PROMOTION OF HEURISTIC LITERACY IN A REGULAR MATHEMATICS CLASSROOM

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What common structure may be found in the following tasks?

Task 1: Find digits A, B and C that fit the following product [1]:

$$\begin{array}{cccc} & A & B & 4 \\ \times & & & 8 \\ \hline 5 & 3 & 9 & C \end{array}$$

Task 2: Prove that for any numbers x and y,

$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

Task 3: Julia should go to bed at 9:30pm, but it is too difficult to stop her playing. Her parents allowed the girl to go to bed when the minute hand met the hour hand. How much additional time did Julia have?

It is easy to see the differences between these tasks, for example in subject matter, context, preliminaries and question formulations. In order to throw light upon a similarity, in this article we will consider how (good) middle-school students approached them. At this point, however, we invite the reader to attempt the tasks before seeing the suggested solutions.

Solution of Task 1: Decompose the problem into two parts. The first step is obvious: C = 2. Then a *division* provides a key for the solution. Dividing 5392 by 8 gives the answer: A = 6, B = 7. A 'direct' solution, namely, multiplication of AB4 by 8 is much more difficult.

Solution of Task 2: The easiest way is to multiply out the brackets, beginning the proof from the right-hand side of the identity.

Solution of Task 3: At 9:30pm the distance AB between the hands was 17.5 minutes (see Figure 1). Obviously, Julia has more than 17.5 minutes of extra time since the hour hand also moves. Suppose that the two hands meet in a point C (the endpoint) in x minutes. Since the hour hand moves from B to C in those x minutes, the minute hand would take $x/_{12}$ minutes to do that same distance. Imagine, therefore, that the minute hand moves from A to C and then counter-clockwise from C to B. This enables us to get the following equation [2]: $AB = AC - BC$ or $17.5 = x - x/_{12}$, $x = 19^1/_{11}$.

When these tasks were given to many middle-school

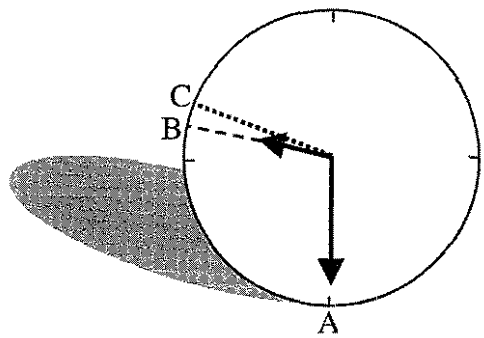

Figure 1: The initial position of the hands in Task 3

students (thirteen- to fourteen-year-old) and to a few pre-service and in-service teachers, we observed similar solution strategies:

- *More-experienced problem solvers* approached the tasks by analyzing problems in directions opposite to those prescribed by the problem contexts: from below in Task 1, from the right in Task 2, and moving from the endpoint in Task 3.

- *Less-experienced problem solvers* followed the contexts directly and faced difficulties or failed.

At a context-free level, these tasks call for a change in the direction of search from forward to backward, getting started from what has been given or from the goal. In other words, the tasks invite the use of the powerful heuristics mentioned in the literature on problem solving: *thinking forward* and *thinking backward* (Larson, 1983; Martinez, 1998; Schoenfeld, 1985; Verschaffel, 1999). For us, heuristics, or a heuristic strategy, is a systematic, mathematical problem-solving strategy formulated in a free-of-context manner.

Applying and adapting a variety of appropriate heuristic strategies is one of the accepted standards of problem solving (NCTM, 2000). Thinking through a solution to a non-routine mathematical task, experts in problem solving call into play many sophisticated strategies (almost) without conscious efforts, while novices need to be taught how to do so (Polya, 1973; Schoenfeld, 1985).

The purpose of this article is to present examples of heuristic-oriented activities (HOA) that invite the use of *working forward* and *working backward* strategies. Following Martinez (1998), we interpret forward and backward heuristic search as problem decomposition from a given to a goal state or *vice versa*. Specifically, we refer to *thinking* *forward* as evaluating if it is worthwhile to use the particular problem solving step before doing it when one starts from the given, and to *thinking backward* when the direction of decomposition is from the goal state.

