A Proposed Framework for Examining Basic Number Sense

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A boy in a classroom was being observed by a visitor. After writing the problem 37 + 25 in vertical form, and drawing a horizontal line, he recorded the answer of 62. "Fine," said the visitor, "tell me how you did that."

"All right" answered the boy hesitantly, "but don't tell my teacher. I said 37 and 20 is 57 and 5 makes 62."

"That's a very good way," commented the visitor. "Why can't I tell your teacher?"

"Because I wouldn't get a mark then. I can't understand the way she tells us to do it on paper, so I do it this way in my head and then write down the answer and I get a mark."

A clerk was serving in a newsagent's shop in England. A customer wanted to purchase two identical diaries, each originally costing 2.50 pounds, but now, in February, marked "half marked price." The customer picked up the two diaries and took them to the counter. "How much please?" asked the customer.

The clerk picked up the first diary and a pencil, wrote the original price, divided by two using the standard written algorithm for long division, and obtained the new price 1.25 pounds. She then picked up the second identical diary, wrote the original price, used the standard written algorithm again, and obtained the new price 1.25 pounds. She then wrote 1.25, underneath it wrote 1.25, added them correctly using the standard written algorithm, turned to the customer and, without a shadow of a smile, said, "That will be two pounds fifty, please."

The boy could not follow the formal written algorithm but understood enough about numbers to invent his own efficient mental method. The clerk showed impeccable performance in the formal written algorithm and yet revealed an alarming lack of awareness of fundamental arithmetic relationships.

One might say that the boy, but not the clerk, exhibited number sense

A little history

As the aims and scope of subject matter change in elementary school, so too does the vocabulary. Initially, one task of the elementary school was to teach arithmetic, by which was meant little other than the multiplication tables and the four rules of number—that is, the correct formal methods of performing written addition, subtraction, multiplication and division. This task is still anachronistically preserved in the title of the elementary school journal of the National Council of Teachers of Mathematics, *The Arithmetic Teacher*.

Gradually the scope of the subject was widened, and became known as mathematics. Elementary school mathematics included measurement, geometry and graphical work, but the central core of the subject (computation) remained largely unchanged.

With the advent of the "New Math", the scope of the subject widened further, and a great many topics, including sets, logic and the arithmetic properties were incorporated. During this period some schools and teachers changed their curriculum and methods of instruction in innovative and mathematically sound ways; some made changes which could best be characterized as isolated additions of content topics largely unconnected. Some schools and teachers did not understand the philosophical changes which led to the recommended content changes, and most were skeptical so their curriculum and their teaching methods remained largely untouched. Soon after this period came the "Back to Basics" call which put restrictions on those schools and teachers making substantive, innovative changes and reinforced the prejudices of those who had not changed

The word "numeracy" was coined in 1959 [Crowther, 1959] to describe quite a high degree of ability to cope with current mathematical demands on the community. However, its meaning, because of its association with "literacy", became debased to mean only an ability to cope with the basic mathematical demands of everyday life. As these demands were not closely reexamined, it again, to most, implied the same range of skills as did arithmetic. In *Innumeracy: mathematical illiteracy and its consequences*, Paulos highlights the reality and dangers of a growing population of people who view mathematics as a mystifying subject beyond their ability to grasp. It is somewhat ironic that many people still view mathematics as facts, rules and formulas in a time when mathematics as a sense-making process is more highly valued for a numerate society.

Indeed, a review of present computational needs of adults reveals that relatively little use is made of formal written computation. Calculators are not only inexpensive, but also a universally available and highly reliable means of calculation. In addition, there is general acceptance of both mental computation and estimation as efficient processes for calculating. These facts have led to a need to examine the role and nature of computation in elementary school mathematics and to consider the increased roles both of choosing a computation strategy and of reflecting on both the process and the result of employing the strategy.

Over the past few years the phrase which has gained wide acceptance as embracing the essence of these changes is "number sense". The origin of this phrase is not clear, although it is clear that it springs largely from a

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desire to replace the word "numeracy" by one which does not have its abstract ring or its association with a conservative and sterile view of mathematical needs. The Cockcroft Report [Cockcroft, 1982] used the phrase "at-homeness with numbers" to describe one aspect of the desirable attributes of a numerate adult. The *Curriculum and evaluation standards for school mathematics* [NCTM, 1989] in the United States and the *National statement on mathematics for Australian schools* [AEC, 1991] both describe the development of "number sense" as a major essential outcome of school mathematics.

