Communications

A reflection on Star and Seifert's operationalisation of flexibility in equation solving

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My interest in flexibility in equation solving comes from a research agenda whose aim is the study of flexibility in the teaching and learning of algebra. Several researchers have proposed operationalisations. One of the most relevant of these is the one proposed by Star and Seifert (2006). In this operational definition, flexibility is knowing multiple solution procedures to a problem and having the capacity to generate new and more efficient procedures to solve it. Even though this definition has had an impact on the research on flexibility (*e.g.,* Xu *et al.,* 2017), there are calls for a more comprehensive account (*e.g.,* Ionescu, 2012) given flexibility's contribution to efficient problem solving.

Here I offer a reflection about flexibility in equation solving that extends the definition by Star and Seifert. To this end, I offer examples of equation solving that suggest that there is a need for another property in the definition, to deepen both the investigation of students' flexibility in equation solving and its fostering in teaching. I add 'connections' because when performing transformational activities, students make a number of procedural connections.

Flexibility in equation solving

As Star and Seifert note, the notion of flexibility is colloquially defined as the ability to change according to particular circumstances. In mathematics, however, it has a specific meaning. Think about the equations

$$4(x + 1) = 8;$$

$$4(x + 1) + 2\ (x + 1) = 12; \text{ and}$$

$$4(x + 1) + 3x + 7 = 8 + 3x + 7$$

and review the solutions presented in Figure 1. What do you observe?

Both students solve the equations correctly. According to Star and Seifert, however, Student A creates more innovative solutions to the problems and completes the three equations using three different solution procedures. Student A divides by 4 as a first step in the first equation; combines the like terms $x + 1$ first in the second equation; and recognises and cancels the like terms $3x + 7$ in the final equation. However, Student B uses the same (standard) algorithm in all cases: expand, combine, subtract from both, and divide. Student A's solutions are more efficient; that is, they require fewer steps. Like Student B, Student A has the knowledge of standard algorithms, but Student A has the additional capacity to use them in non-standard ways in performing particular types of tasks.

For Star and Seifert, flexibility in mathematics, then, can be defined as having knowledge of multiple solution procedures to a problem, a sense of when each way is most efficient, and the capacity to invent or innovate creative new procedures. Qualities such as being able to create multiple and efficient solutions to a given problem make students more flexible thinkers and problem solvers. Star and Seifert accordingly "define a flexible solver as one who (a) has knowledge of multiple solution procedures, and (b) has the capacity to invent or innovate to create new procedures" (2006, p. 282) and consider these two as indicators of flexibility in equation solving. I suggest extending this operationalisation and propose adding a third property to the above definition, namely: (c) the ability to make connections between mathematical ideas and concepts.

In equation solving one of the main ideas is *equivalence*. Examples of mathematical understanding in terms of algebraic expressions and equations include:

> Algebraic expressions can be named in an infinite number of different but equivalent ways. For example: $2(x - 12) = 2x - 24 = 2x - (28 - 4)$.
>
> A given equation can be represented in an infinite number of different ways that have the same solution. For instance, $3x - 5 = 16$ and $3x = 21$ are equivalent equations; they have the same solution, 7 (Charles, 2005, p. 14).

In order to become flexible, students must make contact with equivalent representations of the given expression or equation. Consider this equation used by Star and Seifert (2006, p. 286) to assess flexibility, and typically solved in mathematics classes using algebra: $0.3a + 0.2 = 1.1$. Figure 2 presents a number of different solutions. Based on Star and Seifert's operationalisation, Student B (*less* flexible thinker) would solve the equation following standard algorithms (*e.g.,* Solution 1), but how would Student A, who is a *more* flexible thinker, solve it? Would Student A see the equation as $3a + 2 = 11$ instead and solve it accordingly, as in Solution 4? Or would Student A represent the decimals by fractions and produce something like Solution 6?

