.

We present the stories in three pairs, with each pair united by a common mathematical concept: derivative, distributivity and the binomial theorem. The accompanying analysis of each pair foreshadows two main ideas which are further elaborated in the discussion: 1) we extend the notion of backward transfer to include new facets and settings, and 2) we use the construct of thickening to elaborate on the mechanism of backward transfer. Taken together we add to the conversation on learning mathematics by illustrating a variety of ways in which a new realisation shapes the understanding of familiar concepts and enriches connectivity among them.

Robyn and the rope

Robyn, a mathematics education researcher, presented her students with the following problem many times:

> If a rope is wrapped around the equator of a spherical earth, then the needed length of the rope is the circumference, the length of the equator. Now imagine that this rope is placed on 1m high poles. How much more rope is needed?

Because the circumference of the earth is so large, students often anticipate that a large amount of rope will be needed to complete this task, when only about 6 additional meters are needed:

$$2\pi(r+1) - 2\pi r = 2\pi r + 2\pi - 2\pi r = 2\pi$$

When discussing this problem with a mathematician colleague as part of the data collection for a research project (Yan, Marmur & Zazkis, 2021), Robyn learned a new way of arriving at this answer. As her colleague explained, the answer to this problem was a natural result of differentiation: If the circumference of the earth is given by $C(r) = 2\pi r$, then the instantaneous rate of change of the circumference is given by $C'(r) = 2\pi$. Thus, if the radius of the earth is extended by 1 meter, then the circumference of the earth should increase by 2π meters. Robyn reported that this experience helped her to see a familiar problem from a new perspective.

Cindy and the cylinder

Cindy, a university mathematics educator, was considering a peculiar fact that, for a circle, the derivative of area gives circumference, and for a sphere, the derivative of volume gives surface area:

$$A_{circle}(r) = \pi r^2$$

$$A'_{circle}(r) = C(r) = 2\pi r$$

$$V_{sphere}(r) = \frac{4}{3}\pi r^3$$

$$V'_{sphere}(r) = SA(r) = 4\pi r^2$$

A question occurred to Cindy: Does this relationship—referred to here as the 'derivative relationship'—hold for other shapes? The immediate answer appears negative, considering for example, $A(s) = s^2$ and $P(s) = 4s$, as area and perimeter, respectively, of a square with side length s.

However, the result *does* extend not only to a square but to regular polygons and the platonic solids, with the appropriate choice of parameter (see Zazkis, Sinitsky & Leikin, 2013). For example, for a square, representing area and perimeter in terms of *half* the side length k, $s = 2k$, Cindy found that $A(k) = (2k)^2 = 4k^2$ and $P(k) = 4(2k) = 8k = A'(k)$.

Cindy wanted to explore further. She wondered whether the derivative relationship holds for a right cylinder. For a right cylinder the surface area SA and volume V are given by $SA(h,r) = 2\pi r^2 + 2\pi rh$ and $V(h,r) = \pi r^2 h$, respectively. Since V is a function of two independent variables, it made sense to proceed with partial derivatives.

However, similarly to the cases of a square and other regular shapes, the formulas had to be represented with an appropriate choice of parameters. Cindy rewrote the conventional formulas in terms of *half* the height of the cylinder, $h = 2k$. This gives $V(k,r) = \pi r^2(2k) = 2\pi r^2 k$ and $SA(k,r) = 2\pi r^2 + 2\pi r(2k) = 2\pi r^2 + 4\pi rk$. Computing partial derivatives of V leads to the desired result, demonstrating that the surface area is the sum of the partial derivatives of the volume.

$$\frac{\partial V}{\partial k} = 2\pi r^2$$

and

$$\frac{\partial V}{\partial r} = 4\pi rk$$

$$\frac{\partial V}{\partial k} + \frac{\partial V}{\partial r} = 2\pi r^2 + 4\pi rk = SA(k,r)$$

For Cindy, this was incredibly illuminating. She now knew that the derivative relationship applied not only to regular polygons and the platonic solids, but that it extended to a larger class of geometric objects.

Accounting for derivative applications

In their stories of backward transfer, Robyn and Cindy described thickening understanding related to the concept of derivative. In both stories, new reasoning about a familiar problem or formula emerged by introducing or extending the application of the notion of derivative.

In the Rope Story, we learned about Robyn's interaction with her colleague that caused her to reflect on the rope problem, seeing it from a different perspective. In terms of thickening understanding, we consider this kind of change in

Robyn's perspective to represent the addition of a new layer of understanding. The different solution method led her to see the problem from multiple perspectives: one algebraic, and the other analytic.