The phrase is an excellent one—simple and appealing—but its meaning is as open to different interpretations as was numeracy. In this discussion "number sense" will be used to refer to the basic number sense which is required by all adults regardless of their occupation and whose acquisition by all students should be a major goal of compulsory education. This paper attempts to clarify the term by first presenting a brief description of number sense and by then elaborating on this description by presenting a framework for thinking about number sense

What is number sense?

Number sense refers to a person's general understanding of number and operations along with the ability and inclination to use this understanding in flexible ways to make mathematical judgements and to develop useful strategies for handling numbers and operations. It reflects an inclination and an ability to use numbers and quantitative methods as a means of communicating, processing and interpreting information. It results in an expectation that numbers are useful and that mathematics has a certain regularity

Number sense is widely used in current mathematics education reform documents as it typifies the theme of learning mathematics as a sense-making activity. Like common sense, number sense is an elusive term which has stimulated discussion among mathematics educators, including classroom teachers, curriculum writers and researchers. These discussions have included a listing of essential components on number sense [Resnick, 1989; Sowder & Schappelle, 1989; Sowder, 1991; Willis, 1990], descriptions of students displaying number sense (or lack therof) [Howden, 1989; B.Reys, 1991], and an in depth theoretical analysis of number sense from a psychological perspective [Greeno, 1991].

Number sense exhibits itself in various ways as the learner engages in mathematical thinking. In particular, it is an important underlying theme as the learner chooses, develops and uses computational methods, including written computation, mental computation, calculators and estimation. Number sense plays a role in the use of each of these methods to varying degrees. The invention and application of an invented algorithm calls upon facets of number sense such as decomposition/recomposition and understanding of number properties. As learned paper/pencil algorithms and calculator algorithms are used, number sense is important as answers are reflected upon

The acquisition of number sense is a gradual, evolutionary process, beginning long before formal schooling begins. The young boy in the opening paragraph serves as a reminder that number sense is often evident at an early age as children think about numbers and try to make sense of them. Although evidence of number sense can be demonstrated early, growing older does not necessarily ensure either the development or utilization of even the most primitive notions of number sense, as demonstrated by the actions of the clerk described earlier. Indeed, although many young children exhibit creative and sometimes efficient strategies for operating with numbers, attention to formal algorithms may, in fact, deter use of informal methods. Ironically, as students technical knowledge of mathematics is expanding, their range of strategies may be narrowing. The learned methods (traditional paper/pencil algorithms) become the methods most cherished for some students as they can be executed without having to think. For example, the reaction of a student when asked if a calculation seems reasonable is often to recalculate (generally using the same method as for the initial calculation) rather than to reflect on the result in light of the context and numbers involved. Similarly, when selling three like items priced at $2 19, a clerk reported a total due of $4.88 When the customer responded that the amount seemed too low, the clerk showed no inclination to reconsider. When pressed, the clerk chose to retabulate the items. Only when a different total appeared on the register did the clerk acknowledge an error. While the method of checking (retabulating) is not being questioned, the lack of reflective reasoning is worrisome. Obviously there are a variety of social factors that may influence both the clerk's and a student's reaction, but the lack of logical thought to reconsider a calculation is all too common in both school and out of school situations.

There is evidence that the context in which mathematical problems are encountered influences thinking. Silver (in press) documents this position and argues persuasively for the need to provide students with rich situated activities which not only promote problem solving but stimulate different components of number sense as well. Clearly, number sense is, at times, triggered by the context in which the mathematics evolves. For example, while a student may be comfortable in school with a sum of 514 produced by applying a learned algorithm to the computation of 26 + 38, the same student in a store may demand a reexamination if asked to pay $5 14 for two items priced at 26¢ and 38¢.

Number sense is highly personalized and is related to what ideas about number have been established and also on how those ideas were established. Students highly skilled at paper/pencil computations (often the gauge by which success in mathematics is measured) may or may not be developing number sense. For example, when a sixth grader reports that 2/5 + 3/7 = 5/12 or a second grader says that 40 - 36 = 16, these students are attempting to apply a learned algorithm but are not reflecting number sense. In fact, much of the recent attention to developing number sense is a reaction to over-emphasis on computational procedures which are algorithmic and devoid of the number sense being characterized here.

