Beyond Metaphor: Formalising in Mathematical Understanding within Constructivist Environments

SUSAN E.B. PIRIE, THOMAS E. KIEREN [1]

Background

Much of the power of mathematics comes from one being able to both construct and work with its ideas in ways which are not dependent on physical contexts, are general and not local in nature, and are based on using symbolic forms in creative ways. Ernest [1991] suggests that mathematics done in this way is the "front" of (professional) mathematics, while work with specific local situations, which may serve as motivation for the formal "front" of mathematics, is ascribed a lesser role in its social construction.

In order to facilitate the learning or building up of mathematical ideas, it is inappropriate to simply present the student, even of advanced school mathematics, with formalised ideas. Harel and Tall [1991] have suggested that such an approach leads to disjunctive generalisations which are not related to less formal schemes. In these circumstances, generalisations and related understanding activities are disjoint [Pirie and Kieren, 1992a; 1992b] for many students and can lead to even university level students thinking of mathematics as a rather meaningless mechanical activity [Frid, 1992]. In the case of younger children, Hiebert and Wearne [1987] and Skemp [1987], have warned that once they are taught in the manner described above and develop inappropriate, disjoint, and formal generalisations, these schemes are hard to change. But, of course, as argued by Hiebert and Wearne [1989] or Hart [1988], simply teaching mathematical ideas in a concrete setting and then later "teaching" formal generalisations and rules does not necessarily bring about appropriate, connected, formal understanding either. The question is, how can one provide instruction to bring about connected, formalised understanding in students?

As part of the answer to this question, it is our purpose in this paper to consider the nature of *formalising understanding* in terms of our theory of mathematical understanding and to illustrate such formalising and contrast it with less formal mathematical understanding as seen in the activities of eight and twelve-year-olds functioning in what we have termed *constructivist environments* for mathematics [Pirie and Kieren, 1992b; Kieren 1992]. We have given general accounts of our theory of the growth of mathematical understanding as a whole, dynamic, leveled but non-linear, recursive process, and have discussed some of the features of the theory, such as *folding back*, in a number of places [e.g. Pirie and Kieren, 1989a; Pirie and Kieren, 1991; Kieren and Pirie, 1992]

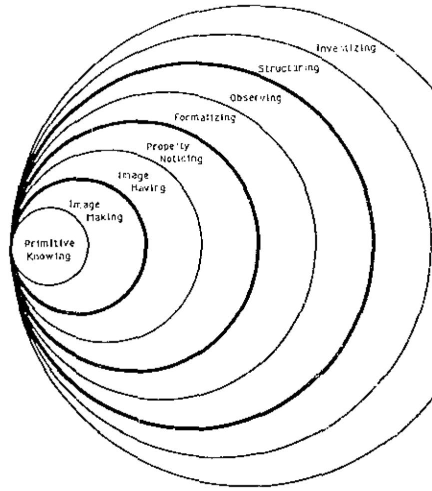


Figure 1
Model of a dynamical theory of the growth of mathematical understanding

We have tried to provide illustrations of the phenomenon in terms of mathematical experiences with children [e.g. Pirie and Kieren, 1989b; Kieren, 1990; Pirie and Kieren, 1992a]. In our theory *formalising* is simply one of the observed levels of modes of understanding. We believe there is no such thing as understanding in the abstract. Mathematical understanding is a process, grounded within a person, within a topic, within a particular environment. For us, formalising is the first formal mode of understanding and, if not disjoint, it will embed less formal understanding levels or modes called primitive knowing, image making, image having and property noticing (see Figure 1). These inner modes of mathematical understanding represent the core of context-dependent, local know-how in mathematics. In growing to formalising the student consciously thinks about informal patterns and abstracts commonalities in a particular way. Having established such formalisations, the student is in a position to reflect on them, *observing* theorems through creating and proving them, and then *structuring* them into a mathematical system. These two subsequent, outer, formal modes of understanding embed the formalising and thus embed the earlier, inner modes of informal understanding. In other writing [Kieren and Pirie, 1991a; Pirie and Kieren, 1992b], we have noted that to be observed to be exhibiting any mode of understanding a student needs to engage in both the *act-*

ing appropriate to that understanding mode and in the reflective *expressing* of and about that action In formalising, we call the action we are considering *method applying*. Here the student acts by generating patterns or symbolic results without necessary reference to their context-based meaning [Pirie and Kieren 1992b], rather than thinking about, reflecting on and describing mathematics in terms of mental images based on physical actions. To complete such understanding the student must engage in *method justifying*; that is, the student must be aware of the method being applied and be able to explain its working.