The Cylinder Story depicts an experience in which Cindy extended her layer of understanding of the 'derivative relationship', a familiar concept in terms of regular shapes in which the derivative of area (or volume) results in a perimeter (or surface area). Cindy described a deliberate and eventually successful effort to investigate whether the derivative relationship was applicable in the case of a cylinder. In doing so, Cindy learned that this derivative relationship was a property that can be applied more generally, and when a volume of solid depended on two independent variables then partial derivatives were in play.

Lonnie and the long multiplication algorithm

While teaching a methods course for pre-service elementary school teachers, Lonnie recognised the standard long multiplication algorithm as an instantiation of the 'FOIL' method for multiplication. 'FOIL'—meaning 'First, Outer, Inner, Last'—is a mnemonic often used in North American schools to help students carry out multiplication of pairs of binomials, where the acronym points to the four terms of the product. According to the FOIL method, when expanding $(a + b) \times (c + d)$, we multiply the First terms of each binomial, yielding ac, then the Outer terms, yielding ad, then the Inner terms, yielding bc, and finally the Last terms, yielding bd.

Lonnie illustrated his epiphany using the example 24 × 35. He explained that before learning the conventional way of writing long multiplication, it is helpful for students to record partial products explicitly, as in Figure 1. Teaching this explicit recording resulted in backward transfer to FOIL, familiar to him from high school.

Lonnie connected the long multiplication algorithm to the FOIL method by interpreting the computation 24 × 35 as (20 + 4) × (30 + 5). Like the FOIL algorithm, long multiplication requires computing partial products and adding those partial products to yield the product of the original numbers. The conventional recording of long multiplication concealed this aspect of the multiplication from Lonnie.

He further indicated a connection between the long multiplication algorithm and the distributive property, even though distributivity is usually presented as $a \times (b + c)$, rather than $(a + b) \times (c + d)$.

Priya and the product rule

While teaching an undergraduate course in differential calculus, Priya wanted her students to give an intuitive explanation for standard derivative rules. While searching for resources to prepare a lesson, Priya realised that the product rule for derivatives could be conceptualised using the area model for multiplication.

For two differentiable functions f and g, and their first derivatives f' and g', the product rule states that

$$\frac{d}{dx}[f(x)g(x)] = \frac{dg}{dx}f(x) + \frac{df}{dx}g(x) = g'(x)f(x) + f'(x)g(x)$$

The area model for multiplication is used to represent multiplication of numbers and polynomials visually, allowing students to consider partial products and compute products as the sum of those partial products, as in Figure 2.

Priya realised that the product rule could be represented visually using a similar model, representing infinitesimal changes in the dimensions of a rectangle as additions to its area, as in Figure 3. In the figure, df and dg represent the changes in the height and the width of the rectangle, respectively. The change in the area, $d(fg)$, is thus given by the sum of the areas $dfg(x) + dgf(x) + dfdg$. In the limit as dx approaches zero, the area $(df/dx)(dg/dx)$ rapidly approaches zero, becoming negligible and resulting in the product rule formula stated above.

Priya recalled, "When planning, I saw the area model [...] I immediately connected back to how powerful that model was in other settings". Although Priya initially thought of the product rule in terms of the distributive property, she reported "seeing it as another instantiation of the same mechanism that underlies our multiplication algorithm, how we multiply mixed numbers, and expand out binomials in algebra".

Accounting for distributivity

In both the Long Multiplication Story and the Product Rule Story, we recognise backward transfer in establishing connections to previous and not explicitly related mathematical ideas. New connections were established involving mathematical topics surrounding the idea of distributivity. We

a

$$\begin{array}{r} 24 \\ \times\, 35 \\ \hline 20 \\ 100 \\ 120 \\ +\, 600 \\ \hline 840 \end{array}$$

b

$$\begin{aligned} 24 \times 35 &= (20 + 4) \times (30 + 5) \\ &= 20 \times 30 + 4 \times 30 + 5 \times 20 + 4 \times 5 \\ &= 600 + 120 + 100 + 20 \\ &= 840 \end{aligned}$$

Figure 1. (a) The long multiplication algorithm with partial products. (b) The FOIL algorithm applied to compute the product 24 × 35.

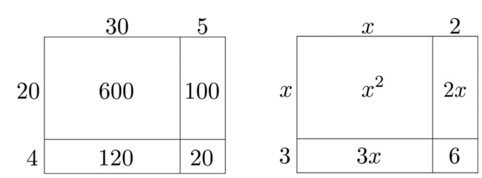

Figure 2: The area model used to model the products 24 × 35 and (x + 3)(x + 2).