In fact, how *more* flexible thinkers would solve the problem is ambiguous. This implies the need to include another

Equation	**Student A**	**Student B**
$4(x + 1) = 8$	$\begin{aligned} 4(x+1) &= 8 \\ x+1 &= 2 \\ x &= 1 \end{aligned}$	$\begin{aligned} 4(x+1) &= 8 \\ 4x+4 &= 8 \\ 4x &= 4 \\ x &= 1 \end{aligned}$
$4(x + 1) + 2(x + 1) = 12$	$\begin{aligned} 4(x+1)+2(x+1) &= 12 \\ 6(x+1) &= 12 \\ x+1 &= 2 \\ x &= 1 \end{aligned}$	$\begin{aligned} 4(x+1)+2(x+1) &= 12 \\ 4x+4+2x+2 &= 12 \\ 6x+6 &= 12 \\ 6x &= 6 \\ x &= 1 \end{aligned}$
$4(x + 1) + 3x + 7 = 8 + 3x + 7$	$\begin{aligned} 4(x+1)+3x+7 &= 8+3x+7 \\ 4(x+1) &= 8 \\ x+1 &= 2 \\ x &= 1 \end{aligned}$	$\begin{aligned} 4(x+1)+3x+7 &= 8+3x+7 \\ 4x+4+3x+7 &= 8+3x+7 \\ 7x+11 &= 3x+15 \\ 4x &= 4 \\ x &= 1 \end{aligned}$

Figure 1. Hypothetical solutions by two hypothetical students (adapted from Star & Seifert, 2006).

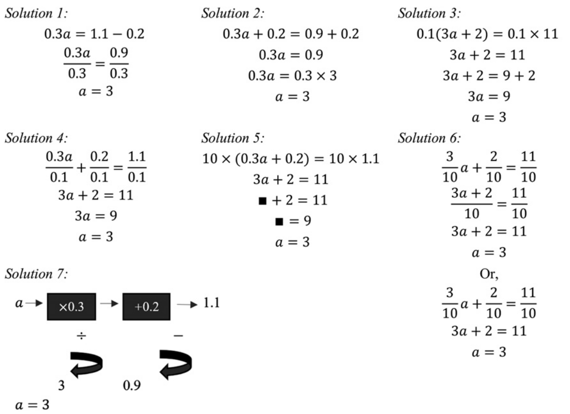

Figure 2. Multiple solutions to the equation 0.3a + 0.2 = 1.1.

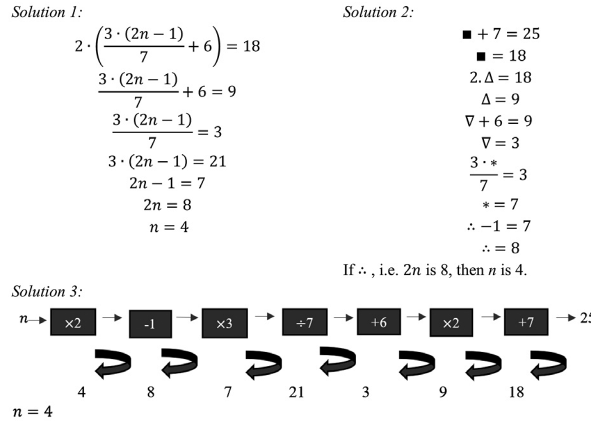

Figure 3. Multiple solutions to the second equation.

property of flexibility, establishing connections. What makes this equation interesting is its richness in connections to mathematical ideas. The terms in the equation, and accordingly the equation itself, can be represented by various equivalents. This is evident in Solutions 1 through 6.

One of the other ways to solve the equation is to use the idea of *backtracking*. This begins by working forwards, *i.e.,* working out what has been done to the variable *a* step by step, then working backwards to undo each procedure. Solution 7 shows the backtracking method, which uses a procedural type of connection (Businskas, 2008).

