KNOWING AND USING MATHEMATICS IN TEACHING: CONCEPTUAL AND EPISTEMOLOGICAL CLARIFICATIONS

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The body of research on teachers' mathematical knowledge for teaching has been growing in importance in recent years in the international research community (Adler & Davis, 2006; Ball & Bass, 2003; Davis & Simmt, 2006; Margolinas, Coulanges & Bessot, 2005). These studies, grounded in empirical data and theoretical reflections about teachers' mathematical knowledge, offer new ways of thinking about teachers' *mathematical* education. They challenge the mathematics teacher education structures prevalent in most universities, where an emphasis is placed on university academic mathematical training – something that has been argued to be quite disconnected from the mathematical practices teachers enact in their classrooms (see, *e.g.,* Moreira & David, 2008; Proulx & Bednarz, 2008).

That said, as francophones who work in the field of *didactique des mathématiques* in Quebec (Canada) [2], we admit to having been surprised by the excitement that the body of research on teachers' mathematical knowledge, pioneered by Ball and Bass, has recently provoked in the scientific community. We were under the impression that this discourse (and the examples given to explicate this new field) had been present for a number of years in francophone communities, at least in Quebec, around the various developments enacted by mathematics teacher educators in pre-service teacher education (see, *e.g.,* Bednarz, 2001; Bednarz, Gattuso & Mary, 1995; Bednarz & Proulx, 2005; Janvier, 1996; Janvier & Hosson, 1999). However, this brought us to realize that these interventions and developments, and their underpinning principles, *had not been* theorized as well and as recent work in the domain of teachers' mathematical knowledge. [3]

We thus perceive an opportunity to explain our conceptualizations about knowing and using mathematics in teaching. We hope to contribute to the current reflections on teachers' mathematical knowledge for teaching and stimulate discussions around this significant component of teachers' knowledge.

We ground our discourse in mathematics teachers' actual practices, using vignettes taken from a collaborative study with a teacher. Our work has been refined through collaborative studies with teachers (see, *e.g.,* Bednarz, 2004, 2009), which enabled us to better understand the knowledge enacted by teachers in mathematics teaching/learning situations. This theorization is also inspired from *a posteriori* reflections on the interventions developed in our secondary-level mathematics teacher education program, established in the 1970s at the Université du Québec à Montréal (UQÀM). Our discussion combines these two complementary axes: (1) theorizations about knowledge teachers enact in their mathematics teaching practices and (2) illustrations, drawn from our program, of efforts to develop this professional knowledge.

Illustrations

The vignettes presented here are drawn from a collaborative research study centered on the elaboration of teaching situations exploring combinatorial problems with Grade 1 students and aimed at developing modeling processes. [4] To illustrate the important elements grounding our conceptualization of knowledge enacted in mathematics teaching, we draw from two data sources. The first comprises excerpts taken from meetings between the teacher and the researcher where they discussed teaching situations, allowing us to understand the teacher's intentions. The second source comprises traces of classroom events, enabling us to describe and analyze what happened in the classroom with students.

Intertwined, mathematical didactical and pedagogical intentions [5]

In this meeting excerpt, the teacher (Roy) refers to a puzzle activity he gave students to solve in groups of six (a puzzle made of six pieces to enlarge to 125%).

The explanations offered by Roy on the significance of this activity – used for the first time with his students – enable us to identify the criteria guiding his choice:

- "Students will have to communicate and agree on a strategy in order for the different puzzle pieces to correspond to each other." [6]

  Here Roy offers a didactical analysis of the task given to students: for him, this task forces students to re-examine their strategies, to explain to each other their strategy for enlarging one piece of the puzzle, and to find a common strategy to have the pieces correspond to one another.

- "It leads students to work together and cooperate. If I use the discourse of the new Quebec program of studies with regard to the development of their personal and social competencies, I would say 'to cooperate and actualize one's potential.'"

Here Roy highlights an institutional dimension, through the program of studies, and pedagogical intentions regarding the importance given to students' cooperation.