Of course, there is considerable other contemporary work on the nature of formal mathematical activity. Herscovics [1992], for example, distinguishes formal mathematical activity as the ability to deal with mathematical schemes in terms of mathematical symbols Notice that in our characterisation of formalising, symbol use does not distinguish formalising from the informal levels or modes of understanding. In formalising *symbols are used differently*: they are not used to describe local patterns but are used without reference to any concrete meaning of the mathematics. In formalising, mathematics is no longer a metaphor for events in a physical or image world It is abstract, although not necessarily expressed in generalised mathematical terms or symbols, as will be illustrated later in this paper. Sfard [1991] is interested in formal mathematical activity, particularly as it pertains to the development of advanced mathematical ideas She implies that moving from an operational view of a notion to a reified or objectified structural view is a difficult metamorphic change. Sfard's work perhaps pertains most to our work when we consider the formal activities of *observing* and *structuring*, but we feel that this is not the only place where such a metamorphic change occurs in mathematical understanding. The change from *image making* to *image having* is a metamorphic transformation, as is that from *property noticing* to *formalising*. We attempt to illustrate this below in the work of children. Thus, this essay discusses the mechanisms which such growing to a structural view of mathematics entails, and considers such mechanisms even in the activities of young children dealing with numerical ideas in the area of fraction learning. In our illustration of the mechanisms of formalising, we will be elaborating and making distinctions in Mason's [1989] ideas of *noticing* which suggest that growth occurs through a delicate shift of attention by the learner. Our theory points to several different kinds of such shifts and we consider one kind of shift here.

At the image having level we see the learners as working with metaphors. For them, mathematics *is* the image that they have and their working with that image Consider the case of eight-year-old children who create fractional parts of a unit (sheet of paper) by successive foldings-in-half. From such activity we have observed that children abstract an image of fractions which might be exemplified by the phrase: "eights [sic] are the pieces I get when I make three folds." Fractions are considered by that child to be certain "pieces". In their speaking or writing about their actions and their reasoning with fractions they are putting fraction symbols or words for the pieces. Such metaphoric thinking allows young children to reason about quite complex fraction situations. The power of the metaphor can be seen in the following incident An eight-year-old child who had already successfully folded and worked with "half-fractions" as described above now tried to fold tenths in a particular way. She burst into tears when she could not get 10 equal pieces. For her, fractions, here tenths, *were* pieces and when she could not get pieces to her specification she could not "get" fractions. For some older children, an equation *is* a balance; adding equal amounts [weights] to both sides does not disturb the equilibrium.

As the attention is shifted from the objects of metaphor—pieces, balances, etc.—to properties of those objects, simile [2] comes into play—"is" becomes "is like". Fractions are now like pieces, but admit the use of equivalence. Equations are now like balances, but admit negative solutions. Integrals are like areas under curves, but admit negative areas.

Thus we observe that understanding at the image having level is fundamentally metaphoric, while understanding at the property noticing level entails the use of similes. A further shift of attention by a learner leads to formalising, which is the principal subject of the rest of this paper.

Illustrations of formalising in a constructivist environment

The transcripts and student activities discussed below are drawn from a series of studies of children, aged 8 and 12, building up their own ideas of fractions. Because we are attempting to study the phenomenon of the growth of mathematical understanding as a lived process it was decided to observe children in situations where they were allowed to construct mathematical ideas as they saw fit, occasioned by an environment where their ideas could guide useful action [Varela, Thompson, Rosch, 1991]. Thus, we studied children in what we term *constructivist environments*. Some of the tenets we hold about such environments [Pirie and Kieren, 1992b] are as follows: the teacher expects children to hold unique and different understandings, reached by different pathways of growth, and these understandings are not achieved states of equilibrium. They are a continuing process, with the child expected, or even prompted, to fold back to less sophisticated activities to add to her experience base. Even when the teacher intervenes with a specific intent, it is the actions of the child which determine the nature and effect of the instruction on her understanding [Kieren and Pirie, 1992].