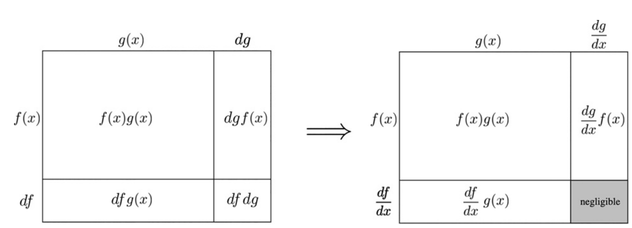

Figure 3. The product rule represented visually using the area model.

consider both stories to be examples of thickening, more specifically by adding a new layer of understanding, which enabled Lonnie and Priya to see familiar concepts from a new perspective.

For Lonnie, engagement with the long multiplication algorithm during teaching helped him connect this algorithm with the FOIL method. Seeing this connection caused a shift in Lonnie's understanding of long multiplication, and it allowed him to reconceptualise both algorithms as different instantiations of the concept of distributivity.

Priya's connection between the product rule and the area model during lesson preparation helped her connect both topics via distributivity. Indeed, the area model is a tool commonly used to represent distribution. The product rule, however, is often taught in a purely symbolic way as a formula to be applied. Priya's effort to present the product rule in a conceptual way enabled her students to see how it describes infinitesimal changes in area.

Chelsea and the choose function

As a student in high school, Chelsea learned the binomial theorem,

$$(x+a)^n = \sum_{k=0}^{n} \binom{n}{k} x^k a^{n-k}$$

She was able to apply the formula (in which $\binom{n}{k}$ is read 'n choose k') to generate the appropriate coefficients in the expansion of binomial powers. However, one aspect of the formula remained mysterious to Chelsea: "What are we choosing?"

When studying combinatorics at university, Chelsea realised that the 'choice' corresponded to what she called 'jumps' when opening parentheses in binomial expansion. She explained her revelation using the example $(x + y)^5$:

> To obtain each term when expanding $(x + y)^5 = (x + y)(x + y)(x + y)(x + y)(x + y)$, one must 'choose' to include one of either x or y from each binomial factor. For example, five-choose-three indicates that, given 5 possible x's, we are counting the number of ways it is possible to choose exactly 3 of them, hence five-choose-three equals 10.

Chelsea accompanied this by drawing a series of arcs connecting x's and y's from each binomial, as shown in Figure 4.

More than just clarifying the concept of 'choice,' this also made it clear for Chelsea why the identity

$$\binom{n}{k} = \binom{n}{n-k}$$

is true: for example, when three x's are chosen, this is equivalent to saying that two x's are *not* chosen, creating what Chelsea called a "nice symmetry".

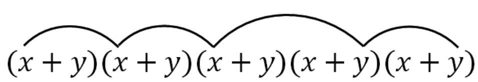

Figure 4. Chelsea's explanation of 'jumps' when multiplying binomials.

Cheng and a change of basis

When Cheng was a PhD student and teaching assistant, his research involved switching between bases for families of orthogonal polynomials. While Cheng was preparing a lesson on the binomial theorem for his undergraduate students, he realised that it could be conceptualised as a method for changing bases of the vector space of polynomials of a single variable, $\mathbb{R}[x]$.

As Cheng was writing out the expansion of a polynomial, he shared having "the familiar feeling of doing something [he] had been doing recently elsewhere". He suddenly realised the binomial theorem could be thought of as "telling us how to expand a vector written in terms of the second basis as a linear [combination] of the vectors in the first basis". In other words, the binomial theorem could be thought of as a way to switch between bases of the vector space, $\mathbb{R}[x]$ just as he had been doing in his research.

Cheng gave the following example. The set B_1 of nonnegative integer powers of x is a basis for the vector space $\mathbb{R}[x]$ of polynomials with real coefficients; the set B_2 of nonnegative integer powers of $1 + x$, constitutes a different basis for the same vector space, $\mathbb{R}[x]$.

$$B_1 = \{1 = x^0, x, x^2, x^3, \ldots, x^n, \ldots\}$$
$$B_2 = \{1 = (1+x)^0, (1+x), (1+x)^2, (1+x)^3, \ldots, (1+x)^n, \ldots\}$$

This means that the binomial theorem can be thought of as taking the vector $(1 + x)^3$ written in terms of B_2 and rewriting it as the linear combination $1 + 3x + 3x^2 + x^3$ of vectors from B_1:

$$(1+x)^3 = \sum_{k=0}^{3} \binom{3}{k} x^k = 1 + 3x + 3x^2 + x^3$$

Cheng reflected favourably on his experience and recollection: "Both things got simpler by virtue of their relation to each other. [I was] happy to have seen it, a bit surprised that I hadn't heard it said that way before".