Figure 3 shows some solutions to another, more complicated, equation:

$$2\cdot\left(\frac{3(2n-1)}{7}+6\right)+7=25$$

According to Star and Seifert's definition, a *less* flexible thinker is supposed to give Solution 1, where standard procedures are followed step by step. How *more* flexible thinkers would solve the problem is again ambiguous.

In solving the equation, procedural types of connections are established, and examples are provided in Figure 3. Solution 2 shows the *cover-up* method. The idea in the background of this method is still working backwards. Here, the whole expression that has *n* in it is covered first to find the value covered up (■ + 7 = 25), and the same is repeated until the value of *n* is found. Solution 3 presents how the equation is solved by the backtracking method.

The examples provided here show that, even though abilities such as being able to create multiple and efficient solutions to a given problem are important flexible behaviours in equation solving, there should be other qualities. These examples imply that linking concepts with broader ideas is one mechanism involved in flexibility. Making connections must, therefore, be considered as a property of flexibility. As such, in addition to having knowledge of standard algorithms to perform relevant tasks and using that knowledge in non-standard ways to do a better job, *more* flexible thinkers have the capacity to connect procedures with the broader big ideas or concepts (*e.g.,* equivalence).

Try the examples given in Figure 4 on your own (or in the classroom) and check how many solutions you create, how efficient the solutions are, and what connections you make. It is my assumption that your (or your students') solutions contain rich information relating to the understanding of flexibility with regard to mathematical connections. This outcome suggests to us that there is value in considering making mathematical connections as one of the indications of flexibility in doing mathematics.

Final remarks

This reflection is by no means exhaustive. Its main purpose is to contribute to the development of the current conceptualisation of flexibility. Specifically, it makes evident that the two properties in Star and Seifert's operational definition need to be extended. I propose incorporating an important quality of flexibility, making connections, which is fundamental for understanding mathematical concepts. I believe that this extension will allow a better analysis of flexibility in research and will contribute to teaching and learning, especially given that making connections is frequently used by mathematics teachers during their teaching.

This reflection opens potential routes to study the issue further. First, studying whether and how making connections is a property of flexibility may provide a more profound way to define and analyse flexibility in transformational activities and may also help to foster the use of mathematical connections in teaching. Second, it is important to fine-tune this new property into more specific types of mathematical connections in transformational activities. Importantly, it is seen that, as Ionescu (2012) indicates, a number of variables play a role in flexibility. As well as the ability to make mathematical connections, knowledge in the relevant content domain (De Gamboa *et al.,* 2020; Zazkis & Mamolo, 2011) may influence flexibility, or this knowledge may impact both on the ability to make connections and on flexibility.

1. $4(x+3)=16x$
2. $15x+10=5x+20$
3. $5(x+3)+10x=35+5x$
4. $2(x+3)+4x+8=4(x+2)+6x+2x$

Figure 4. Additional examples (from Star and Seifert, 2006, p. 286).

The complex relationships among these three variables needs further research. Finally, I believe that these types of future reflections or investigations would expand the understanding of the concept and properties of flexibility in performing mathematical tasks, as this communication only focuses on a small number of examples.

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Quadratic equations without a quadratic formula

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One can safely say that the quadratic equations are not overly popular. The US talk-show personality Stephen Colbert describes them as "an infernal salad of numbers, letters, and symbols" (2007, p. 120). Given this, any simplification in methods of solving quadratic equations is a step in the right direction. One such simplification was offered by Savage (1989), but it remained largely unnoticed until Po-Shen Loh (2019) developed it, apparently independently, some 30 years later. As we will show later, it may be much older.

Students are usually taught to find the roots x_1 and x_2 of a general quadratic equation

$$ax^2 + bx + c = 0 \tag{1}$$

using the standard formula

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{2}$$

Before learning the formula, students often learn to factor equations (usually with a = 1) by looking for two numbers whose sum is $-b$ and whose product is c. In general, the sum of the roots is

$$x_1 + x_2 = \frac{-b}{a} \tag{3}$$

and the product is

$$x_1 x_2 = \frac{c}{a} \tag{4}$$

These formulae are named after François Viète (1540-1603), who may have been the first to discover them. They are central to the method described by Savage and Loh.

