# GOOD-ENOUGH UNDERSTANDING: THEORISING ABOUT THE LEARNING OF COMPLEX IDEAS (PART 2)

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*In Part 1 of this article (Zack and Reid, 2003) we offered two examples of students operating with good-enough understandings in mathematics, and related their understandings to features of good-enough understanding identified by Mackey (1997) in the context of reading. Mackey contends that the ability to read further, on the basis of a very imperfect understanding of the story to that point in time, is a vital and profoundly undervalued skill in complex reading. Opting for a temporary decision which is good enough for the time being is not only a good move, it is one which we make all the time when in the midst of learning.*

*Mackey proposed that learners work with 'good enough for the moment' ideas as placeholders. When confronted by many complex ideas for the first time through, they make many tentative, temporary decisions, and keep diverse and sometimes contradictory possibilities 'in the air' waiting at times to the end to make sense of what has transpired. Similarly, students engaged in mathematical activity behave 'as if' their understandings are sufficient, as long as they do not fail in some way. Good-enough understanding means 'making do' on occasion, and moving on. Being good enough is not a weakness to be overcome, however. In fact, we contend that being good enough is all we can do. In this second part of our article, we will consider a case where a student's understandings continue to be good enough in spite of operating with two contradictory understandings.*

The mathematical activity described here occurred in May 2001 in Vicki's fifth grade (ten- to eleven-year-old children) class. The students had been working on the *Count the squares* problem: finding how many squares there are in an $n$ by $n$ grid. They had solved the problem by counting for several specific cases and had observed that the number of squares is equal to $1^2 + 2^2 + \ldots + n^2$. They had a need for an 'easier' way, however, and Vicki gave them a formula: $n(n + 1)(2n + 1) \div 6$, which they came to call the Johnston Anderson formula after the author of the paper in which Vicki first saw it (Anderson, 1996). The students then protested that just giving a formula without explanation was insufficient. David came in late May to present some possible explanations.

Here we are concerned with the explanation he presented on May 22, the tri-pyramid proof, and the discussion that occurred subsequently. In the tri-pyramid proof the sum $1^2 + 2^2 + 3^2 + 4^2$ is represented by a pyramid of squares (see Figure 1A opposite). Three of these pyramids are then assembled to make a 'quasi-box' of dimension 4 × 5 × 4 5 (see Figure 1B opposite), which gives rise to the general formula $n(n + 1)(n + 1/2) \div 3$. A variant is to put two quasi-boxes together (see Figure 1C opposite). This leads to the Johnston Anderson formula (See part one of this article (Zack and Reid, 2003) for more background details [1, 2].)

As in the first part of our article, we speak at times as individuals and at other times in unison.

## David: Leo and 2A + 1

Leo developed his formula, A(A + 1)(2A + 1) ÷ 6, on May 22, 2001 in conversation with me and with some of his fifth-grade peers, in particular Alice. His understanding of the expression (2A + 1) in that formula provides a clear example of a case where understandings shift, while always remaining good enough. I will initially present Leo's work with me and with his peers during the large group discussion on Tuesday, and then present him at work two days later on Thursday, first with his partner Seth and then in the group of four (Leo, Seth, Alice, Jane). In the subsequent discussion, Vicki and I will further develop our discussion of good-enough understanding.

Towards the end of my explanation to the class on May 22, Leo proposed putting two "almost cubes" together to create a more regular shape. Leo was the first person among the students in his class and those of previous years (1998-2001) to suggest this idea. Leo had clearly been thinking about the possibility of putting together two of the 'odd' shapes (what I have referred to as the quasi-box is what Leo calls here a "not really a cube" and later an "almost cube", see Figure 1B opposite); we see this in the third of Leo's formulae recorded on his worksheet [3]:

| | |
|---|---|
| X(X + 1)(X + 1/2) ÷ 3 | 1/4 pyramid |
| X(X + 1)(X + 1/2) | not really a cube |
| X(X + 1)(X + 1/2) × 2 | cubic rectangle |