- A mathematical dimension also implicitly surfaces, through: (1) the choice made concerning the enlargement factor [a percentage]; (2) the proportional reasoning the activity requires from students; and (3) the choice of registers with which students have to work [geometrical register through the enlargement of various figures, numerical register through the enlargement factor].

Choosing activities is a key element of a teacher's practice; it is at the heart of planning. On what grounds are these "choices" made? The preceding analysis highlights a multiplicity of criteria for these choices, as well as their simultaneous and nested character: an institutional dimension (in reference to the program of studies); a didactical dimension (in the activity's analysis, concerning what it can provoke); a mathematical dimension (through the reasoning and properties it engages, and the various registers of representation); a pedagogical dimension (in learning to work with others). We therefore see, in this example, how the teacher simultaneously works on diverse intertwined components when choosing an activity, illustrating the various nested comprehensions enacted.

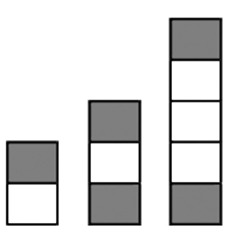
Exploiting activities in class: knowledge enacted in action

The data below are taken from three teaching sessions, coming from a sequence conducted with two different groups of students (enabling us to see the teacher's adaptation *in situ,* as various adjustments are made in the action of teaching). To facilitate the analysis, we group them into episodes.

Episode 1. Student groups working on problems
The following problem was given to students:

As a team, find all possible towers of a certain height that can be built with white and red blocks, where no blocks of the same color can be placed side by side. (Students have access to blocks.)

- How many towers of 5 blocks can be built?
- How many different towers of 6 blocks can be built?



- If we conduct this activity with another group of students, without counting each time, is it possible to find a way to find the number of towers that can be built, and that would work for towers of any height? Explain how you know.

We focus below on the action of the teacher in relation to what happened in each group of students.

Group A: Students work on the problem and a number of questions rapidly surface.

1. Questions in relation to the task's constraint: Roy *is forced to restate that no blocks of the same color can be placed side by side (but leaves it to students to interpret this constraint in their solving process, as he only mentions it anew to them).*
2. Questions concerning towers' symmetry: To students who ask if a tower WWWWR is the same as RWWWW, Roy *suggests imagining a real tower, with floors painted in red and white.* He asks: "Do we obtain the same tower if we color each level differently?" Then adds: "If you say yes, why? If you say no, why?"
3. Questions on the possibility of building towers using only one color. Roy *asks them if these towers respect the condition that no two blocks of the same color can be placed side by side.*

Group B: Students quickly get to work and different solutions emerge.

Walking from team to team, Roy *realizes that the third question is difficult for students. He decides to ask them to consider towers of height smaller than 6 blocks that can be built with 1, 2, 3 and 4 blocks.* [In relation to what students have done with towers of 6 blocks, Roy will explain in a subsequent research meeting, "it is better to have them work on smaller cases in the hope that they find some kind of regularity."]

Some students seem to appreciate the idea of working with smaller towers. Some teams begin formulating a pattern that would explain a way of obtaining the number of towers that can be built for any height.

We see here, through the excerpts we have italicized, what Mason and Spence (1999) call *knowing-to act in the moment* – that is, a knowledge that is enacted in the action, on-the-spot, in relation to students' diverse ways of engaging in the activity. The teacher here has to invent – in the action – ways of responding, of reacting to what happens. There is not a set of pre-determined interventions, something that is obvious in comparing the groups.

That said, what is this "knowing-to act," which is not necessarily explicit, that is enacted by the teacher *in situ* in reaction to students' engagement?

- *This knowing-to act refers to an implicit "reading grid"* (for interpreting what happens with students) *enacted in action.* This grid guides Roy in distinguishing which elements to address (*e.g.,* when acting upon Group B's difficulties with the third question of the activity) and which to leave for students to take care of (e.g., when restating the task's constraint in Group A). This reading grid represents, in a way, Roy's criteria for answering students' questions, that is, for guiding him in what he can or cannot say. This grid is generated in action, in response to students' questions, solutions, difficulties, or ways of engaging in the problem.