The level of number sense necessary for children and adults today may be greater than in the past. For example, today both students and adults encounter a greater range of

Number Sense: A propensity for and an ability to use numbers and quantitative methods as a means of communicating, processing and interpreting information. It results in an expectation that numbers are useful and that mathematics has a certain regularity (makes sense).	1. Knowledge of and facility with NUMBERS.	1.1 Sense of orderliness of numbers	1.1.1 Place value 1.1.2 Relationship between number types 1.1.3 Ordering numbers within and among number types
		1.2 Multiple representations for numbers	1.2.1 Graphical/symbolic 1.2.2 Equivalent numerical forms (including decomposition/recomposition) 1.2.3 Comparison to benchmarks
		1.3 Sense of relative and absolute magnitude of numbers	1.3.1 Comparing to physical referent 1.3.2 Comparing to mathematical referent
		1.4 System of benchmarks	1.4.1 Mathematical 1.4.2 Personal
	2. Knowledge of facility with OPERATIONS.	2.1 Understanding the effect of operations	2.1.1 Operating on whole numbers 2.1.2 Operating on fractions/decimals
		2.2 Understanding mathematical properties	2.2.1 Commutativity 2.2.2 Associativity 2.2.3 Distributivity 2.2.4 Identities 2.2.5 Inverses
		2.3 Understanding the relationship between operations	2.3.1 Addition/Multiplication 2.3.2 Subtraction/Division 2.3.3 Addition/Subtraction 2.3.4 Multiplication/Division
	3. Applying knowledge of and facility with numbers and operations to COMPUTATIONAL SETTINGS.	3.1 Understanding the relationship between problem context and the necessary computation	3.1.1 Recognize data as exact or approximate 3.1.2 Awareness that solutions may be exact or approximate
		3.2 Awareness that multiple strategies exist	3.2.1 Ability to create and/or invent strategies. 3.2.2 Ability to apply different strategies 3.2.3 Ability to select an efficient strategy
		3.3 Inclination to utilize an efficient representation and/or method	3.3.1 Facility with various methods (mental, calculator, paper/pencil) 3.3.2 Facility choosing efficient number(s)
		3.4 Inclination to review data and result for sensibility	3.4.1 Recognize reasonableness of data 3.4.2 Recognize reasonableness of calculation

Figure 1
Framework for considering number sense

numbers (e.g. government budgets in the trillions of dollars, athletic events timed to the thousandths of a second), in more varied contexts (e.g. graphs, surveys), utilizing new tools (e.g. computers and calculators) than was the case a generation ago. Indeed in a technological age, it might be said that the possession of number sense is one major attribute which distinguishes human beings from computers. There is every reason to believe that the 21st century will introduce additional reasons for an increased focus on developing and maintaining number sense.

The NCTM *Curriculum and evaluation standards* state that children with good number sense: have well understood number meanings, have multiple interpretations/representations of numbers, recognize the relative and absolute magnitude of numbers, appreciate the effect of operating on numbers, and have developed a system of benchmarks to consider numbers [NCTM, 1989]. Although the description of number sense offered in the *Standards* is helpful as an overview, it is important that further clarification be undertaken, debated and discussed. For example, it is not clear what it means to "have well understood number meanings" or how to obtain such a goal. In fact, it might be argued that each of the components included in the *Standards* description can be subsumed in "having well understood number meanings." After studying the brief but rich literature on number sense, a proposed framework was organized and is presented in the remainder of this paper.

A framework for basic number sense

Although lists of components of number sense and attributes of students who possess it are evident in the literature, how these components fit together has not been described. In fact, Greeno [1991] suggests that "number sense is a term that requires theoretical analysis, rather than a definition." The framework suggested in Figure 1 is an attempt to articulate a structure which clarifies, organizes, and interrelates some of the generally agreed upon components of basic number sense, many of which have been conjectured by different people over many years. It would be futile to delineate all possible components of number sense as number sense should grow and expand thoughout secondary school and beyond. Even if it were possible to identify and assess all components, a person's number sense may be inadequately reflected by the individual components. It is likely that the whole of number sense is greater than its parts. Nevertheless, the proposed framework is an attempt to identify key components and to organize these key components according to common themes. A careful review of the framework together with systematic research efforts on number sense will likely reveal additions and deletions, or a reorganization of the proposed framework. Such reviews are welcome and necessary to establish dialogue for further refinement and development of a framework.

