# A FRAMEWORK FOR ANALYSING TEXTBOOKS BASED ON THE NOTION OF ABSTRACTION

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Mathematics textbooks play an important role in the teaching and learning of mathematics (Valverde *et al.,* 2002). They mediate curricular expectations (Ball & Cohen, 1996), particularly in those systems with mandated curricula and accompanying texts (Howson, 1995). They shape what students learn and have a positive influence on student learning (Healy & Hoyles, 1999; Riordan & Noyce, 2001). Independent of curricular intentions, they have been shown to reflect culturally located teachers' perspectives on mathematics and their expectations for mathematical learning (Haggarty & Pepin, 2002), frequently leading, particularly in unregulated systems, to substantial variation in what students are expected to learn (Tornroos, 2005). They can frame how teachers present mathematics and attain the achievement of goals beyond which teachers would typically aspire (Brown, 2009). They offer a source of didactical ideas, problems and exercises (Howson, 1995) and, more generally, have been implicated in teachers' professional development, particularly at a time of curricular reform (Remillard, 1999).

Research on secondary school mathematics textbooks has analysed problem types (Zhu & Fan, 2006), problem solving procedures (Fan & Zhu, 2007), procedural complexity (Vincent & Stacey, 2008), cognitive demand (Jones & Tarr, 2007) or concept treatment (Cai, Lo & Watanabe, 2002). Most textbook analyses focus on tasks, such as worked examples and exercises. Few studies have analysed *illustration* or *exposition* (Shard & Rothery, 1984) and, thus, research does not provide a picture of the intended learning trajectories in textbooks.

Take the topic of Pythagoras' theorem, for example: Figure 1 shows the first page in one edition of a Taiwanese textbook (Nani, 2011, p. 90). Before *illustrating* this theorem, the squares generated from each side of a specific right-angled triangle are represented with a visual diagram and underlying grid lines, and the area of the hypotenuse square and the sum of the areas of the other two squares are respectively asked. Next, the mathematical terms, hypotenuse and leg, are *exposed* with pictures of right-angled triangles. Then, the relationship between the areas is *illustrated* with the original diagram, a linguistic formula and a symbolic formula. Lastly, the question, "Do other right-angled triangles possess the same property?" is asked.

Previous studies lack any analysis of the essence of learning that is necessary to enter the intended learning trajectories in textbooks. In the example shown in Figure 1, the textbook asks learners to generalize a property of a specific right-angled triangle through connecting with previous knowledge, introducing the names of the three sides and focusing attention to the squares of the three sides. I am interested in how learners are guided to develop mathematical ideas and how such abstract ideas are organised to be accessible to learners. This situation leads me to focus on abstraction, an essential process in the construction of mathematical knowledge and a key adaptive mechanism of human cognition (von Glasersfeld, 1990). What is abstraction? How can we conceptualise it so as to encompass the rich meanings of abstraction found in diverse perspectives? In order to deepen our understanding of the quality of learning and teaching, it is important to conceptualise the construct of abstraction and develop a framework that can be used to investigate the abstraction intended in textbooks. In this article, I endeavour to develop such a framework, by integrating different perspectives on abstraction.

## What is abstraction?

I will examine the definition of abstraction from a constructive-empirical and a dialectic perspective. The majority of work on abstraction in the development of mathematical understanding from a constructive-empirical perspective draws on Piaget's (1985) description of three forms of abstraction: empirical abstraction, pseudo-empirical abstraction and reflective abstraction. When someone is acting on objects in the external world and deriving knowledge from the properties of physical objects, empirical abstraction is exploited. For example, children accumulate everyday concepts from their increasing classification of everyday objects, which allows them, for example, to discuss a cup in abstract terms without referring to a concrete example of a cup. A focus on actions and their relationships contributes to pseudo-empirical abstraction, Piaget's second form of abstraction. For example, children may count a set of cups in different orders and realise that the counting always results in the same number, which is a property of all countable objects. However, elementary mathematical concepts require learners to go beyond simple classification and internalise not only the physical actions on the everyday objects but also mental actions on the properties of relations within or between physical actions and non-physical or mental objects, such as numbers. These further examples of constructions are reflective abstraction, Piaget's third form of

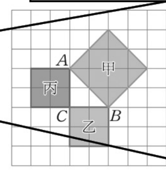
**Unit 1**

溫故啟思

1. 右圖方格紙中，每個小方格的邊長都是 1。分別以直角三角形 *ABC* 的三個邊向外做正方形。

(1) 正方形甲的面積為＿＿＿＿。

(2) 正方形乙、丙的面積和為＿＿＿＿。




Review the old and enlightened thoughts.