- *This knowing-to act refers to manners of doing.* For example, Roy *suggests a possible visualization* as a help to solve the problem (in response to the question about towers' symmetry in Group A); he *suggests using smaller numbers* to facilitate the generalization of a pattern (Group B); and he *restates the problem's constraint* (to counteract the idea of having towers of only one color in Group A).
- *This knowing-to act concerns aspects considered important in his teaching.* This appears when, through his questioning, he requests that students justify their answers (*e.g.,* point 2 in Group A). Implicitly, this also communicates his vision of mathematics and the ways in which mathematics is done (Bauersfeld, 1998), centered here on justifications.

Episode 2. Making sense of solutions and deciding which to pursue

> Roy asks, some twenty minutes before class ends, that students write up their solutions and the problem. While walking around the classroom, Roy *conducts a first scan* of what students are doing, in relation to the strategy and justifications used. After class, Roy *selects some solutions* to offer to the group the next day. This selection *is done rapidly to attempt to highlight the "right" and "wrong" solutions, the variety of strategies used* and *by privileging, in the case of similar strategies, the ones coming from students who have not shared them yet in class.* Indeed, Roy prefers that different students get chosen, since, as he explains, "we don't want to focus excessively on some of them," since some students who have previously contributed in class have been given nicknames by others (*e.g.,* one was given the nickname 'Fibonacci' because he had developed a solution that made a link between the task's numbers and the Fibonacci sequence).

Here is the heart of another key component of a teacher's practice: analyzing and investigating students' solutions. This selection can be done either *a posteriori* when solutions are gathered, or in action on the basis of what is observed in class when students work on problems. The teacher here makes choices to organize how the reinvestment of students' solutions will be managed and exploited for the next class.

That said, what guides this selection? The preceding analysis highlights important criteria guiding Roy: *mathematical* (validity or not of the solution offered), *didactical* (making apparent a wide variety of strategies and comprehensions of the problem, adequate or not), and *pedagogical* (concerns about equity in not always choosing the same students, and with classroom management regarding name-calling).

Episode 3. A reinvestment of students' solutions in class

> *Roy begins his reinvestment* (about a problem other than the tower one) *by resituating students* through recalling the handshake problem:
>
> In an international event, 10 persons coming from different countries were meeting for the first time. Everybody shook hands with each other and introduced themselves.
>
> (a) How many different handshakes will be shared at this meeting?
>
> (b) What will happen if there are more than 10 people? Can you find a way to compute the number of different handshakes, which works for any number of people present at the event? Explain how you know.
>
> *Roy reminds* students that many have found this problem easy, and explains that this is great *because what he wants to see are their various solutions and not only their answers. Roy explains that he wants to understand how they arrived at their answers.*

We see here Roy enacting ways of reengaging students in the problem, by reminding them of the problem's context. This reengagement is grounded in both a *pedagogical intention,* in resituating students in the learning situation and preparing them for the task, and a *didactical intention,* in insisting on the aspects he considers important in the activity (explaining their solutions and making sense of them). Here as well, implicitly, this communicates to students Roy's vision of how mathematics is done, focusing on explaining and sharing solutions.

> Roy invites a first team, Carla and Claudia, to come to the board and share their solutions. For question (b), Carla explains that, for example, with 20 persons one has to do 20 times 19 and then divide by 2. Another student, Bernard, asks why with 10 persons you get 45 handshakes and with 20 persons you obtain 190 handshakes. Roy *initiates the classroom discussion on this by asking: "What do you think?"*
>
> Two contradictory arguments are then offered by students:
>
> Bernard: "The calculations are wrong since, like for 10 persons, you need to consider the fact that the number of handshakes is decreasing."
>
> Pascal: "With 20 persons, we should have double the number of handshakes than with 10, so 90 and not 190."
>
> *Roy then asks students to consider the case of 5 persons:* "What do we get? 22.5 handshakes?" Some students offer 15 as a response, and Carla mentions that "22.5 is impossible."