Figure 1 differentiates three areas where number sense plays a key role, namely number concepts, operations with number, and applications of number and operation. Figure 2 illustrates interconnections among the major components. These interconnections suggest a monitoring process which links number sense with metacognition. A person with good number sense is thinking about and reflecting on the numbers, operations and results being produced. This reflective thinking will at one time or another involve any of the framework components shown in Figure 1.

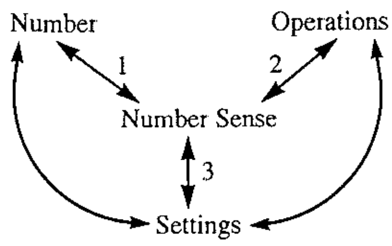

Figure 2
Interconnections of major components of number sense

In order to clarify the components in Figure 1, each of them will be briefly discussed. Specific questions or examples that might provide insight into each particular component of number sense will also be illustrated. The numbers in brackets refer to those in Figure 1.

Knowledge of and facility with numbers (1)
For many years educators have argued that meaningful understanding of the Hindu-Arabic number system, including an appreciation of its structure and regularity, is fundamental. [Brownell, 1935; Heibert, 1984] Various methods have been advocated to help children develop conceptual understanding of number, including the use of manipulatives and/or models, the use of number lines, a study of various number bases, and units of instruction targeted at "place value." While these methods are undoubtedly important, the number sense framework proposed here is organized not by instructional "topics", but by a collection of "understandings" a learner is likely to exhibit/utilize. For the area of number, these include: a sense of orderliness of number; multiple representations for numbers; a sense of relative and absolute magnitude of numbers; and a system of benchmarks.

Sense of orderliness of number (1.1) Number sense implies an understanding of how the Hindu-Arabic number system is organized and how this organization aids in reviewing and considering numbers. The system of place value, including its application to whole and decimal numbers, is an important component of this area. Understanding rational numbers, including how they are represented, is also included in this component. An understanding of the number system helps the learner mentally organize, compare, and order numbers encountered in a mathematical environment. For example, a young child learning to count beyond 20 comes to appreciate the patterns identified both orally and in written form inherent in the number system. Once identified, these patterns provide a powerful source of support for extending the counting sequence. In the same way, a fifth grader explores decimal numbers by counting (with the aid of a calculator) from zero to 10 by tenths (or from zero to one by hundredths). As with the young student, the counting sequence (in particular, the calculator display) provides a powerful tool for helping the student recognize, identify and repeat patterns that emerge. Another example makes use of a number line to help a sixth grade student understand relationships between decimal numbers:

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                         P
0------------------------•-----------------100
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Ask the student to name the number marked at point P. Now change the endpoint of the number line from 10 to 10 and ask the same question. Next change the endpoint from 10 to 1 and finally from 1 to 0.1, after each change asking for the number represented by P. As number sense develops students detect patterns and regularities between P and the various endpoints As understanding of the orderlinesss and regularity of the number system develops, students begin to use this knowledge. For example, a middle grade student exhibiting a sense of orderliness would respond "yes" to the question, "Are there any numbers between 2/5 and 3/5?" and be able to give several appropriate examples.

Multiple representations for numbers (1.2) Numbers appear in different contexts and may be expressed in a variety of symbolic and/or graphical representations. Number sense includes the recognition that numbers take many forms and can be thought about and manipulated in many ways to benefit a particular purpose For example, recognizing that 2 + 2 + 2 + 2 is the same as 4 × 2 is a useful conceptual connection between addition and multiplication. Recognizing 30 cents as being a quarter plus a nickel or 3 dimes, or recognizing 30 minutes as 1/2 an hour, would also be useful in certain situations At a later grade, number sense would be reflected by recognizing different symbolizations, such as 3/4 = 6/8 or 3/4 = 0 75 or 3/4 =75%. The knowledge that numbers can be represented in many different ways, together with the recognition that some representations are more useful than others in certain problem solving situations is both valuable and essential for developing mathematical power.