In the right grid, each side length of each grid square is 1. Off each of the sides of the right-angled triangle ABC, construct a square.

（1）The area of the square 甲 is

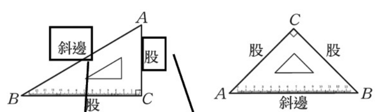
（2）The sum of areas of the square 乙 and square 丙 is

**Unit 2**

1 勾股定理的發現

Discovery of Pythagorean Theory

小學時，我們曾學過直角三角形，例如：我們常用的三角板，就是直角三角形。如下圖，在直角三角形 *ABC* 中，角 *C* 為直角（簡記為∠*C*＝90°），直角所對的邊稱為斜邊，另外兩邊稱為股。




hypotenuse

leg

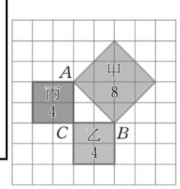
When we were in elementary school, we had learned right-angled triangle. For example, the triangle plate which we often use is a right-angled triangle. As shown below, in the right-angled triangle ABC, angle C is a right angle (denoted as ∠C=90°). The side opposite to the right angle is named *hypotenuse*, and the other two sides are named *legs*.

**Unit 3**

由溫故啟思中，我們看出股長為 2 的等腰直角三角形，其斜邊上的正方形面積會等於兩股上的兩個正方形面積和，即

正方形甲面積 ＝ 正方形乙面積 ＋ 正方形丙面積，

也就是 $\overline{AB}^2=\overline{BC}^2+\overline{AC}^2$。




In the unit of “Review the old and enlightened thoughts”, we observe that for the isosceles right-angled triangle with leg 2, the area of the square off the hypotenuse is equal to the sum of the areas of the two squares off the two legs, i.e.,

The area of the square 甲 = The area of the square 乙 + the area of the square 丙.

That is, $AB^2 = BC^2 + AC^2$.

**Unit 4**

其他直角三角形也會有這樣的性質嗎？

“Do other right-angled triangles possess the same property?”

*Figure 1.* *The presentation of Pythagoras’ Theorem in a Taiwanese mathematics textbook (Nani, 2011, p. 90). Data source © 2011 Nan I Publishing. All rights reserved.*

abstraction, in which the mental operations themselves become new objects of thought. Accordingly, action and thought are interpreted in terms of the thinker's purpose (Cobb, Nicholls, Yackel, Wood & Wheatley, 1990).

Like Piaget, Skemp (1986) also stated that abstraction is linked to the construction of common properties, classes or relationships. He regarded abstraction as a mental activity through which humans become aware of similarities in their experiences, which are then classified to create classes of experiences, which can then be used for comparing and assimilating new experiences. The term experience used by Skemp is deeper than the empirical abstraction described by Piaget (Mitchelmore & White, 1995). Moreover, Skemp describes these classes of experiences as concepts, which fall into two forms, primary and secondary; the former are derived directly from experience, while the latter are abstracted from primary concepts. From Piaget's and Skemp's view, abstracting or identifying the essential properties or underlying structures of physical or mental experience, including actions and objects, is required for the process of concept formation. In this constructive-empirical view [1], abstraction is considered as "higher-order knowledge which consists of 'classifications' and 'generalisations' arising from the recognition of commonalities isolated in a large number of specific instances" and "an ascending developmental process from the concrete to the abstract" (Ozmantar, 2005, p. 79). For example, we see different polygons on a sheet and try to classify them according to some attributes. After we recognise that they all have straight edges, are non-self-crossing and enclose a region, we perceive these attributes as properties which we then abstract to form a definition of a (non-self-crossing) polygon. This classification is a process of reflective abstraction. We might not notice that they are all planar, and not include that attribute initially. The initial understanding of these shapes is more concrete than the final understanding because the abstraction of shapes increases as the quantity and the quality of recognised attributes increases. As Mason (1989) noted, abstraction lies in "a delicate shift of attention", for example, "from seeing an expression as an expression of generality, to seeing the expression as an object or property" (p. 2) for constructing arguments.