The studies took the form of whole class "teaching experiments" in which one of the authors of this paper (Kieren) acted as the teacher. Students acted and worked on fractions, their understanding occasioned by folding activities and activities using kits with various fractional pieces Records of class activities were video or audio taped. Research assistants tracked the work of class subgroups and made transcripts of audio tapes and field notes. The student-made records of task work were collected. These various tapes, transcripts, and artifacts were then studied to identify activities which allowed us to test, elaborate, and develop constructs and the properties of our theory. This form of research allowed us, following Bruner [1992, p. 64], to consider the meanings of children's formalising by specifying "the structure and coherence of the larger contexts in which specific meanings are created" by the children.

What does a person's formalising look like and how is it distinguished from earlier, inner mathematical understanding? The scope and length of this paper do not allow us to consider all aspects of this question. [3] Formalising, however, is not restricted to bright students. In this article the first example below contrasts the inner and the formalising understanding of a twelve-year-old girl considered to be "average" in mathematics. The second example looks at the work of three eight-year-old boys responding to a fraction problem. One of the boys is provoked to formalising understanding while the other two are not.

Example 1

Tanya, a twelve-year-old girl, assessed as having average ability in mathematics, built up an understanding of addition by physically covering fractional pieces with other fractional pieces from her kit. For her, fractions were physical amounts, or chunks, which could be covered. For example,

$$\frac{2}{8}+\frac{1}{12}+\frac{2}{6}=\frac{2}{3}$$

because two eight pieces and one twelfth piece fit on one third while two sixths fit on the other. From this it appears that her image of addition could be characterised as "if I can find a piece on which the addends fit then I can find the sum." While such an image of fraction addition allowed Tanya to cope with many complex additive situations, it also had its shortcomings. For example when faced with adding one-half, one-third, and one-fourth, Tanya said that while she knew the amount would be more than one, she could not find the amount; there was no piece or pieces on which the three addends would "fit". The teacher in this class suggested to Tanya and other students that they now think about relationships among fractional amounts differently: given a set of different fractions, say thirds, fourths, and sixths, the students were to find fractions of "one kind" which would fit on all of the fractions in the test set (here twelfths or twenty-fourths would do). This activity appeared to help Tanya change her old image of addition of fractions. She moved beyond the need to physically cover pieces by using her property of equivalence. Fractions had now become *like* chunks but numerically represented. Thus Tanya, applying a class notation relating to equivalent fractions to express her own "new" thinking, worked 1/2 + 2/3 as follows:

(A) "Well twenty fourths could cover both so:

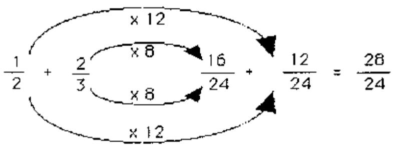


Figure 2

Notice that Tanya's reasoning refers to her new covering image of addition, but that she now can work numerically on each fraction addend separately. This understanding allowed Tanya to add, in specific cases, any number of fractions based on her kit (halves, thirds, fourths, sixths, eighths, twelfths and twenty-fourths), and allowed her to add fractions with imaginary kits as well (e.g. halves, fourths, fifths, tenths, twentieths). But such a powerful understanding was still attached, through simile, to Tanya's physical models and physically-based images of fractions.

(B) A short while later the following exchange took place between Tanya and her teacher:

T: What do you think about addition (fractions) now?

Tanya: It's easy! You just make fractions that work for them all. Say you had 2/3, 4/7 and 5/6. Well, sixths would work for 2/3 and 5/6, so you'd have to make forty-seconds.

T: Why is that?

Tanya: Well I just know that forty-seconds will fit because sixths times sevenths make forty-seconds

T: So?