Decomposition/recomposition involves expressing a number in an equivalent form as a result of recognizing how this new equivalent form facilitates operating on the recomposed numbers. Suppose for example, a person is checking out of a market and has a bill for $8 53. The person could pay with a $10 bill and get $1.47 change. Another person might pay the check with a $10 bill and three pennies. The change would be $1.50 In each case, the total amount paid is the same. However, in the latter example the person wanted to avoid carrying extra coins and recomposed the amount of payment to $10.03 to keep the change received more manageable. Decomposing $8 53 into $8.50 + $0.03 provided the rationale which led the buyer to pay with what appears to be an odd amount, yet results in dealing with fewer coins. For a younger student decomposition/recomposition often manifests itself as the learner "invents" ways to solve arithmetic prblems. For example, a first grader might recognize that one could add 25 and 27 by decomposing 27 (thinking about it as 25 + 2), then recomposing the new problem, 25 + 25 + 2, adding the 25's to make 50 then adding 2 to produce the sum of 52. This student manifests some important intuitive understanding about number and addition in both the invented procedure and in the ability to carry out the procedure.

Another component of number sense in this area, "comparing to benchmarks", refers to the use of common "anchors" in our number system which are often helpful in making judgements. For example, when considering the fraction 5/8, one could think about it graphically (as part of a circle or on a number line) or in an equivalent fraction or decimal form. An equally important representation is the sense that 5/8 is "a bit more than 1/2" or "between 1/2 and 3/4." Here, one-half serves as an anchor (or benchmark) to represent and/or compare other numbers.

Sense of relative and absolute magnitude of numbers (1.3) The ability to recognize the relative value of a number or quantity in relation to another number and the ability to sense the general size (or magnitude) of a given number or amount is a behavior that develops with mathematical maturation and experience. For example, what notion does a third grader have about the size of 1000? Asking students questions, such as, "How long does it take to count to 1000?" or "Have you lived more or less than 1000 days?" provides them an opportunity to think about 1000 in a personal context, thus helping them better understand the size of 1000 in a variety of contexts.

System of benchmarks (1.4) Just as a compass provides a valuable tool for navigation, numerical benchmarks provide essential mental referents for thinking about numbers. Numerical benchmarks are generally powers of 20, multiples of powers of 10, or midpoints such as 1/2 or 50%, although any value for which the learner has a confident understanding can serve in this capacity. Benchmarks are often used to judge the size of an answer or to round a number so that it is easier to mentally process Examples include recognizing that the sum of two 2-digit numbers is less than 200, that 0 98 is close to 1, or that 4/9 is slightly less than one-half. In each case, benchmarks are numerical values devoid ofcontext, which have evolved from experience and/or instruction.

Benchmarks may also evolve from personal attributes or encounters. For example, a person weighing 50 kg may use this information in estimating the weight of another person. Similarly, a child who attends a baseball game where the attendance is 50,000 may at a later time use this as a referent for judging the size of other crowds. The variety and complexity of the benchmarks in making decisions about numbers and numerical contexts is a valuable indicator of number sense.

Knowledge of and facility with operations (2)

Much of present day school mathematics is dedicated to helping students understand operations, including how they are performed. For example, in elementary school a conceptual foundation for the operations of addition, subtraction, multiplication and division is provided, together with the development of specific skills necessary to perform each operation by a paper/pencil procedure. Over the course of schooling, the operands change from whole numbers to fractions, decimals and integers in elementary school to polynomials and matrices in secondary school.

Although some old models are utilized (e.g. "joining together" for addition) some new models are also introduced (e.g. a number line is utilized to facilitate the development of subtraction of integers) and some familiar models are modified (e.g. the rectangular array model is modified to help students realize that the product of two fractions each between zero and one produces a product smaller than either factor).

Key components of an ability to understand and use operations are: an understanding of the effect of operations, an awareness of mathematical properties of operations, and an awareness of the relationship between operations. In the near future it is likely that the emphasis traditionally placed on paper/pencil computation will be dramatically, if not completely, eliminated as calculators emerge as the common computational tool. However, the emphasis given to each of the components of number sense described in this section should not diminish. In fact, it is likely that more attention will (and should) be given to these understandings as they are at least as important to a calculator user as to a paper/pencil computer.