Leo stated that his third formula might be different from those of others. In presenting his idea, he suggested that if one could put two of the cubic shapes together, one would get "a rectangle which fits" (see Figure 1C opposite). In making his proposal, Leo seems to have been thinking in terms of the geometry of the situation, motivated by a desire to create a nice shape. His original formula for the dimensions of the resulting "cubic rectangle" (X(X + 1)(X + 1/2) × 2) shows no awareness that the combination of two "almost cubes" of height (K 1/2) would produce a solid of height (2K + 1). However, the discussion of how to encode the dimensions of this shape in an algebraic formula was underway, when Alice, making reference to her formula (K(K + 1) (K 1/2)) ÷ 3, suggested, for the third dimention of the "cubic rectangle":

> It could be like mine: A and a half times 2.

Leo countered by saying:

> You could take away the A and a half and do something else which would be an easier way to do it

At this point both Alice and Leo were focused on the (K 1/2) portion of Alice's formula (K(K + 1)(K 1/2)) ÷ 3) written on the classroom chalkboard That Leo didn't specify the something else suggests that he had not worked out the length of the unknown side of the cubic rectangle. It is interesting, however, that he wanted to replace the "A and a half" with "something else", given the challenges he faced two days later in working with the "half" in Alice's formula (as we have described in part 1, Zack and Reid, 2003).

When I asked the students to focus on that dimension, namely the length of the long side, Leo proposed (2A + 1), suggesting that he was now seeing the third dimension as a single quantity, not two times the height of the "almost cube". The (2A + 1) might have meant a number of different things to him at this point, however, and we cannot be sure exactly how he interpreted it. I stroked along the long side as I asked about its length, which put the focus on the length of one square edge (4), and then that of the next square (4), and then I emphasised that there is one more unit at the end The expression (2A + 1) might have come from Leo generalizing from this single case, where the result was (2 × 4 + 1). Another possible origin for the expression is Leo seeing the two almost-cubes of height (A + 1/2) as being combined to make a shape of height (A + 1/2 + 1/2 + A) Such a visual image would have scaffolded the interpretation of (2A + 1) as two times the (K 1/2) or (X + 1/2) part of Alice's and Leo's expressions.

It was only at the close of the class discussion on May 22 that Leo delightedly became aware that his idea, and the algebraic expression that had been constructed during the discussion based upon his idea, had produced the Johnston Anderson formula, $n(n + 1)(2n + 1) \div 6$ While the formula Leo wrote on his sheet for the "cubic rectangle" was a variant of Johnston Anderson's formula, he did not see it as such until one had been built (from the two quasi-cubes) and its third dimension had been found to be (2A + 1)

Two days later, on May 24, the students worked in pairs and then in groups of four on a sheet that asked them to discuss and explain parts of their peers' algebraic expressions (see Figure 3).

Leo seems to have had a different understanding of (2A + 1) at this point [4] His first effort at explaining why it occurred in his formula to his partner Seth involved combining two staircases, not two "almost cubes". I had used combining two staircases in the whole class explanation as an introduction to my presentation of the second proof, as I see the process of combining pyramids to find the sum of squares as analogous to the process of combining staircases to model the Gauss procedure for finding the sum of the positive integers. Leo explained the "2" as the result of using two "staircases", and the "+ 1" as the additional row or column that results when two staircases are combined (see Figure 2, p 24).

This conflates two readings of the same situation If the two staircases are seen as two staircases, then the number of cubes involved is $2T_n$ (where $T_n$ is the $n$th triangular number). In this reading no addition of 1 occurs. On the other hand, if the two staircases are seen as a rectangle, then the number of cubes involved is $n(n + 1)$ No multiplication by 2 occurs in this case. After their discussion, both Leo and Seth wrote answers on their sheets based on this understanding. Leo wrote, as part of his answer to the invitation on the worksheet to "Explain each part of the formula":

> when you put one staircase on top of another, you're doubling the blocks, and adding one more row or column.

Seth wrote in answer to the question, "In Leo's formula why is there (2A + 1)?":

> When you put one staircase on top of another your doubling the blocks and adding one more row or column.