Tanya:

$$\frac{2}{3}+\frac{4}{7}+\frac{5}{6}$$
$$=\frac{4}{6}+\frac{4}{7}+\frac{5}{6}$$
$$=\frac{28}{42}+\frac{24}{42}+\frac{35}{42}$$

And like that!

In contrasting "A" and "B" above, we have "caught" Tanya crossing the *don't need boundary* from property noticing to formalising. The evidence for this is two-fold. Firstly, Tanya used a method, multiplying sixths and sevenths, which works between the addends and does not reference concrete images of each fraction piece separately. She is understanding addition of fractions as a method based on the forms of the number. In this understanding Tanya uses numbers in their own right. They are not *like* anything. Secondly, Tanya now has a method for adding, which is not confined to fractions she know physically, but which works for any fractions. Although in both cases she is using the notion of equivalence, in "B" her understanding is not based on physical pieces but on a purely numerical process. She deliberately chooses a fraction, four sevenths, which she doesn't know, physically or image-wise. For her, a seventh is an abstract quantity even though numerically expressed. She is applying a method which she thinks works for all fractions.

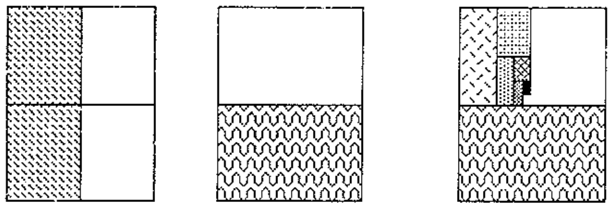
Example 2

A class of eight-year-olds has been working with a kit containing ones, halves, fourths, eighths and sixteenths. They were challenged with a problem:

> *A fraction is missing. It is bigger than one fourth and smaller than three fourths. Can you solve this missing fraction mystery?*

This problem was given to ascertain whether the children had an image of fractions as combinable amounts. But, as suggested by our expectations for constructivist teaching, it was the actions of each child that determined for them the nature of the intervention. Twenty of twenty-two-eight-year-olds in a regular grade three class gave a wide variety of correct solutions to this "mystery". Several gave more than one. All were able to describe, draw and explain their actions. Here are solutions, pertinent to the distinctions we wish to draw, offered to the whole class from the chalkboard by three boys:

Don gave three solutions:

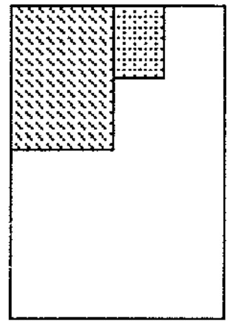


$\frac{2}{4}$ $\qquad$ $\frac{1}{2}$ $\qquad$ $\frac{1}{2}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}+\frac{1}{256}+\frac{1}{512}$

Figure 3

His first two solutions showed that he knew, from his image of fractions as amounts, that two quarters and one half fit between one fourth and three fourths His third solution is interesting. Don had noticed a property of half fractions which might be characterised by an observer as follows: "You can get a fraction which is half as big as the one you have by cutting its space in half." Here he was applying this property and justifying his lengthy series by appeal to its physical model He made it clear to his colleagues that he knew he could go on dividing up the remaining space into smaller and smaller pieces by halving. It is interesting to note here that Don's drawing did not reflect the actual shapes for the fractions in his kit. His diagram depends on halving the visible shape he has drawn. He does this by cutting the remaining space into halves alternately horizontally and vertically. This is not how the kit was constructed and is thus a strong indicator that he was working with his mental images for the fractions and not with the original concrete representation. Don was illustrating that he could generate and record new, broad, complex mathematical properties using his image of fractions, and his classmates appreciated and understood what he had done. About half of the children reproduce this thinking in their own later work. We interpret this as an indication that these children's images were sufficiently rich to allow Don's actions and explanations to provoke a broader understanding in them.

Sam took another approach Rather than acting additively, Sam, unlike anyone else in class, thought of the solution subtractively and wrote:



$$1-\frac{1}{4}-\frac{1}{16}$$

Figure 4

Sam wrote his sentence on the board and showed in his drawing why his solution "worked". Again the children followed his argument and several children imitated his thinking in their own later work.