The constructive-empirical view of abstraction accounts for the abstracting process from the initial understanding of experience or objects, which is more concrete than the final understanding. A different view of abstraction suggests that the relationship between the abstract and the concrete is dialectic. This view is influenced by Hegel's proposition of the dialectic relation between the acting subject (individual or collective) and the object of the activity (Leont'ev, 1978). For example, a child (the acting subject) interacts with several toy machines (the activity) to gradually conceptualise the concept of robots (the object). The dialectic relationship between subject and object implies that both are constituted with each other and exist only in relation to one another. Taking this relationship into account, "abstraction can never produce meaningful insights in the concrete world, unless there is some inner relationship between the concrete and the abstract" (van Oers, 2001, p. 287). In Davydov's (1990) view, abstraction ascends from a fragmentary, undeveloped structure (abstract) to a consistent, highly structured universal (concrete). For example, Roth reported his experience of learning vectors which were first known as representing lines in space and then as collections of objects in many types of situations (Roth & Hwang, 2006). His sense of the concreteness of vectors increased as the learning or application of vectors continued. Roth and Hwang (2006) further reinterpreted the dialectic view of abstraction as the double development between concrete and abstract. The two movements, like the dialectic relationship between subject and object (Leont'ev, 1978), are constituted with each other.

One difference (though not contradiction) between the constructive-empirical and the dialectic views of abstraction is their different focuses in describing abstractness. The constructive-empirical view attends to the relationship between the specific experience from which something is to be abstracted and the generality abstracted, and assumes that the latter is more abstract than the former. Moreover, this relationship can be discriminated by the relative degree of abstractness of objects, and can be recognised or constructed by the individual, perhaps with the help of social discourse. The dialectic view focuses on the transactional relationship between subject and object, assuming a two-way relationship between the concrete and the abstract. The dialectic view uses concrete/abstract to analyse both the material aspects of subjects' actions on objects and ideal aspects of objects which are recognised or constructed by subjects in activity. In the constructive-empirical view of abstraction, the relative abstractness of two mathematical objects can be distinguished by their hierarchical relations in the broad cognitive map (Skemp, 1986) or cognitive structure (Piaget, 1985). Hence, abstraction ascends from concrete specificity to abstract generality. In the dialectic view of abstraction, the subject's sense of what the object is all about may change from abstract to concrete, and the perceived nature of the object may be developed from concrete to abstract. Thus, abstraction can be portrayed as a dialectic development to and fro between the concrete and the abstract (Ozmantar, 2005).

Whether or not abstraction ascends from the concrete to the abstract or vice versa, mathematical objects can inevitably be constructed or interpreted by subjects with different forms of representations (Duval, 2006) or semiotic systems (Ernest, 2006). In the constructive-empirical view, Mitchelmore and White (2000), drawing on Skemp's notion of concept formation by abstraction, designed manipulative materials and visual models to develop students' understanding of mathematical angles from specific situations, through general contexts to abstract generality. The underlying idea is that models can be a case for reasoning about mathematical objects in addition to a way of representing mathematical objects or their meaning (Cobb, 2002). In the dialectic view, artefacts, mediating the relation between subjects and objects, are involved in the process of abstraction (Hershkowitz, Schwarz & Dreyfus, 2001). Vygotsky (1978) proposed that artefacts can either be material or immaterial, and operate on subjects, objects and the relation between them. Thus, artefacts can be emphasised as one key component of "the formation of abstractions and the use of formed abstractions" (Ozmantar & Monaghan, 2007, p. 91). In both views, models or artefacts can be understood as semiotic tools to develop different levels of understanding or various interpretations of mathematical objects.