We see here various "knowings" enacted in action. At the *didactical* level, Roy throws back the question to students in order for them to take responsibility for the validation of solutions (we see here again his insistence on students' justifications). At the *mathematical* level, Roy finds and offers a counterexample – "do 5 people give 22.5 handshakes?" – to counter the 'double' strategy. This counterexample is also a teaching intervention that has a *didactical* intention, since Roy's goal is to have students reflect on the solutions put forth and to create a doubt. Moreover, Roy will make an explicit link with the program of studies, at the *institutional* level, in a subsequent research meeting: *"'Using counterexamples' is in the program of studies; I am glad that students have understood this counterexample."* We see here that resorting to a counterexample as a teaching strategy had various intentions, drawing its source in mathematical knowledge (for finding a counterexample to argue on the non validity of a proposed solution), didactical knowledge (for promoting reflection on the solution and creating doubt) and institutional knowledge (for relating this use of a counterexample to the program of studies).

Owing to space constraints, we end our vignettes here. We now examine these episodes for the mathematical knowledge enacted.

"Knowing and using mathematics in teaching"

To explicate our conceptualization of teachers' enactment of mathematical knowledge in teaching, we discuss its nested character and then address three of its fundamental dimensions: (1) its nature, closer to knowledge-in-action, a "knowing-to act," than to factual knowledge; (2) its situated character; and (3) its emergence and unpredictability, requiring a capacity to react in the moment.

Nestedness

As Ball and Bass (2003) made clear, mathematics teachers engage with mathematical situations in their teaching in ways quite different from mathematicians. The above analysis highlights this difference. In exploiting a problem (*e.g.,* the towers one) a mathematician would certainly pay attention to, for example, the combinatorial models, the general formula used to find all possible tower combinations, and to the extension of this specific problem to a more general class of problems in which various constraints could be defined (different colors, different arrangements, *etc.*). A teacher, engaging in this problem in practice, would approach it from a different perspective. It would be linked to the students, to their various strategies, to the spontaneous models they generate and from which the teacher will reinvest the problem and its solving in relation to specific intentions. As the data above illustrates, the teacher is not interested in permutation-arrangement models, but in students' spontaneous models, their justifications, their validity and their evolution.

For a teacher, a mathematical situation is always grounded and interpreted in a teaching/learning context. The interpretation, as we have discussed above, mobilizes *simultaneously* diverse resources: didactical, pedagogical, mathematical, and even institutional. These dimensions are constantly, even if tacitly, taken into account in the teacher's understanding of the situation. A "mathematical situation" in the classroom is therefore always simultaneously "solved" and "exploited" in regard to various intertwined components. These various components are not enacted in isolation, since each influences the other in the choices made. Each component of course can be distinguished in the analysis, as we have done above, but they are indissociable in a teacher's practice. For example, the teacher's reading grid is simultaneously tainted/enriched with pedagogical, didactical, mathematical, and even at times institutional dimensions. The above example of the use of a counterexample also illustrates imbrications of mathematical, didactical and institutional understanding/intention. In that sense, even if called teachers' *mathematical* knowledge for teaching, it is never purely mathematical, as it is simultaneously mathematical, didactical and pedagogical/institutional. Thus, at the heart of a teacher's practice we find *a very specific knowledge, composed of intertwined and articulated dimensions.*

Other conceptual and epistemological clarifications

A. As we have seen in Roy's case, a teacher's mathematical knowledge for teaching represents *knowledge developed in the action of teaching* (a specific reading grid; criteria; ways of doing; *etc.*) in relation to the tasks in which a teacher is engaged (*e.g.,* choice of problems; observing students' ways of solving the problem and answering questions; analyzing students' productions in order to select reinvestments; orchestrating this reinvestment; *etc.*). These tasks, and the teacher's professional activity in relation to them, constitute the anchor points of this knowledge enacted in the action of teaching – which is constantly being developed/ refined through other activities conducted with students or when the same activity is conducted anew with another group. This is in line with recent studies analyzing teachers' practices, which highlight the professional gestures developed by teachers (see, *e.g.,* Butlen, 2006; Robert, 2001; Roditi, 2005). This knowing is far from being some factual and "static" knowledge that one could gather and appropriate independently of the practice in which it takes its meaning.