Later they revised the wording of their answers because of a comment made by Vicki, who reminded them that in this case they are talking about combining pyramids, not staircases. Their only revision was to replace the word "staircase" with the word "pyramid" however, resulting in an answer that does not in fact relate to putting one "almost cube" on top of another (although Vicki and I interpreted it that way when we first read it). At the same time the change of wording disguised their original meaning, making it easier for others to understand it differently.

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Johnston Anderson: n(n+1)(2n+1)÷6

Formulae that were suggested when we were using pyramids

Alice: K(K+1) K½ ÷3

Leo: A(A+1)(2A+1) ÷ 6

Tom: [N(N+1)(N.5)] ÷ 3

Mariel: ((H(H+1)H) + (H(H+1) ÷ 2)) ÷ 3

Adele: [(H(H+1)H) + H(H+1) − ½(H(H+1))] ÷ 3

Group A (based on Leo's idea): (2H+1)(H+1)(H) ÷ 6

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*Figure 3: A collection of formulae for counting squares*

There is an important difference between Leo's understanding in this situation and his understanding of the half-layer (described in part 1, Zack and Reid, 2003). In both cases his understanding was partial, but also good enough. However in the case of the half-layer he knew he did not understand something, and here in the case of the (2A + 1) he thought he did understand In regard to his dealings with the (2A + 1) idea, we see Leo keeping diverse, contradictory possibilities in the air.

It is worth noting that Leo and Seth were interested in using correct words for the things they were talking about. They corrected some of their answers, for example replacing "side" with "dimension", when they became aware of a word that better expressed their meaning. They also launched an inquiry into the proper term for what they had been calling the "cubic rectangle", their "double-almost-cube", which they later named a "box" (see Figure 1C). As a pair they invented the term "almost cube" (replacing Leo's "not really a cube", May 22, see Figure 1B, p. 24) and they used it in their other answers. If they had been thinking of the almost-cube, not the pyramid or the staircase, when they wrote their answers, it seems likely that they would have revised their answers to use that term once they had invented it. They did not at that time, nor did they later when Leo read out his answer to Alice and Jane.

At some stage Leo came to understand (2A + 1) differently. When he first tried to explain it to Alice and Jane while in the group of four, he used their staircase (half-layer) to complement the half-layer of an almost-cube. He may have already been thinking in terms of two almost-cubes, and leaving the rest of the second almost-cube to their imaginations. On the other hand he may still have been thinking in terms of staircases, in which case he would have seen no need for another almost cube.

When Leo then repeated his explanation with two almost cubes, his words can be interpreted to refer to one dimension of the box being (2A + 1), or he might still be indicating the two staircases that are combined in the centre. Whatever Leo meant, it is entirely possible that Alice interpreted his explanation as referring to the dimension of the 'box' If that is the case, Leo's understanding of the origin of the term (2A + 1) was good enough for him to provide Alice with an explanation that allowed her to understand (2A + 1) in terms of the combination of two almost-cubes.

It is difficult to know when he was speaking to Alice and Jane whether Leo was still thinking in terms of staircases or was thinking of "almost cubes" When he read his answer to his group of four he did not correct "pyramid" (which he had used to replace "staircase") with "almost cube", which suggests that he did not see a mismatch between the words he was using and the image he had. However, there is a mismatch between his gesture and his words. He uses gestures that suggest putting together the two almost-cubes, both with the actual cube interlock structures and with gestures alone.

Thus, although at one point he was saying "pyramids", and he held a pyramid in his hand, the sweep of his arm echoed his former moves with the almost-cubes and cohered with his focal idea of Tuesday, May 22, that of clicking the two almost-cubes together. At other times he pointed to the middle layer made up of the two half layers of the almost-cubes when he was explaining the '+ 1' in (2A + 1) But here again there is ambiguity, because he might be indicating the whole-layer formed, or just one row of it as he did earlier when he made a similar gesture while showing Seth how (2A + 1) came from the joining of two staircases [5].

Six months later (December 6, 2001), following a review with Vicki of some of the investigations done with David on May 22 and 23, Vicki asked both Leo and Seth to look at their written answers on the sheet from May 24 Seth did not comment. Leo, on the other hand, immediately stopped at the word "pyramids" and said:

> Leo: Oh, pyramid. Was that a mistake? What I meant was two almost cubes, plus one in the middle.