## Components of abstraction arranged by textbooks

Based on the constructive-empirical and dialectic views of abstraction, I define mathematical abstraction as a mental or social activity through which subjects intentionally identify, reconstruct or apply new mathematical objects represented or mediated by using semiotic tools (models, artefacts or multiple representations). I will argue that this definition integrates the two perspectives on abstraction discussed above. First, the attributes of the subjects and objects behind textbooks can be analysed individually [2] in the constructive-empirical view or dually in the dialectic view. Second, semiotic tools have two functions. One is to reveal subjects' understanding of objects or the features underlying objects, and the other is to mediate the interaction between subjects and objects under conventionalized constraints. This definition of abstraction could include disjunctive, reconstructive and expansive generalization (Harel & Tall, 1991), which corresponds to the identification, reconstruction and application of new mathematical objects. The definition is consistent with the idea, that "an abstract concept then is not so much a reproduction of reality, but actually establishes a point of view that guides our thinking" (van Oers, 2001, p. 284).

Furthermore, this definition helps us focus on the three key components: subjects, objects and semiotic tools. The three elements also correspond to Mason's comment on mathematical abstraction which "refers to a common, root experience (subjects): [...] a delicate shift of attention from seeing an expression (semiotic tools) as an expression of generality (objects), to seeing the expression (semiotic tools) as an object or property (objects)" (Mason, 1989, p. 2, parentheses added). The latter object is abstracted by subjects based on the understanding of the former object and through semiotic tools. Although the three components have been applied to analyse interactions between subjects and objects with semiotic tools, they have not been applied to analyse textbooks. In textbooks, subjects are assumed to have prior experience relevant to the current object, and are idealised to interact with the current object mediated by semiotic tools. How could textbooks support the idealised subjects to engage in the process of abstraction? I assume that learning, along with textbooks, can cause physical or mental actions of subjects or initiate interactions between subjects and objects through semiotic tools, and then lead to the emergence of new objects or new relationships among subjects, objects and semiotic tools.

## Essential attributes underlying the three key components

To analyse textbooks according to the three components of abstraction, we further extract operational attributes underlying the three components of abstraction, based on mathematics education literature about learning and teaching of abstraction.

According to Skemp (1986), each concept (object) can be composed of some subordinate concepts (processes or models), *i.e.* sub-objects of the object. The abstraction may progress from observing specific situations or examples, through recognising properties from various situations or generic examples, to identifying the essence of these sub-objects (Mason, 1989; Mitchelmore & White, 2000). The level of abstraction of these sub-objects can be distinguished by their relative generalities. Mathematical sub-objects could be instances, properties, applications or views of mathematical concepts, procedures and models. Take the object of fractions as an example. Part-whole and measure are two sub-objects of fractions. The part-whole view is less abstract than the measure view because the part-whole view is less convenient to demonstrate the making of improper fractions, and the measure view is found to be powerful for learning the abstract meaning of rational numbers (Kong & Kwok, 2003). Thus, the measure view is more general than the part-whole view for constructing the concept of rational numbers.

With respect to the subject component, White and Mitchelmore (2010) argue that "without a strong link between fundamental mathematical concepts and students' experience, any abstraction approach is likely to falter" (p. 209). This resonates with Hazzan's (1999) comment that, when learning new concepts, students make unfamiliar ideas familiar based on previous experience. In textbooks, subjects' experience may be considered from either a transmission or a construction perspective. The two perspectives view subjects' experience as necessary knowledge for absorbing a new mathematical object rather than integrating with it, and for further developing a new mathematical object. In other words, subjects' experience can be considered as inactive or active. I identify the connectivity feature of subjects' experience as one attribute underlying the subject component.