B. These last aspects lead to another important characteristic of this knowledge: *this knowledge is situated* (Lave, 1988), as it develops in a specific context linked to a practice of mathematics teaching. This knowledge is not independent of students' learning, of the classroom – of the real context in which it is enacted. We are thus talking about a knowing-to act activated in a real classroom situation, where this action-knowledge emerges, develops and is constantly refined. We can speak in that perspective of a context that acts as a structuring resource (*ibid.*) for the teacher, structuring the practice and the knowledge enacted in it. This knowledge is reorganized from one situation to another, through the teacher's experience in relation to the viability of enacted interventions, and the principles and criteria that guided those actions.

This notion of viability, borrowed from constructivism (*e.g.,* Glasersfeld, 1984), contrasts with a view of a teaching knowledge that would be universal and applicable to all teaching situations. Teachers' mathematical knowledge for teaching can be seen as adapted responses to teaching/learning situations in which it is enacted. As Schön (*e.g.,* 1983) explains, the professional develops his own knowing-to act through various experiences, and it is this body of experiences that allows the teacher to orient actions in teaching situations. This knowledge is developed in close relation to practice and invokes a dense network of understandings in regard to intentions, emergent situated meanings, students solutions and reactions to tasks, and consequent teacher adapted responses/reactions to (what Schön calls the "back-talk" of the situation). This action-reaction process emerges from the teacher's body of mathematical, didactical and pedagogical/institutional knowledge, elaborated *in situ* in the context of teaching.

Thus, the role of context appears particularly important in this conceptualization. The development of teachers' mathematical knowledge for teaching is grounded in a teaching/learning situation and in its "tasks." We distance that perspective from a standardized conception of teaching practices that would be based on a technical rationality paradigm in which the tools provided to teachers (textbooks, situations, didactical material, *etc.*) aim at offering ready-made solutions to problems encountered in practice. By grounding the development of this teaching knowledge in the context of teaching practice, we base our conceptualization on a different conception of practice, more in line with the practical rationality paradigm developed by Schön (*e.g.,* 1983) – a practice where undetermined situations and unpredictability play an important role and require a judgment-in-context from the teacher.

C. This conceptualization leads to a third fundamental aspect, in what Mason and Spence (1999) call *knowing-to act in the moment.* This teaching-knowledge is adapted in "real-time" to the event; the teacher needing to adapt responses as the dynamics of the classroom and of the situation prompts drift from the planned script. We speak here of knowledge produced on-the-spot. The teacher must constantly reflect on possibilities, offer and invent new avenues and representations in reaction to students' actions, think of additional explanations to clarify or resituate the tasks offered, choose to emphasize some aspects and not others, know that this or that explanation or representation is related to what the student offered as a solution and may eventually benefit *this* student's understanding, *etc.* Thus, this knowledge cannot be considered as a pre-established knowledge designed in advance for reacting well to situations, but mainly as *knowledge developed in context,* as *knowing-to act that is adapted* (to a situation) and *deployed on-the-spot* in reaction to an event (a student solution, an unexpected answer or error, a change in the unfolding of a task, *etc.*).

To illustrate what this conceptualization might entail in a teacher education context, we offer below illustrations of initiatives developed in our teacher education program.

Promoting teachers' mathematical knowledge

In the 1970s, a group of UQÀM mathematics teacher educators developed interventions to educate future mathematics teachers. Instead of entering into the program's specifics (see Bednarz, 2001; Bednarz, Gattuso & Mary, 1995; Bednarz & Proulx, 2005; Janvier, 1996; Janvier & Hosson, 1999), we offer illustrations of tasks and activities.