Illustration of the Savage/Loh method

Let us use the method to solve a quadratic equation given by Muḥammad Ibn-Mūsā al-Khwārizmī:

$$x^2 + 10x - 39 = 0 \tag{5}$$

We can easily calculate the sum of the roots:

$$x_1 + x_2 = -10 \tag{6}$$

and the product:

$$x_1 x_2 = -39 \tag{7}$$

The key point in the Savage/Loh approach is the observation that since sum of the roots equals −10, then the average of the roots is −10/2 = −5. The roots differ from this average by some value z.

$$x_1 = -5 + z,\ x_2 = -5 - z \tag{8}$$

Using Viète's product formulae we obtain

$$(-5 + z)(-5 - z) = -39 \tag{9}$$

This leads us to

$$(-5)^2 - z^2 = -39 \tag{10}$$

In other words, z^2 equals 25 + 39 = 64. Thus, z equals either 8 or −8. Substituting either of these values into the equations in (8) gives the two roots 3 and −13.

A non-standard example

Consider this unusual quadratic equation (Gashkov, 2015):

$$x^2 + x = 1111111122222222 \tag{11}$$

Gashkov's solution depends on a hint, that one of the roots equals x_1 = 33 333 333. Using the quadratic formula requires a calculator that can handle 16 digit inputs, within the capacity of smartphones, but still impossibly large for many calculators allowed for use in schools. Using the Savage/Loh method it can be solved without Gashkov's hint, and (with some effort) without a calculator.

As a = 1 and b = 1, the sum of the roots is −1 and the average is −0.5. This gives the two equations

$$x_1 = -0.5 + z,\ x_2 = -0.5 - z \tag{12}$$

The product c/a is simply the number on the right side of Equation 11, which picks up a negative sign when the equation is put into standard form.

$$x_1 x_2 = -1\ 111\ 111\ 122\ 222\ 222 \quad (13)$$

Combining Equations 12 and 13, we have

$$(-0.5+z)(-0.5-z) = -1\ 111\ 111\ 122\ 222\ 222 \quad (14)$$

or

$$(-0.5)^2 - z^2 = -1\ 111\ 111\ 122\ 222\ 222 \quad (15)$$

Hence $z^2 = 1\ 111\ 111\ 122\ 222\ 222.25$. Taking the square root (which could even be done by hand if necessary) yields $z = 33\ 333\ 333.5$. Substituting this into Equation 12 gives correct values of the roots.

Benjamin's example

Benjamin (2015) states, "for the equation $x^2 + 9x + 13 = 0$ our best option is to use the quadratic formula" (p. 39). Using it, Benjamin arrives at the roots

$$x_{1,2} = \frac{-9 \pm \sqrt{81-52}}{2a} = \frac{-9 \pm \sqrt{29}}{2} \quad (16)$$

Benjamin goes on to claim, "There are a few formulae that you need to memorize in mathematics, and the quadratic formula is certainly one of them" (p. 39). Next we will show that the Savage/Loh method makes it unnecessary to remember the quadratic formula.

Because $x_1 + x_2 = -9$, the average of the two roots is $(x_1 + x_2)/2 = -4.5$. This gives us two equations for the roots in terms of the average and some value z:

$$x_1 = -4.5 + z,\ x_2 = -4.5 - z \quad (17)$$

The product of the roots is

$$(-4.5-z)(-4.5+z) = 13 \quad (18)$$

so,

$$20.25 - z^2 = 13 \quad (19)$$

and

$$z = \pm\sqrt{7.25} \quad (20)$$

Hence, $x_1 = -4.5 + \sqrt{7.25}$ and $x_2 = -4.5 - \sqrt{7.25}$. The result coincides with Benjamin's using the quadratic formula, as it should be. While it is useful to remember the quadratic formula, its memorization is by no means necessary.