The forthcoming examples of HOA were designed and implemented in the framework of heuristic training, over a period of five months in classrooms, aimed at the development of heuristic literacy, which may be defined as the use of heuristic vocabulary in discourse, the enrichment of one's heuristic arsenal, and awareness of heuristics used in problem solving (Koichu, 2003). The HOA are related to the topics 'Quadrilaterals', 'Abridged multiplication formulas' and 'Quadratic equations'.

Examples of heuristic-oriented activities

We present five examples of HOA related to the promotion of thinking forward and thinking backward strategies. HOA 1 was used at the beginning of the five-month period with the purpose of incorporating a heuristic vocabulary into the classroom mathematical discourse. HOA 2 and 3 were in use in the middle of the period, when many difficult and unconventional problems were proposed for both classwork and homework. These activities included the option of help provided in terms of the vocabulary of the heuristics in use. Articulation of the strategies' names is of less importance in HOA 4 and 5, which were used in order to promote the use of the intended strategies implicitly.

The first activity

HOA 1: There is a 4 × 4 maze (see Figure 2) with the only exit from square A4. A piece of cheese is located near the exit

There are doors between the squares and some of these doors are open and others are not. A mouse, striving for the cheese, ran the following path: up, to the right, down, to the right, to the right, up, to the right, cheese [3]. Where was the mouse at the beginning of its journey? [4]

HOA 1 consisted of four parts In closing each part, the teacher conducted a short discussion of the strategies that the students had used.

Part 1: No constraints. The students try to find out the answer using all the available tools. We found that the following discussion was typical in the two classes who took part in the sequence of lessons and in many additional workshops.

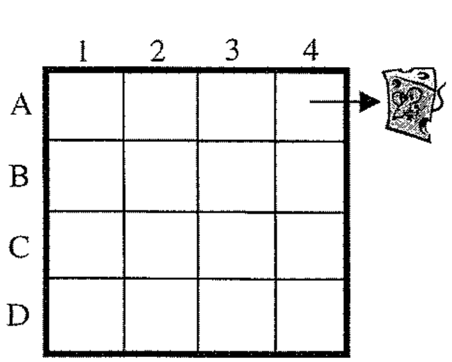

Figure 2: The 4 × 4 maze.

Teacher:	So where was the mouse?
Students:	B1, B1, B1, B2, B1, B1 . .
Teacher:	I see most of you got B1 How did you get it?
Avi:	It's easy. I drew all the steps from the end. Instead of 'right', 'left', instead of 'up', 'down' and so on
Students:	Me too . I did the same
Olga:	I also got to B1 but in another way I started from any point . . from C2 ... and drew the mouse's way step by step. When I missed the cheese by two squares, I moved the entire path one square up and one square left Then I got to B1
Teacher:	Very interesting! How can we describe the strategies that you've used?
Avi:	'From the end to the beginning'.
Boris:	'To change a direction'
Olga:	'To guess and fix'
	[The teacher lists on the board all the students' suggestions.]
Teacher:	Actually, you thought about the problem backwards, from the end, or forwards. I see that most of you used 'thinking backward' or 'from the end to the beginning', and Olga thought by means of 'thinking forward' or 'guess and fix'

Remark: HOA 1 was designed and implemented for the purpose of introducing a common vocabulary of heuristics into the students' problem-solving discourse. The (initial) names of the strategies arose from the classroom discussion above. It was important that the names of the strategies came from the students, as well as from the teacher. Confrey (1994) called this teaching technique *close listening* and advocated it as a powerful tool for reflection and analysis of the students' ways of thinking.

Part 2: Two games in pairs, no constraints Students work in pairs where one of them is a 'teacher' who offers their partner a description of the mouse's path. Afterwards the roles are changed

Part 3: No writing. The teacher (or one of the students as a teacher) offers a new path orally, repeating it up to three times. The students are not allowed to use pencil and paper The maze is still on the class board

Fragment of a (typical) discussion after Part 3:

Teacher: Many of you got the answer; there were no deviations by more than one square Wonderful! How did you think without writing?