The third component, semiotic tools, includes natural language, technology and various representations, such as symbols, figures, pictures, *etc*. Semiotic tools are regarded as not only "embodiments of ideas or concepts" (Janvier, Girardon & Morand, 1993, p. 81) but also instruments which enable subjects to think about ideas or concepts (Radford, 1998). Different types of semiotic tools are treated as a window into subjects' abstraction of mathematics. However, (the designers of) textbooks are assumed to consider semiotic tools as one component of abstraction that makes abstraction possible through reading and writing, as well as through interacting with others. Coordination among multiple representations of mathematics is required for deep understanding (Duval, 2006), although the use of multiple external representations holds potential disadvantages in learning mathematics (Nistal, Van Dooren, Clarebout, Elen & Veschaffel, 2009). Thus, the analysis of textbooks needs to examine the multiplicity of representations in particular and semiotic tools in general.

Besides extracting the attributes relative to each component of abstraction, we need to attend to the relationships between any two components of abstraction taking the above mentioned dialectic view of abstraction into consideration. First, one important attribute underlying the relationship between subjects and objects is subjects' needs for new objects. According to Vygotsky (1978), objects encapsulate subjects' motives for action, while subjects transform the purpose of objects. Drawing on Leont'ev's work on activity theory, Hershkowitz, Schwarz and Dreyfus (2001) proposed that the need for a new structure is the first stage of the genesis of an abstraction and emphasised the importance of the conceptual, affective and social factors for formulating the need. Students' needs for constructing new objects are also related to their motives. Skemp (1986) described motivation as "a description we apply to behavior which is directed towards the satisfaction of some need" (p. 123), and supposed that "questions about motives are usually, in disguise, questions about needs" (p. 123). Accordingly, the subjects' needs for abstraction are identified as one attribute, and are classified into conceptual, affective and social needs.

Next, one important attribute underlying the relationship between objects and semiotic tools is the transformability of semiotic tools to connect undeveloped, fragmentary objects (abstract) and developed, whole objects (concrete). Transformations of semiotic tools are viewed as one mechanism of the evolution of objects (Hershkowitz *et al.,* 2001) as well as one source of cognitive difficulty in the comprehension of mathematical process (Duval, 2006). Transformations within or between types of semiotic tools relative to the same object are required when solving problems. For example, when subjects are asked to interpret one statistical diagram in textbooks, I assume that the transformability from diagrammatical tools into linguistic tools may be observed.

Lastly, one important attribute underlying the relationship between subjects and semiotic tools is the (assumed) subjects' purposes for using semiotic tools in context. Hershkowitz *et al.* (2001) explain that "the context of an activity is not only an external, objective description of the material conditions of the activity but also includes subjective components" (p. 199). Furthermore, context is interpreted as the purpose of the semiotic tools that are conveyed to or used by subjects, because subjects "choose to

| Emerging Source (Attribute) | Description | Example (see Figure 1) |
|---|---|---|
| Objects (O) (Generality) | the connotation and the extent of objects | Units 1 and 3 provide a specific right-angled triangle. Units 2 and 4 refer to generic right-angled triangles. |
| Subjects (S) (Connectivity) | inactive or active connections of subjects' experience | In unit 1, subjects' experience is inactively connected due to no mention of the relationship between the squares. In unit 2, subjects' experience is actively connected for generating the identification of new mathematical terms. Based on units 3 and 4, subjects' experience is actively connected for generating a new object. |
| Semiotic Tools (T) (Multiplicity) | types of semiotic tools irrelevant or relevant to objects | In unit 1, natural language, geometrical language and geometrical figures are three types of the semiotic tools relevant to Pythagorean theory. |
| O-S (Needs) | conceptual, affective or social needs for abstraction. | In unit 1, no need for abstraction is explicitly initiated because students are just asked to calculate areas of squares. In unit 4, a conceptual need of reconstructing a new concept is generated. |
| O-T (Transformability) | the changes in semiotic tools relative to the same object | In unit 1, subjects are asked to calculate areas (geometrical measure) from information represented by natural and geometrical language, and geometrical figures. |
| S-T (Purposes) | the assumed purposes for subjects to use semiotic tools in context | All of the four units are set in a geometrical context. The purpose of unit 1 is to reproduce the identification of areas, and the purpose of unit 3 is to articulate the relationship of the sides. |

*Table 1. Six attributes underlying the three components of abstraction.*

carry out actions that seem relevant to them in the given context" (p. 199). These purposes may include any thought-demanding activities, such as, for instance classifying, articulating viewpoints, experimenting, making predictions, finding evidence or errors, making conjectures, justifying those conjectures, explaining results, and discussing (Perkins & Unger, 1994).