Einarsson's Example

Einarsson (2005) discusses the quadratic equation

$$1.6x^2 + 100.1x + 1.251 = 0 \quad (21)$$

He notes, "Four significant figure arithmetic when using the standard formula gives the roots $x_1 = 62.53$, and $x_2 = 0.03125$. The correct solution is $x_1 = 62.55$ and $x_2 = 0.0125$. We can see that in using the standard formula to compute the smaller root we have suffered from cancellation, since $4\sqrt{(b^2 - 4ac)}$ is close to $(-b)$" (p. 57). Obviously, direct use of the quadratic formula can result in an inaccurate solution.

Let us now see what we obtain using the Savage/Loh approach. The sum of the roots is

$$x_1 + x_2 = \frac{-100.1}{1.6} = 62.5625 \quad (22)$$

So the average is 31.28125. The two roots are

$$x_1 = 31.28125 + z,\ x_2 = 31.28125 - z \quad (23)$$

The product is

$$x_1 x_2 = \frac{c}{a} = \frac{1.251}{1.6} = 0.781875 \quad (24)$$

Combining Equations 23 and 24 gives us

$$x_1 x_2 = (31.28125 + z)(31.28125 - z) = 0.781875 \quad (25)$$

So

$$(31.28125)^2 - z^2 = 0.781875 \quad (26)$$

From this we can calculate z^2

$$z^2 = (31.28125)^2 - 0.781875 = 977.7347265625 \quad (27)$$

and hence $z = 31.26875$. This gives us the exact solution of the roots, $x_1 = 62.55$ and $x_2 = 0.0125$. We have seen that this way of solving the quadratic equation avoids the numerical errors that come with using the quadratic formula.

Priority

Consider now the question of priority. It appears that method described by Savage and Loh was known earlier. Flusser (1981) describes a problem from the book *Arithmetica* by the third century mathematician Diophantus of Alexandria, "Find two numbers such that their sum is 20 and the sum of their squares is 208" (p. 389). Diophantus starts by thinking of the two numbers being (in modern notation) $10 + z$ and $10 - z$. Note that 10 is the average of the numbers. This is precisely what we have done above to solve quadratic equations. Because the sum of their squares is 208 one can write (again we have modernized the notation)

$$(10+z)^2 + (10-z)^2 = 208 \quad (28)$$

from which Diophantus finds $z = 2$, and so the sought numbers are 8 and 12. While this is not exactly the Savage/Loh method, it is closely related to it and relies on the same key insight.

A second example belongs to the Florentine mathematician Antonio de'Mazzinghi (1353–1383). De'Mazzinghi posed a problem similar to the one solved by Diophantus,

> Find two numbers such that multiplying the one into the other makes 6 and their squares are 13, that is adding together the squares of each of them we obtain 13. We ask [what] are the numbers. (Katz & Parshal, 2014, pp. 198–199)

De'Mazzinghi suggested to let "the first number be one *cosa* minus the root of some *quantità* [and] the other be one *cosa* plus root of some *quantità*" (Katz & Parshal, p. 265). Franci (1988) notes "In solving many problems Antonio uses two unknowns, one called *cosa* and the other *quantità*. As far as I know, Antonio is the first algebraist to use two unknowns" (p. 246).