Avi: This time it was difficult to think from the end. I could hardly memorize the path [in order to use 'from the end to the beginning' strategy without writing, Avi first tried to memorize the way and then to inverse it].

Students: I did ... I could .

Olga: My strategy worked again! I still could imagine the entire path from any point and then fix it!

Boris: Yes, I also used 'thinking forward'.

Teacher: I see this time 'thinking forward' or 'guess and fix' was more popular than 'thinking backward' Was it?

Remark: There are no new strategies in the above fragment It is clear, however, that a requirement of 'no writing' yields an essential change in the distribution of the previously incorporated strategies in the students' performance. It is also important that, at this stage, the introduced strategies' names helped the students to express themselves Some of the students changed away from the strategies that they had previously used, probably by being faced with the strategies of other students. We interpret such changes as the beginning of the enrichment of students' heuristic arsenal and as introducing new meta-discursive rules into classroom communicative activities (Sfard, 2000).

Part 4: No writing, no visualization The teacher offers the same starting game as in Part 3, using a new path. The drawings of the maze are erased from the board.

Fragment of a (typical) discussion after Part 4:

[The path of the mouse was, to the left, up, up, up, to the right, to the right, cheese.]

Teacher: Good! D4 is the answer. How did you find it? Thinking backward or thinking forward?

Michael: Not really. You said "up" three times in succession. Therefore, the mouse couldn't start from rows A, B and C. It must be D ...

Eva: Yes, this was what I did And besides, "to the left" was neutralized by "to the right". There wasn't a choice for the mouse. Only D4.

Teacher: Very interesting. I think this is a new strategy. How can we call it? This is not 'backward' or 'forward' ...

Michael: Maybe, 'to reject what is impossible', or 'rejection'.

Teacher: 'Rejection of possibilities'?

Eva, Michael: Yes. OK.

Afterwards, during the same lesson, the teacher asked her students to recall mathematical situations when they used 'thinking forward', 'thinking backward' (or 'from the end to the beginning') and 'rejection of possibilities' strategies in problem solving.

Remark: It was important that the strategies' names helped students to express themselves both in non-mathematical and mathematical contexts Perkins and Salomon (1988) have mentioned such *bridging* (i.e. creating bridges between different contexts by using a unified context-free terminology) among useful techniques of teaching for transfer

The second activity

HOA 2: Diagonals of a parallelogram ABCD intersect in a point O The points M, N, K and L are the intersection points of the angle bisectors in each of the triangles ABO, BCO, CDO and DAO, respectively What can you say about the quadrilateral MNKL? Formulate a conjecture and prove it [5]

This problem, a difficult one for these thirteen- to fourteen-year- old students, was offered to pairs for solution. A set of hints was available for each pair. The students could get the hints one at a time with not more than two hints to be taken every 5 minutes. The hints belonged to several clusters relating to content with more general and more specific hints in the clusters. Formulations of the hints were grounded on our expectations of students' difficulties and were checked with a pilot group of similarly aged students. At this point, we again invite the reader to try the problem before reading the following hints.

Hint 1: A (good) picture can help! Draw a large picture using a ruler. Consider: do you really need to draw all twelve bisectors mentioned in the problem?

Hint 2: Look at the picture (Figure 3) and assign the points [6].

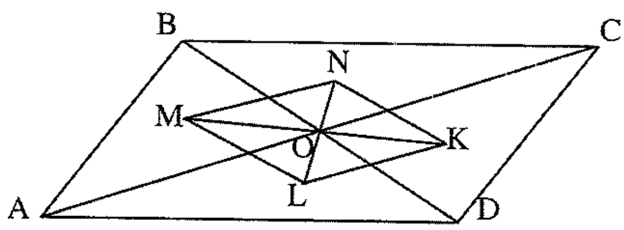

Figure 3: Diagram for HOA 2.