Accounting for the binomial theorem

As previously, we recognise both stories as examples of backward transfer. While Chelsea presented a story in which she gained deeper insight into a relationship she already knew existed within the binomial theorem, Cheng told us about how he connected his understanding of the binomial theorem with his understanding of changing bases by seeing one as an example of the other.

In terms of thickening, Chelsea's story indicates expanding an existing layer of knowledge. As her story made clear, Chelsea was already aware that there was a connection between combinations and the binomial theorem, as it is explicit in the binomial formula. The insight she described in her story helped strengthen the connection. Recognising "what are we choosing" resulted in a thicker understanding of the theorem.

We consider Cheng's thickening of understanding as adding a new layer. Cheng described seeing an entirely new connection between the binomial theorem and the notion of switching between bases in a vector space. In his newly acquired view he considered the binomial theorem as a concrete formula for changing between bases of the vector space of polynomials.

Accounting for thickening understanding via backward transfer

In the preceding sections, we presented six stories of backward transfer. In each story, we identified aspects of thickening understanding and described each instance of thickening as either adding a new layer of understanding or expanding an existing layer. In this section, we revisit the stories and elaborate on each kind of thickening.

We claimed that a new layer was *added* to existing understanding when two seemingly (to the learner) disconnected topics were suddenly understood to be connected. Layers were added to existing understanding in four of the stories presented: Robyn's, Lonnie's, Priya's and Cheng's. For example, in Robyn's story, she learned of a new solution to one of her favourite problems that allowed her to realise a connection to calculus.

In each of Lonnie's, Priya's, and Cheng's stories, two pieces of mathematics were seen as connected, and in each pair, one mathematical concept was seen *as an instantiation of the other.* Lonnie's added layer of understanding in the Long Multiplication Story helped him see long multiplication as an example of the FOIL algorithm, just with a different representation. Priya, in seeing a connection between the product rule and the area model, was able to see the product rule as modelling a change in area. Cheng, as he explained in his story, saw the binomial theorem as a specific instance of a change of basis for the vector space of single-variable polynomials. These insights gave our participants the ability to conceptualise some aspect of previously familiar mathematics in terms of a perspective they had not previously considered, thickening their understanding of that topic through connection to other topics.

We claimed that an existing layer of understanding was *expanded* when a previously understood idea was seen to apply to a broader domain of examples than previously known or when a familiar connection between topics was strengthened. Cindy and Chelsea expanded existing layers of understanding in the Cylinder Story and the Choosing Story, respectively.

For Chelsea, realising what is 'chosen', thickened her layer of procedural knowledge of the binomial theorem. The familiar connection between the formula and combinations was enriched by appreciating *why* the connection exists at all. For Cindy, the familiar derivative relationship—connecting the concepts of volume and surface area to that of derivative—was extended to include cylinders. This necessitated the use of partial derivatives, as volume and surface area of a cylinder rely on two independent variables.

Discussion

The main contribution of our work is in expanding and refining the notion of backward transfer by highlighting the variety of instances of the occurrence in different contexts and elaborating on the mechanism of backward transfer using the construct of thickening.

We collected stories of participants' personal experiences gaining richer understanding of familiar mathematics through engagement in mathematical activities, and we considered their stories as examples of backward transfer. Using the notion of thickening understanding, we accounted for each story in terms of expanding a conceptual layer (in Cindy's and Chelsea's stories) or adding layers (in Robyn's, Priya's, Lonnie's, and Cheng's stories). A notable aspect common to all these experiences is their memorability for each individual. Despite the variability in content, context, and intentionality, these discoveries have left an indelible mark on their respective storytellers. We note that it is the *experience* of suddenly seeing previously understood mathematics in a new way, rather than the content, that makes these stories meaningful and memorable. As Liljedahl, Sinclair and Zazkis (2006) noted when describing thickening, this process often has an 'affective' nature. This affective nature made these experiences so significant for our participants that they remember the moments of realisation, and particular situations in which the realisations appeared, years after the fact.

The limited prior work on backward transfer pertains to students in the midst of their formal mathematics learning (Bagley, Rasmussen & Zandieh, 2015; Moore, 2012), with a particular focus on authentic mathematics classrooms (Hohensee, 2014). We expand this notion of backward transfer to include any individual actively involved in a mathematical activity. We note that backward transfer occurred for some participants when deliberately seeking solutions and explanations, such as when investigating mathematical relationships (Cindy) or preparing for instruction (Priya). For others it occurred accidentally, during lesson preparation (Cheng), in the middle of teaching (Lonnie), conducting a study in mathematics education (Robyn), or while taking an undergraduate course (Chelsea). This suggests that contexts for the occurrence of backward transfer are quite broad.