Each attribute is summarised in Table 1, and illustrated with reference to the example shown in Figure 1. The generality of objects is related to their connotation and extent of applicability. The connectivity feature of subjects is concerned with inactive or active connections made to subjects' experience. The multiplicity of semiotic tools refers to types of semiotic tools that may be irrelevant or relevant to objects. The subjects' needs for learning objects, the transformability of semiotic tools for objects, and the assumed purposes for subjects to use semiotic tools in context are identified as the other three attributes. It is my view that these six attributes, due to their emergence from the abstraction-related literature, represent a warranted framework for analysing textbooks. In addition, semiotic tools are discriminated as being provided by the textbooks or the assumed subjects.

## Feasibility of the framework

To show how this framework can be operationalised, I suggest procedures and principles for its application to the analysis of textbooks. First of all, the objects of interest are identified in general. Next, the sub-objects are identified relative to each object according to the views of the object and the types of text. For instance, part-whole and measurement views of fractions, procedural, conceptual or applicable views of the statistical concepts of median, mode and mean. Types of text may be classified as expositions, introductions, examples or exercises, peripheral writing and signals (Shard & Rothery, 1984). Each sub-object can then be analysed with respect to the six attributes. It is better to construct the criteria or categories of each attribute in advance. For example, generality could be classified as particular, generic and formal within or outside mathematical situations. The conceptual factor could be classified into several categories, such as conceptual conflict and conceptual insufficiency. The affective factor could be classified into categories, such as fun or interest, appreciation, and encouragement. And the social factor could be classified as, for example, interaction with peers, with teachers and with others outside the classrooms.

## Potential of the framework

I have drawn on the constructive-empirical and dialectic views of abstraction and extracted six attributes for analysing how textbooks arrange the three components of abstraction which comprise and intertwine objects, subjects and semiotic tools. The framework mainly stands on the shoulders of these theories; however, the scope of this framework not only integrates the two views of abstraction but also moves a step forward in applying them to analysing didactic materials. When textbook analysis is deepened into mathematical abstraction, the framework serves as a way to analyse the intended abstraction behind textbooks without student data.

The framework treats textbooks as playing an active role: they influence students' learning and the interactions among students, teachers and others. In addition, this framework makes it possible to manipulate some attributes in order to develop principles for the design of effective textbooks. For example, the sequence of sub-objects can be arranged differently to design two versions of textbooks. Students' learning can then be investigated for the two sets of materials. How can different learning results through using the two materials be interpreted? Based on this framework, we can analyse how each attribute varies with the change in the sequence of sub-objects. This analysis may lead to thinking of multiple ways to explain the learning results of students and then to suggest principles for the design of textbooks in order to increase students' opportunities to learn.

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## Notes

[1] The term constructive-empirical view of abstraction denotes that abstraction is developed from physical or mental experience, and different from the transmission of ready-made knowledge from outside to the individual mind.

[2] Analysing the characters of subjects and objects denotes that we consider the prior knowledge of subjects and the abstractness of objects individually, and does not imply there is no relationship between subjects and objects.

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Of course, comparison is both possible and necessary, and it is what I and others of my persuasion spend most of our time doing: seeing particular things against the background of other particular things, deepening thus the particularity of both. Because one has located, one hopes, some actual differences, one has something genuine to compare. Whatever similarities one might find, even if they take the form of contrasts … or incomparabilities … are also genuine, rather than abstract categories superimposed on passive "data," delivered to the mind by "God", "reality," or "nature." […] Theory, which is also both possible and necessary, grows out of particular circumstances and, however abstract, is validated by its power to order them in their full particularity, not by stripping that particularity away. God may not be in the details, but "the world"—"everything that is the case"—surely is.

From Geertz, C. (2000) *Available Light: Anthropological Reflections on Philosophical Topics,* p. 138. Princeton, NJ: Princeton University Press.

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