At this stage the teacher conducted a short classroom discussion using the question, "What can you say about the inner quadrilateral?" After starting with the conjecture, "It is at least a parallelogram ", most of the students found that the conjecture, "MNKL must be a rhombus" was worth checking.

Hint 3: Try to plan a solution Read the problem again and think forward! What do you need to know in order to prove that MNKL is a rhombus? Consider which way seems to be more promising: to prove that the four sides are equal or to prove something about the diagonals?

Hint 4: Have you chosen to think about the diagonals? Excellent! Why are you sure that the diagonal MK passes through point O? Prove it!

Hint 5: Remember that OM and OK are angle bisectors! Can you use this fact in order to prove that angle MOK = 180°?

Hint 6: Explore the symmetry: conduct the same proof for points N, O and L.

Hint 7: You have proved that MK and NL pass through point O. Decompose the rest of the solution: what additional two things, related to the diagonals, would help you to prove that MNKL is a rhombus?

Hint 8: You have proved that MK and NL pass through point O. Now you need to prove that:

i MK ⊥ NL

ii MK and NL bisect each other

Hint 9: You almost proved (i) when you thought about vertical angles and bisectors. For (ii), how can you prove that two segments are equal? Think about congruency of triangles.

Hint 10: If you still cannot see a solution, remain calm and think backwards! Try to put in good order all the stages you have done Re-read the problem What information haven't you used yet?

Hint 11: Raise your hand and call, "Help me!"

Remarks: In the two classes nobody cried, "Help me!" Actually, the task provoked some kind of competitive behavior Many students tried to solve the problem using a smaller number of hints than their 'rivals'. Most of the pairs used between four and ten hints in order to solve the problem. The teacher's assistance was minimal. Using Sfard's (2002) terms, previously introduced heuristic names ('draw a picture', 'think forward', 'think backward', 'explore the symmetry', and 'decompose the problem') played a role of *metalevel intimations* [7] when offered as a source of help Sfard (2002) noted that metalevel intimations are vital both for mastering discourses (including discourses with oneself) and for the process of learning.

The third activity

HOA 3: There are two large (L_1 and L_2) and three small (S_1, S_2, S_3) mathematical tests in the last quarter of a school year. The order of the tests is S_1, L_1, S_2, S_3 and L_2 The weight of each small test in the final grade is 12%; the weights of L_1 and L_2 are 24% and 40% respectively. Jenny got 80 (out of 100) in S_1 and 60 (out of 100) in L_1.

1. Can Jenny, in her current situation, get a final grade of 85? Explain

[The answer is, "Yes". Jenny's maximum final grade may be computed in two ways Firstly, directly:

$$80 \times 0.12 + 60 \times 0.24 + 2 \times 0.12 \times 100 + 0.4 \times 100 = 88$$

or by computation of the points that she lost in the two first tests:

$$100 - 20 \times 0.12 - 40 \times 0.24 = 88]$$

2. Jenny decided not to overstrain herself and to be content with the final grade 56. Theoretically, can she completely ignore test L_2?

[The answer is, "No" For example,

$$100 - 12 - 40 < 56]$$

3. What is the minimum grade that Jenny has to get in tests S_2, S_3 and L_2 in order to obtain 56 as a final grade?

[Denoting s_2, s_3 and l_2 as S_2, S_3 and L_2 tests' scores respectively,

$$0 \le s_2, s_3, l_2 \le 100,$$

we obtain the equation,

$$s_2 \times 0.12 + s_3 \times 0.12 + l_2 \times 0.4 = 56 - 24$$

which has a large number of integer solutions, for example,

$$s_2 = s_3 = 0, l_2 = 80.$$

Carefully reading the problem, it is possible to discover an additional piece of information:

$$s_2 = s_3 = l_2$$

Consequently, the answer is 50].