Clearly then, at this point Leo's understanding has returned to the idea he had on May 22, after a long journey through other possibilities, without it ever failing to be good enough for him to proceed with the mathematical activity in which he was engaged.

**Vicki and David: Discussing Leo's understandings**

Thus we see that in the process of constructing meaning and juggling new concepts, Leo is at times aware and at other times not aware. He entertained two different explanations and seemed unaware that they were inconsistent. He introduced the idea to the class of combining two almost-cubes Then two days later he interpreted (2A + 1) as referring to staircases, and then pyramids Then, in the group of four, he sometimes seems to indicate one understanding and at other times another. He reread his "two pyramid" version a number of times, but never explicitly rejected it, even though he used gestures that suggest that he may have also understood it in terms of two almost-cubes as well as (or possibly instead of) two staircases Six months later, however, he immediately saw a problem with what he had written about pyramids and changed it using the correct term, "almost cube"

Leo was working with what Mackey calls *partial information* Mackey speaks of placeholders, of the need for provisional understandings, and for working through and against misunderstandings (Mackey, 1997, p 428). There is a lot coming at Leo (and of course at all the children): two proofs and the crucial ideas inherent in them, as well as the diverse formulae constructed by his peers. There is a lot for him to sort out, much for him to grasp.

In the case of his understanding of (2A + 1), he understood where it came from when it arose in the half-class discussion, but a different (and to us incoherent) understanding was good enough for him in his discussions with Seth. Given that he knew he had understood the idea before, perhaps it was not important for him to carefully check to see if he still understood it. The sentence that he wrote on his sheet, originally about staircases, was sufficient to stand in place of a detailed understanding of the (2A + 1) while he struggled with other crucial ideas. The final pulling together, or synthesis, can wait, and seems only to have happened at some point after the end of the classroom experience.

## On good-enough understanding and learning

Mathematics is often portrayed as sequential. Complete understanding of underlying concepts is assumed to be necessary before new concepts can be learned. We would suggest that this is not the case.

When we looked closely at the videotapes of the discussion at the close of David's sessions with the students in May 1999 and in May 2001, we marvelled that the children could understand so much given that the ideas were complex. The students exhibited a sense of accomplishment at the close, and their answers to post-session follow-up questions attested to their control of some of the ideas. As we considered and delineated the various elements (see part 1, Zack and Reid, 2003), it became clear that there was indeed much to challenge or confuse. Thus our question was how *did* the students come to understand even a part of what had transpired.

Our investigation revealed that much of what we had marvelled at was in fact incomplete, tentative, and sometimes inconsistent. Yet at the same time the students were able to continue to work on the problem, to explore, and even to explain things to each other that they seemed not to fully understand themselves. The students do not understand some of the complex ideas when they are first presented; at times they hold them in abeyance and wait, at other times they ask questions. As in the case with Mackey's students, the students create, extend, evaluate, and sometimes discard their initial assumptions and questions (*cf.* Mackey, 1997, p. 430).

There are messy, loose ends during the conversations, and tentative and unfinished components in the learners' responses. Under the pressure to progress and try to get a picture of the whole, learners often resort to making good-enough decisions, contingent and temporary and based upon less than complete information (Mackey, 1997, p. 429). They are holding a few ideas in mind at the same time, perhaps thinking, 'it could be this, or it could be this'. Keeping in touch with a number of the possibilities is good enough for now.

Clearly for these students what was needed was not complete understanding, but a good-enough understanding. In the cases of both Maya and Leo, for example, we thought at first that they understood the divisions by three and the expression (2A + 1). Later events and closer examination of the videotapes revealed gaps in their understandings around these ideas. Yet later, after much time had passed with no further explanation from us, and no apparent consideration of the problem by the students, these two students had come to see these ideas in a way that seems to us to reflect very deep and stable understanding. Clearly, not all students attain the level of synthesis in evidence in Maya and Leo's understanding. However, all the students are seen to press to make sense of complex ideas. The untidy and inevitably partial nature of the students' work perhaps signals lack of understanding to some, but to us is part and parcel of the process of coming to understand. The students' disposition to proceed on the basis of an incomplete grasp is, we contend, an essential component in complex problem solving.