Understanding the effect of operations (2.1) Fully conceptualizing an operation implies understanding the effect of the operation on various numbers including whole and rational numbers. Models are often used to help students understand the action of the operation. For example, modeling multiplication as repeated addition provides a concrete way of helping children think about multiplication as well as to carry it out. It is important that various models for multiplication be explored so that students see both the power of a model as well as its limits. For example, thinking of multiplication as repeated addition may lead to incorrect generalizations (e.g. "multiplication always makes things bigger") A variety of models such as a number line or an array model are helpful as children see multiplication in a variety of contexts and models

Investigating the change in answer as the size of operands varies in an operation contributes to number sense. For example, what happens when two numbers less than 1 are multiplied? How can this situation be modeled? What does the general model imply? What happens if one of the factors is less than 1 and the other is greater than 1? Reflecting on the interactions between the operations and numbers stimulates high level thinking and further enhances number sense.

Understanding mathematical properties (2.2) Mathematical properties, including commutativity, associativity and distributivity, have long been included in school mathematics programs Unfortunately they are often thought of as formal rules, and often viewed as a statement of the obvious. For example, the statement $3 \times 4 = 4 \times 3$ is often perceived as trivial and of little practical importance. In fact, many young students memorize 5×2 and 2×5 as two unrelated facts, failing to take advantage of commutativity in learning the multiplication fact. Much later, when multiplying matrices in secondary school, students learn that if A and B are matrices, $A \times B$ is not necessarily equal to $B \times A$, and their concept of a commutative operation takes on greater significance

Number sense often manifests itself as students, both young and old, intuitively apply arithmetic properties in inventing procedures for computing. For example, when multiplying 36×4 mentally, a student might think of 4×35 and 4×1, or $140 + 4$ or 144. This solution applied commutativity, as it changed the order of the factors to 4×36, and it also used the distributive property in recomposing 4×36 as $(4 \times 35) + (4 \times 1)$. This student might also recognize and use other equivalent forms such as $(4 \times 40) - (4 \times 4)$ or $(30 \times 4) + (6 \times 4)$, and these multiple solutions would be further evidence of number sense. The main intent here is to illustrate the value of linking practical applications to the development and understanding of fundamental mathematical properties Students with good number sense have typically made these connections and are comfortable in applying the properties in a variety of different situations

Understanding the relationship between operations (2.3) Connections between operations provide more ways to think about and solve problems. For example, as a student considers this question: "How many wheels are on 8 tricycles?" they may think about and apply a counting procedure (count by ones each wheel), they may apply repeated addition (adding the number of wheels on each tricycle: $3 + 3 + 3 + 3 + 3 + 3 + 3 + 3$), they may add by grouping (make 4 groups of 2 tricycles each: $6 + 6 + 6 + 6$) or apply multiplication (8×3 or 4×6). Each of these solutions reflects a slightly different way of thinking about the problem as well as differing degrees of attention to efficiency

The inverse relationship between operations is another valuable connection in that it provides the learner yet another way of thinking about a problem. For example, when asked to decide the quotient of $480 \div 8$, a person might view this as $8 \times ? = 480$ rather than as a division problem. This does not mean the person is unable to perform the division, but rather that the person knows the inverse relationship exists between division and multiplication and is comfortable in using the relationship to conceptualize and solve the problem.

In order to understand the relationship between operations, it is essential to first understand each operation. The relationships between operations grow as the operands are expanded from whole numbers to rational numbers. As rational numbers are being explored, it is natural to explore and utilize further relationships, such as those between mulltiplication and division. For example, to multiply by 0.1 is equivalent to dividing by 10; and dividing by 0.1 is equivalent to multiplying by 10. These relationships which connect multiplication and division, when understood (and perhaps discovered) by the learner further expand the range of strategies.