4. Recommend to Jenny a few learning strategies for the rest of the tests that would enable her to get a final grade of 80 or more Take into account that she has never got more than 95 in the past

After 20 minutes of working on the problem in pairs, the students turned to check what the other pairs had done Where they found discrepancies, they resolved them in discussion. If one of the pairs did not solve some item and another pair did, the more successful students were asked to help their classmates with one or more heuristic hints, starting with more general ones and moving to more specific ones. Afterwards, these hints were presented to the whole class and discussed.

Remarks: This 'real-context' problem invites both thinking forwards and thinking backwards at two levels. The first one is prescribed by the context. The students helped Jenny to plan her future learning strategies and to distribute her efforts among the tests under the proposed constraints. The second level is related to different ways of solving the problem Questions one to three may be solved in terms of 'thinking forward' or 'thinking backward' strategies. Use of the 'guess and fix' strategy was also possible The open-ended item four invited the use of all the above strategies as well as 'exploration of extreme values and cases'. Students' discussions of any discrepancies and their experiences in formulating (heuristic) hints promoted the further articulation of the strategies' names.

The fourth activity

When the students learned and memorized classical mathematical formulas, the teachers discovered that the students still could not use them properly in non-trivial exercises. The teachers were disappointed by the results of the test which was given after a period of conventional drill and practice. This experience led to the design and implementation of the following HOA.

Name	Formula	Code
Difference of squares	$a^2 - b^2 = (a - b)(a + b)$	DS
Square of a difference	$(a - b)^2 = a^2 - 2ab + b^2$	SD
Square of a sum	$(a + b)^2 = a^2 + 2ab + b^2$	SS
Taking out a common factor		CF
Collecting like terms		LT

HOA 4: With the help of students, the formulas and the names of commonly used algebraic transformations were coded as follows:

After a short explanation, the students were given a set of exercises These exercises were not to be solved but the students were asked to code possible ways of solving the problems. They were encouraged to imagine and to code as many of the steps of the solutions as they could. For example, the exercise,

Factor $(x + 3)^2 - (2x + 5)^2$

was coded in the following ways:

Dalia: SS, SS, LT [meaning $(x^2 + 6x + 9) - (4x^2 + 20x + 25) = -3x^2 - 14x - 16 = ?$]

Iris: DS, LT [meaning $(x + 3 + 2x + 5)(x+ 3 - 2x - 5) = (3x + 8)(- x - 2)$]

Eli: DS [meaning $(x + 3 + 2x + 5)(x + 3 - 2x - 5)$ and may be also the next step orally].

The exercise, simplify:

$(1 - a)(1 + a)(1 + a^2)(1 + a^4)(1 + a^8) \times \dots \times (1 + a^{1024})$

was coded as follows: DS, DS, ..., DS.

At the next stage of the lesson all of the students' suggestions were discussed. Then everybody chose any coding scheme they liked in order to solve the exercises

Remarks: According to the teachers, the next test they gave to the students was more successful. More interestingly, the teachers used the introduced coding scheme in related topics such as equations, and extended the idea to some additional topics without our assistance. According to Zazkis (2000), informal coding sometimes indicates incomplete understanding At the same time, she noted that:

> [...] use of informal code by a teacher should be a pedagogical choice, rather than a symptom of a lack of proficiency with the mathematical code itself. (p. 42)

We like HOA 4 since it promotes practical experience in the 'thinking forward' strategy without mentioning it by name HOA 4 led to shorter and more elegant solutions. We suggest that preliminary coding puts into order students' heuristic searches and helps them to plan solutions prior to diving into technical work. It only slightly reduced the number of technical mistakes made, but made all the exercises acceptable to weaker students Besides, preliminary coding introduced an intermediate open-ended stage into these algebraic exercises with one correct answer Zaslavsky (1995) has argued for the importance of open-ended tasks as a source of "rich and powerful learning situations" (p. 15) for students as well as for teachers' professional development.