The development of knowing-to act, enacting multiple nested resources

In their first course [6], future teachers prepare three consecutive lessons, one of them to be conducted in front of their peers and a practicing schoolteacher, then analyzed by all involved, and then readjusted – the cycle repeated for another lesson on another topic. In this context, each student teacher is confronted with multiple reading-grids: the didactical-mathematical analysis of the mathematics teacher educator on school content, key reasoning put into play, interaction with students' solutions, *etc.;* the professional analysis of the experienced schoolteacher in relation to its practice, to classroom management, to real students' interactions, engagement and reactions to a task, to the value of the tasks offered, to the program of studies, *etc.;* the peer analysis based on their developing teaching knowledge. These multiple points of view provide future teachers the opportunity, in the action of teaching and in its a *priori* and *a posteriori* phases, to develop their teaching knowledge (criteria, reading grids, interventions, *etc.*) in which multiple resources come into play (didactical, mathematical, pedagogical, institutional) to deepen their professional-personal reference framework on mathematics teaching. [7]

A situated knowing-to act, using multiple nested dimensions

The teaching cycle is also reinvested in a *real classroom context* with secondary-level students at various points in the program through three practica: one in the first secondary grades, one in the last ones, and one with special students (difficulties, adults, high-achievers, multilingual, *etc.*) where they have to elaborate and tailor mathematical interventions to specific students and milieus. In planning for these situations future teachers conduct deep analyses of mathematical topics involving the various nested mathematical, didactical and pedagogical/institutional dimensions.

These "conceptual analyses" take the form of written documents. On one hand, they contain a mathematical/ institutional dimension about the meaning of a concept, its key reasoning, as well as its place in curriculum relative to other concepts, grades, and disciplines. On the other hand, these analyses contain a didactical dimension in regard to anticipations of (1) possible understandings, errors and difficulties; (2) explanations and representations that could promote understanding and key reasoning; and (3) selection of problems and activities with these details in mind. Those documents are widely used in the courses, where students consult them in regard to the course content, use them as reference to build their teaching in their school practicums, and, in the last year, create their own on a chosen topic, under the supervision of a mathematics teacher educator. The intention behind this 'evolving usage' is to allow for an appropriation of these conceptual analyses, sensitizing future teachers with their significance and richness, and finally to enable production of such analyses involving nested resources.

Articulating the courses with classroom reality

The concern with classroom reality is present in the emphasis placed on learning situations, through real students' solutions or teaching vignettes. We offer some examples here.

As an example of a task around *knowing-to act rooted in the context of classroom practice,* the following task situates the future teacher in a 7th-grade classroom for teaching multiplication of fractions.

> You are teaching in 7th grade. You want to work on multiplication of fractions, using the following numbers:
>
> (a) 10×3 (b) $10 \times {}^{3}/_{4}$ (c) $10 \times 1\,{}^{1}/_{5}$ (d) ${}^{10}/_{11} \times 1\,{}^{1}/_{5}$
>
> - Create a problem using an everyday context, accessible to students and easily visualized, that uses the repeated addition sense for multiplication;
> - Prepare an illustration that works and that you could use for all numbers to help students visualize the operation;
> - Show, for each case, with the illustration and specific explanations, how one can make sense of (c) from the answer obtained in (a).

The presence of specific constraints in the task (multiplication sense, visualisation, different numbers) comes from a didactical intention, that is to have student teachers realize the possible meaning gaps that can happen from natural to rational numbers, as well as from students' conceptions and difficulties.

As an example of *knowing-to act in the moment,* the following example concerns decimal numbers and its teaching.

> As a 7th-grade teacher, you offer this task to your students:
>
> > Arrange the following numbers from the least to the greatest:
> > 2.46 2.254 2.3 2.052 2.32
>
> Many of your students have written:
> 2.052 2.3 2.32 2.46 2.254
>
> And others have written:
> 2.052 2.254 2.32 2.46 2.3
>
> Complete the following steps:
>
> a. Describe and make sense of the error(s) committed by students;
>
> b. Find a similar task in which the students' reasoning would lead to the same error, confirming their strategy;
>
> c. Find a similar task in which the students' reasoning would lead to a right answer;
>
> d. How would you intervene on these difficulties?

This task requires analysis of students' difficulties and reasoning, as well as diagnosis-in-action to confirm the highlighted errors (questions b and c). In (d), the future teacher is prompted to develop "on the spot interventions" that would lead students to reflect on their understandings.