Furthermore, prior research focused on topics that are closely sequenced, such as students' reasoning about linear functions after being exposed to quadratic functions (Hohensee, 2014) or understanding the notion of accumulation from introductory calculus course after exposure to more advanced calculus (Moore, 2012). Therefore, 'backward' can be interpreted in terms of conventional curricular sequence and relative difficulty of the mathematical concepts. This proximity of exposure and content is considered *near* transfer (Barnett & Ceci, 2002).

Our work extends the notion of backward transfer to include *far* transfer (Barnett & Ceci, 2002) in terms of mathematical content (*e.g.,* content from different branches of mathematics) and amount of time between exposures to related ideas.

First, we shared instances of backward transfer related to topics in mathematics that are typically not successive in a curricular sequence. These stories connected seemingly unrelated topics, such as Priya's connection between the area model, which is often used in elementary mathematics, and the product rule found in differential calculus.

Second, in prior research students were exposed to more advanced topics (*e.g.,* quadratics) close in time to their learning of a less advanced topic (*e.g.,* linear functions). However, Lonnie's backward transfer moved back over a decade, recalling his high school experience when in graduate school.

Moreover, our interpretation of backward is more fluid. In prior studies the directionality of 'backward' coincides with a conventional curricular sequence, where learning of more advanced topics influences reasoning about previously learned and therefore less advanced topics. Thus, when considering 'backward' the timeline of reported experiences coincides with the relative difficulty of the mathematical concepts. Some instances of backward transfer shared in our work connect engagement with less advanced topics backward to previously known advanced topics. One clear example of this can be seen in Cheng's story, where lesson planning for the binomial theorem reminded him of changing bases in polynomial vector spaces. While binomial theorem conventionally exemplifies less advanced mathematics than the notion of change of basis, the 'backward' here applies to the previous time exposure to the mathematical content, rather than to its relative difficulty.

In summary, we extend the notion of backward transfer by considering several novel aspects: various kinds of engagement with mathematics and connecting seemingly unrelated topics, cognitive distance between topics and time between exposures, and the possibility of connecting backward from less advanced material to previously encountered, more complex topics. Each aspect results in thickening personal understanding, either by expanding a conceptual layer or adding a new layer.

Looking forward

Mason (1998) suggested that the most significant products of research in mathematics education are, first, the transformations in the researchers' personal development and, second, the stimuli to other researchers and teachers to test out conjectures for themselves in their own environment. Our work began as an exercise in the latter, an exploration of the memorable connections formed in experts' understandings of mathematics. This exploration ultimately resulted in enriching our own understandings of ideas in mathematics. Such enrichment/expansion could be the case for some readers, which is an additional contribution of this study.

We suggest that the insights from backward transfer ultimately result in changes in one's teaching or research. For a teacher there is a potential desire to either share the novel realisations with students or to occasion an instructional situation where the realisation may happen and thicken students' understanding. For a researcher, personal mathematical realisation may result in new directions to explore.

Looking forward, we hope that this work provides a stimulus to readers to reflect on their personal stories and experiences of making connections and sudden realisations via backward transfer, as well as on the effect of these realisations on their mathematical being.

Note

[1] All names are pseudonyms, chosen to remind readers of the story.

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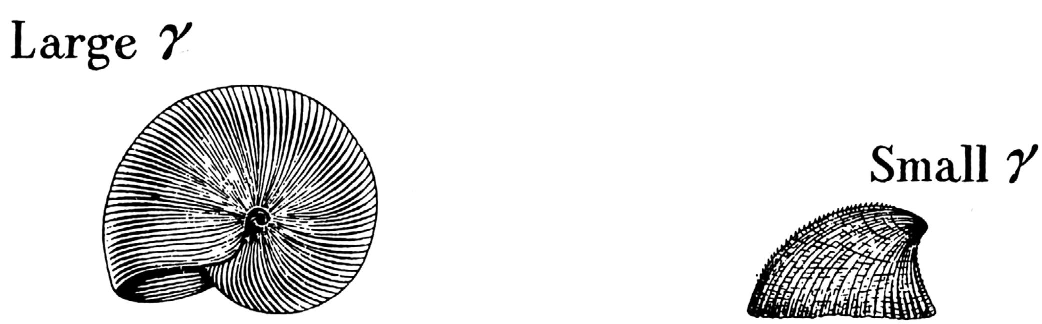

—From 'On Growth and Form' by D'Arcy Wentworth Thompson