The fifth activity

HOA 5: The following four-part task was given in a geometry lesson

Part 1 (initial problem): Work carefully in accordance with the following algorithm:

1. Draw a parallelogram (1) that is not a rectangle or a rhombus.
2. Divide the parallelogram (1) into a rhombus and a parallelogram (2), which is not a rhombus You may need to modify the parallelogram (1).
3. Now you have a rhombus and the parallelogram (2), which is not a rhombus Divide the parallelogram (2) into a rhombus and a parallelogram (3), which is not a rhombus. You may need to modify the entire picture
4. Do the same for the parallelogram (3), which is not a rhombus. You should obtain a parallelogram (4) that is not a rhombus

Given that the sides of the parallelogram (4) are 1 and 2 cm, find the sides of the initial parallelogram (1).

Part 2 (collecting the answers): A teacher listed the students' answers on the board Surprisingly for many students, who confidently solved this simple problem, there was more than one answer Some of the students' answers are represented in Figure 4.

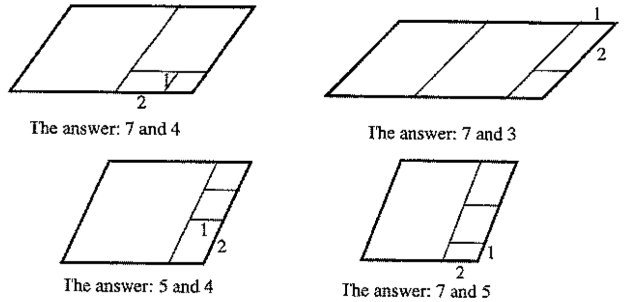

Figure 4: Examples of the students' answers to part 2 of HOA 5.

Part 3 (a situation of uncertainty): The students checked each other's solutions and discovered that there were many correct answers. A natural question appeared: "How many answers fit the problem?" On many occasions, an immediate response of the students (and teachers) to the above question was, "an infinite number of answers". Then some sceptical voices doubted this extreme statement, initially without explanations. A situation of uncertainty evolved at this stage.

Part 4 (resolving the uncertainty): This uncertainty was resolved by the students' suggestion to think about the problem "from the end to the beginning". It is possible to carry out the algorithm given in Part 1 starting from the last step to the first one, adding the rhombuses to the parallelogram instead of cutting them off. There are exactly two options (up to symmetry) of ways to add a rhombus to a 'smaller' parallelogram at every step of the algorithm. Since the initial parallelogram was divided three times, there are exactly $2^3 = 8$ different numerical solutions, as shown in the following tree (Figure 5):

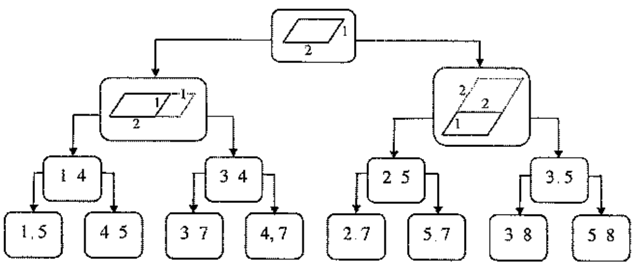

Figure 5: A tree of the solutions

Inverting the algorithm, the students usually started with drawings of the parallelograms, indicating their sides. After one or two steps, most of the students used an algebraic representation only, discovering a recursive pattern: substitution of one of the sides of a parallelogram by the sum of its sides.

Remarks: This task was designed in order to create a situation which implicitly invites the use of the 'thinking backward' strategy, and it did work in many workshops with pre- and in-service teachers, as well as with the thirteen- to fourteen-year-old students. There are other interesting facets of HOA 5:

1. The students faced the fact that 'any parallelogram' may be drawn in different ways. Fischbein (1993) might say that the 'parallelogram' was enriched by different figural representations.

2. HOA 5 includes a stage of uncertainty with respect to different (correct) answers and to an overall number of correct answers. Zaslavsky (forthcoming) treats such feasible uncertainty as a good opportunity for meaningful learning.

3. This geometrical task enables a teacher to incorporate into a lesson the issue of recursive reasoning. A switch from the figural representations to the numerical one requires the use of 'thinking forward'. The search for algebraic patterns promotes the ability to generalize. Besides the cognitive potential, HOA 5 has a mathematical one. It provides an unconventional way of introducing Fibonacci numbers (see the right branch of the tree in Figure 5).