Applying knowledge of and facility with numbers and operations to computational settings (3)

Solving real world problems which require reasoning with numbers and/or applying operations to numbers involves making a variety of decisions including: deciding what type of answer is appropriate (exact or approximate), deciding what computational tool is efficient and/or accessible (calculator, mental computation, etc.), choosing a strategy, applying the strategy, reviewing the data and result for reasonableness, and perhaps repeating the cycle utilizing an alternative strategy. This process involves several different types of decisions. First, it involves understanding the relationship between the problem context and the necessary computation. Second it requires an awareness of a range of possible strategies for performing computation and an inclination to choose an efficient strategy. Finally, it includes an instinct to reflectively review the answer, and to check it both for indications of correctness and for its relevance to the original problem context.

Understanding the relationship between problem context and the necessary computation (3.1) The problem context provides clues not only for appropriate operation(s), but also for the numbers to be used in these operations and whether an exact or an approximate solution is appropriate. Consider for example, the following information:

> *Skip spent $2.88 for apples, $2.38 for bananas and $3.76 for oranges.*

Many different questions could be raised regarding this situation, and how these numbers are treated depends on the question asked. For example, if the question is "How much did Skip spend for this fruit?" the prices need to be totaled to produce an exact answer, and any one of several different computational methods (mental computation, written computation or calculator) could be applied. On the other hand, suppose the question is, "Could Skip pay for this fruit with $10?" In this case, estimation can be used to decide rather quickly and confidently that $10 is enough to make the purchase

Awareness that multiple strategies exist (3.2) Number sense involves recognizing that different solution strategies often exist for a given problem. When an initial strategy appears to be unproductive, formulating and applying an alternative strategy is an appropriate response. This tendency to pursue a problem by exploring it different ways often allows comparisons of different methods before making a final judgment or pursuing yet another vantage point. This metacognitive reflection is sometimes difficult to identify because it often occurs quickly and sometimes without conscious thought. Here, the emphasis is on the general awareness that different strategies exist rather than the metacognitive process of choosing, executing, and reviewing the various outcomes

Inclination to utilize an efficient representation and/or method (3.3) Awareness that some strategies and/or computational tools are more efficient at times than others is also an indicator of number sense. For example, a competent second grader asked to add 8 + 7 would likely dismiss the strategy of counting on by ones, choosing rather to mentally recompose the problem (as 7 + 7 + 1, based on knowlege that two sevens equals 14, or as 8 + 2 + 5, using the knowledge that 8 + 2 = 10.)

A corollary of this element is that the child or adult with little number sense often uses a more difficult method of calculation. The reasons for this vary, but often result from habits established from long practice of a particular method of calculation, a lack of confidence in alternative methods of calculation, and/or lack of knowledge of such alternatives.

Inclination to review data and result (3.4) When a solution is produced, people with number sense examine their answer in light of the original problem (considering the numbers included as well as the questions asked) to determine if their answer "makes sense." This reflection is generally done quickly, naturally, and becomes an integral part of the problem solving process. This metacognitive review of the problem context might involve a reflection of the strategies that might have been used as well as an evaluation of the particular strategy selected, and finally a check to determine if the answer produced was sensible.

Students often omit this checking precisely because the result (indeed the problem itself) is not important to them. A striking example of this is given by an unmotivated group of secondary students. To stimulate interest their teacher asked what they would like to do. They replied that they would like to build a boat. "Fine" said the teacher. "Go away and calculate how much wood you need to build a boat." The group reappeared shortly with a list of dimensions. "Right," said the teacher "I'll go and get the wood." It was only then that the students realized the teacher was serious. This was not just an artificial exercise: they were actually going to build a boat. They immediately asked for the list of dimensions; it was two whole days before they returned with thoroughly tested and checked calculations.

Where to from here?

Number sense is a topic of great interest in school mathematics. It is also nebulous and difficult to describe, although it is recognizable in action. Continued productive discussion of number sense (by researchers, teachers and curriculum developers) must at some stage be based on a definition, characterization, or model which portrays number sense in a clear yet comprehensive manner. The more clearly number sense is understood, the more likely there will be progress made in research, as well as in curriculum development and instruction

In this article, we have presented a model for characterizing basic number sense. We are aware of some imperfections (for example, is the third component of our framework the same as problem solving? If so, are number sense and problem solving different?) and some areas of overlap (in particular, "benchmarks" appear in both the first and second components of the framework). We do not hold any illusions that this framework will or should be accepted as a definitive model. We do however feel that it provides a useful starting point and welcome continued dialogue.

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