In fact, his partner Bart imitated Sam immediately. But the idea of subtraction provoked Bart into applying a formal method for solving the problem:

$$\frac{10}{1}+\frac{10}{2}-\frac{5}{1}+\frac{1}{1}-\frac{10}{1}-\frac{0.5}{8}-\frac{1}{4}$$

Notice how Bart was disregarding his image of fractions related to the kit and was simply adding and subtracting combinations of fractions as formal entities such that their combination was less than three fourths It is crucial to notice at this point that formalising does not necessarily entail algebraic symbolising. It is dependent on thinking in generalities. The similarity between Bart's writing and that of Don and Sam does not reveal the underlying differences in understanding. While both Don and Sam reflected on their correct solutions and justified them, either by acting on the fraction kit pieces or through drawings reflecting images and noticed properties, Bart did not do this. He said that he would just add things (fractional numbers) and subtract things until he got a little below three fourths. Bart did not refer to any physical image of fractions, either in applying his method or in reflectively expressing his explanation of it Although his solution was offered to the whole class, no one imitated Bart's formalising methods in any further missing fraction mysteries. We interpret this as an indication that the other children's informal understandings were not sufficient to allow Bart's actions and explanations to provoke their understanding to grow to the outer level of formalising. Bart, however, now applied such a formal method exclusively to all subsequent problems. The inability of his peers to follow Bart's method is consistent with another tenet of our theory of the growth of mathematical understanding: in formalising a solution to this problem, Bart had crossed a "don't need" boundary. He did not need to think of fractions as amounts or quantities, but could treat them as formal entities. His classmates, however, had not crossed this boundary with respect to fractions. While some children could comprehend and use

Don's or Sam's rich and new solutions to the "mysteries" because they were still image-based, no children, including even Don and Sam, could comprehend what Bart was going It appears to an observer that they still needed to refer even their property-rich thinking to images based on previous actions Their mathematics had a metaphorical flavour. Bart's went beyond metaphor

Concluding remarks

What *is* formalising in mathematical understanding? It would be easy to think of it in terms of using mathematical symbols in a thoughtful way to do mathematical tasks. But both our theory and the above examples suggest otherwise. As we follow a child who grows to formalising we see that it is not the use of symbols which distinguishes formalising from the image making, image having or property noticing layers of understanding. It is the abstract meanings given to the symbols—for Tanya a seventh had the quality of an "n"th—and the generating of mathematics no longer based on concrete images and properties which show formalising Formalising is characterised by a sense that one's mathematical methods work "for all" relevant examples. Children who are formalising do not need the physical actions and images which brought them to the point of formalising. Mathematical action and expression are no longer necessarily metaphorical. Such actions can be statements that are made and justified in their own right without being put for "real events".

It is frequently thought that formalising understanding is only observable in the work of advanced secondary school students. Our evidence shows that formalising as a mathematical activity is available to younger children. How can such formalising be brought about? It cannot necessarily be provoked by the interventions of a teacher or, indeed, of another pupil. That is, it is not sufficient to be "told" or "given" a formal method by the teacher. This may well be true even when the formalising is assumed by the teacher to be meaningful for the children because they have previously worked with related physical materials. In order that it not be disjoint, formalising must grow from the child's personal mathematical structures and unfold from less formal image-based mathematical understandings. That is, formalising understanding is not an add-on to previous informal mathematical activity or understanding. The children, as we have argued in the case of Tanya and Bart above, must recognize the patterns in their informal activities which allow them to come up with and justify methods which, to an observer, are now independent of the physical actions of images upon which they were originally based. Such growth appears to be fostered by circumstances where children give multiple and varied responses to questions, where children are encouraged to use mathematical language in describing, justifying, and expressing their actions, regardless of their sophistication, and where there are rich opportunities to make and use patterns.

Notes

[1] Research for this paper was supported in part by Social and Humanities Research Council (of Canada) Grant 410 900 738

[2] The authors of this paper are troubled by the exclusive use of the language of metaphor, in the mathematics education literature, for any student or teacher thinking which relates to physical objects. We believe that there is a world of difference between simile and metaphor.

[3] We have in other places looked at this transition to formalizing one gifted eight-year-old [Pirie and Kieren, 1992a, 1992b].

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