Concluding discussion

Heuristics are associated with many non-routine mathematical problems. The presented HOAs were designed in order to uncover built-in heuristics and to promote, explicitly and implicitly, heuristic literacy. 'Thinking backward' and 'thinking forward' strategies were promoted along with the other heuristics by means of tasks primarily related to a standard mathematics curriculum, in real classroom conditions.

Alongside the focus on the intended heuristics, the above tasks may be considered in terms of many additional pedagogical perspectives, namely, enrichment of classroom discourse (e.g. Sfard, 2000), encouraging reflection and indirect instruction (e.g. Confrey, 1990), *close listening* (Confrey, 1994), seeking similarities in different mathematical contexts by means of free-of-context vocabulary (Perkins and Salomon, 1988), use of open-ended tasks (Zaslavsky, 1995), creation of feasible uncertainty (Zaslavsky, forthcoming), and switching representations and coding schemes (Zazkis, 2000).

The above remarks about HOAs are focused on the students' heuristic literacy. It is also interesting to see briefly what happened to the teachers who took part in the teaching experiment [8]. At the beginning of the experiment, Anna and Larisa [9] (the teachers who volunteered to take part in the teaching experiment with heuristics) did their best in following the plans we developed for them. They began to suggest some mathematical tasks for HOAs from the fourth week of our cooperation.

After three months, Anna and Larisa prepared about 50% of the HOAs by themselves. They also learned to distinguish many strategies in students' problem solving through reflecting on classroom activities. In the words of Anna:

> During our work together, I activated the [heuristic] strategies in my head, and I understood them better

We observed that, through the intervention, the teachers passed from consuming the materials that we developed for them to co-designing, from discovering heuristics in their own problem-solving experiences to heuristic literacy

As authors of this paper we apply Anna's words to ourselves. With respect to heuristic literacy, the co-operation was mutually effective Echoing Cobb (2000) we can state that in the teaching experiment, theory emerged from practice and fed back in to guide it

In closing, we would like to turn to the question posed in NCTM (2000):

> How should these [heuristic] strategies be taught? Should they receive explicit attention, and how should they be integrated with the mathematical curriculum? (p. 54)

We believe that one of the possible answers is as follows: promotion of students' heuristic literacy may be an effective tool, in combination with teachers developing heuristic literacy, induced either by personal problem solving experience, or by learning through teaching of the regular curriculum with deliberate emphasis on a heuristic approach in problem solving

Acknowledgements

The research of the first author was supported by the SAKTA RASHI Foundation (budget No 765-076-00) and by the Technion Graduate School. The research of the second author was supported by the Fund for Promotion of Research at the Technion.

We are grateful to the teachers of "Mofet", the association which enabled us to gather data for the study and especially to Mrs Anna Koltin and Mrs Larisa Marko for their enthusiasm, professionalism and co-operation

Notes

[1] This task is adapted from the Scholastic Aptitude Test - Mathematics (SAT-M)

[2] Even if the equation AB + BC = AC is preferred, implying only clockwise movement, you begin from a consideration of the endpoint C

[3] There must be a new path of the mouse for additional uses of the game Obviously, every path must end by '. , to the right, cheese'.

[4] We heard of this problem in a colloquium talk by Professor Ron Aharoni of the Department of Mathematics at the Technion, describing his work with elementary school students

[5] HOA 2 originated in the Olympiad problem taken from Zubelevich (1971).

[6] In this article we assigned the points on the diagram, but we did not in the lesson.

[7] According to Sfard (2002):

> Metalevel intimations are ideas for discursive decisions induced by interlocutors' tendency to behave in a regular rather than accidental way that is an accord with metadiscursive rules that seem to regulate discourses (p. 337)

[8] Following Cobb (2000), we refer to a *classroom teaching experiment* as a process of the interlacing of teaching and research when practicing teachers are members of the research and developmental team

[9] The teachers authorized us to use their real names.

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