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## Notes

[1] This article is based on collaborative research supported by the Social Sciences and Humanities Research Council of Canada (SSHRC grants #410-94-1627, #410-98-0085 and #410-98-0427).

[2] David: The day after I explained the formula using the tri-pyramid proof, I offered the students an alternative explanation. Although no one in these classes seemed to have a problem with a three-dimensional object (the quasi-box) representing the sum of two-dimensional objects (squares), I wanted to offer an explanation that avoided this possible problem. The alternative explanation is also based on a proof in Nelsen (1993). In it two sets of squares are laid out to form an irregular staircase, and a third set of squares is disassembled into 'sticks' whose lengths are the odd numbers that constitute the squares. These sticks fill in the spaces in the staircase to form a rectangle measuring $(2A + 1)$ on one side, and $(1 + 2 + 3 + \dots + A)$ on the other side. Previously we had established that $(1 + 2 + 3 + \dots + A)$ is $A(A + 1) \div 2$, so this explanation gives the formula: $(2A + 1)(A(A + 1) \div 2) \div 3$. This explanation reinforced both the occurrence of the $(2A + 1)$ factor (while explaining it differently) and the Gauss formula for the sum of integers.

[3] A note on notation: Different letters were used by the children in developing their formulae. A, K, N and X were all used at various times. This seemed not to cause any confusion for the children. In the following, the letters used in particular contexts are maintained, which means equivalent formulae sometimes appear in different notations.

[4] Vicki: In posing the question "In Leo's formula why is there (2A + 1)?" on the worksheet given to all the students on May 24, I fully expected that Leo would respond correctly. Our goal was to see what the other students would do with this question. When observing the group of four (Alice, Jane, Leo and Seth) in discussion together on May 24, and when reading Leo's written answers on his sheet when the session was over, it seemed to us, to me and David, that Leo was solid in his grasp of the (2A + 1) idea.

It was only when I subsequently looked at the videotape recording of the pair, Leo and Seth, in conversation on Thursday, that I became aware that, contrary to our impression, Leo did not have a good grasp of the idea - that is, of his own idea - when he first began his work with Seth. Thus this episode served as a good example of the saying "you'll see it when you believe it," since in Leo's case, because it was he who on Tuesday had offered the idea which mirrored the Johnston Anderson expression, I believed that he was still thinking that way on Thursday, and it coloured what I heard and read.

[5] Vicki: David and I had differing opinions about when Leo shifted understandings. I felt that once Leo was working in the group of four, he was working with the idea of connecting the two "almost cubes" and not the staircases. David felt that it was still possible to interpret Leo's talk as referring to the linking of the two staircases, and he put forth a case for both possibilities. We did not reach a consensus on this issue, and felt comfortable maintaining divergent views. This situation resonated with Luttrell's (2000) observations that we might think of "good enough" methods in researching, since: "Good enough researchers accept rather than defend against healthy tensions in fieldwork" (p. 515). Luttrell's article provides yet another 'take' on the idea of "good enough"; in speaking of "good enough" methods for ethnographic research, she shows how she thinks not of absolutes, but of "sustaining multiple and sometimes opposing emotions, keeping alive contradictory ways of theorising the world" (p. 516).
David: I have written elsewhere on the value of maintaining multiple contradictory perspectives in research (Reid, 1996). The key is not only to disagree but also to develop mutual understanding of the other's point of view. Our interpretations are enriched by being set in a context that shows their limitations and potential blind spots. In this case Vicki's focus on Leo's gestures pointed out to me the limits of my methods of searching for parallels and differences in the verbal expressions Leo used each time he explained the factor (2A + 1). On the other hand, my analysis of the words used suggested that relying on gestures also has limitations. It may seem that this approach leaves us knowing less, which is true in a sense, but it also leaves us knowing more about what we do not know.

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