The preceding tasks offer a brief look at some of the *didactique* des *mathématiques* components within our courses. They illustrate how we are concerned with articulating the multiple nested dimensions (mathematical, didactical, pedagogical/institutional) that a teacher enacts in everyday practice. These tasks put the student teacher in a context of on-the-spot intervention, because teaching constantly requires capacities for rapid reaction to and interaction with students around their difficulties, questions, , solutions, *etc.*

Conclusion

We have attempted to illustrate how our conceptions of how a teacher "knows and uses mathematics in teaching," elaborating on theoretical aspects of this knowledge (multi-dimensional character, situated, in action, in the moment). This conceptualization has been an important source of inspiration for us in structuring and organizing our interventions, as mathematics teacher educators, in order to educate professionals of mathematics teaching who draw on multiple intertwined resources (mathematical, didactical and pedagogical/institutional). As Ball (2000) explains, we should not think of this teaching knowledge, and therefore our teacher education programs, as an accumulation of isolated bits of knowledge, where it is left to student teachers to articulate them. We believe it is important to conceptualize this knowledge as a multidimensional knowing-to act that is situated and reactive to a mathematics teaching/learning situation, using actual teachers' practices as reference in order to promote our student teachers' development of this *specific* teaching knowledge.

Notes

[1] Une version française de cet article est disponible sur le site web de FLM au http://flm.educ.alberta.ca/.

[2] To understand some of the particularities and multiplicity of perspectives in Quebec's *didactique des mathématiques,* see Bednarz (2007).

[3] This said, it would be a mistake to blindly associate and combine as one all studies that claim to belong to this body of research. There appears to be important differences among researchers who work in this field; perspectives that sometimes contrast with the initial theorizations put forth by the original authors, namely Ball and Bass's research group.

[4] See S. Barry (2008) *Analyse des ressources mises à contribution par enseignant et chercheur dans l'élaboration de scénarios d'enseignement en dénombrement visant le développement de la modélisation en secondaire 1,* unpublished doctoral thesis, Montréal, QC, Université du Québec à Montréal.

[5] The "pedagogical" component makes reference here to the classroom as a micro-society, with its social rules, its norms and its functioning; it concerns anything related to the establishment of relations with students in general and classroom management. The "didactical" component situates the intended mathematical knowledge at the center of the classroom teaching/learning situations; it is interested in the advancement of students' mathematical knowledge. The technical connotation often used in English for "didactical" is not the one we refer to here.

[6] The sentences in quotation marks refer to the teacher's discourse; translated from French to English by us.

[7] We refer here to courses of mathematics education, called *didactique des mathématiques.*

[8] For details on how this framework develops and is being enacted by future teachers in this program, see J. Proulx (2003), *Pratiques des futurs enseignants de mathématiques au secondaire sous l'angle des explications orales,* unpublished masters' thesis, Montréal, QC, Université du Québec à Montréal.

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In a mathematics test students were given the problem,

Let $f(x)$ be a function defined for all $x \neq 1$. If

[1] $\lim_{x\to 1}[(x^2 - 1)f(x) + 3x - 7] = 10,$

then what can you deduce about

[2] $\lim_{x\to 1}(x - 1)f(x)$?

a) What do you think the examiner intended by setting this problem?

b) A student responded as follows:

Let $h(x) = (x - 1)f(x)$, and assume that $\lim_{x\to 1}h(x) = \alpha$. Then according to the assumptions of the problem, and the properties of limits, we have,

$$10 = \lim_{x\to 1} [(x + 1)h(x) + 3x - 7] = 2\alpha + 3 - 7, \text{ and so } \alpha = 7.$$

i) Evaluate the student's solution.
ii) Evaluate the student's reasoning.
iii) How would you respond to the student?

(adapted from Biza, I., Nardi, E. and Zachariades, T. (2007) 'Using tasks to explore teacher knowledge in situation-specific contexts', *Journal of Mathematics Teacher Education* **10**, pp. 301–309)