In modern notation, de'Mazzinghi began with

$$st = 6 \tag{29}$$

$$s^2 + t^2 = 13 \tag{30}$$

He then substituted for s and t

$$s = x - \sqrt{z},\ t = x + \sqrt{z} \tag{31}$$

which is not very different from the substitution made by Diophantus of Alexandria or by Savage and Loh. Substituting these into Equation 30 he obtained

$$13 = s^2 + t^2 = \left(x - \sqrt{z}\right)^2 + \left(x + \sqrt{z}\right)^2 = 2x^2 + 2z \tag{32}$$

Isolating z from this equation gave him

$$z = 6.5 - x^2 \tag{33}$$

Substituting this back into the equations in 31, de'Mazzinghi obtained:

$$s = x - \sqrt{6.5 - x^2} \tag{34}$$

and

$$t = x + \sqrt{6.5 - x^2} \tag{35}$$

Putting these values back into Equation 29 gave de'Mazzinghi

$$6 = \left(x - \sqrt{6.5 - x^2}\right)\left(x + \sqrt{6.5 - x^2}\right) = x^2 - 6.5 + x^2 = 2x^2 - 6.5 \tag{36}$$

with

$$x^2 = 6.25, \text{ or } x = \pm 2.5 \tag{37}$$

Finally,

$$s = 2.5 - \sqrt{6.5 - 6.25} = 2 \tag{38}$$

$$t = 2.5 + \sqrt{6.5 - 6.25} = 3 \tag{39}$$

Because of these historical precedents it appears justified to refer to this method of solving quadratic equations as the Diophantus/Savage/Loh method.

Reflections on learning

Using the Diophantus/Savage/Loh method, we think that even Stephen Colbert could solve quadratic equations without using the quadratic formula. In fact, the quadratic formula is not absolutely needed. Students can apply this conceptual approach to solving problems, and the historical links allow them to delve more deeply into how it functions, rather than simply memorizing a formula. As Pimm (1983) puts it

> Surely a teacher's role is to help clarify, and not hide, how a piece of work was done. Yet in published mathematics the methods of work are entirely absent and too smooth a path is given. If we never have to struggle with an idea or problem, we feel useless and outraged in the face of a real problem, one outside the narrow confines of what we have seen done. (p. 14)

There was a time when having a formula that could solve any equation of a certain type was an essential part of the knowledge of any user of mathematics. Now, of course, we have software that can solve any equation expressed symbolically. What is needed is insight into how to think logically through a problem, applying basic principles as needed.

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A discussion with two five-year-old girls about similarities and dissimilarities in two paintings

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Some time ago, we ran across a study by Blanton *et al.* (2018) on six-year-old American students' understanding of the equal sign. They reported on the operational understanding of this symbol which had developed prior to formal instruction. This led us to wonder how children might attend to various relations in everyday life. We know that some children have a spontaneous tendency to focus on quantitative properties of their environment (*e.g.,* Hannula, Lepola & Lehtinen, 2010). So, we designed an experiment where five-year-old children explored and discussed two paintings (Figures 1 & 2). The aim of our experiment was to explore how children relate different elements in the paintings to one another using the concepts of similarity and dissimilarity.

Similarity and other equivalence relations are not self-evident notions for young learners. According to Mazur (2008), a source for the challenge is that the 'same' object can be presented in different ways. Perceiving the sameness requires a more advanced underlying structure. One can only wonder what kind of mental processes it takes to equate two objects in the real world when none of them are exactly identical in all details. Nevertheless, research in children's perception and understanding of similarities and dissimilarities has shown that already six-year-old children can successfully perceive even small differences and categorise similarities among facial expressions.

The experiment

The five-year-old girls who participated in the experiment are from a Swedish kindergarten where they have art class twice a week. The number of students in the whole class is 21. These two girls were selected among four pairs that volunteered to participate. The reason for selecting just these girls was that they were more talkative than their peers.

The discussion with the girls was conducted as a semi-structured interview. The interviewer did not ask if any specific details were similar or dissimilar to something else, leaving all relations to be discovered by the children. In addition to explaining their findings verbally, the girls were encouraged to draw or in some other way to visually express how they experienced similarities and dissimilarities. The discussion was video-recorded.

Our findings

It will not surprise anyone who has worked with young children that the girls were able to point out more than ten pairs of elements in the paintings that are dissimilar with respect to some property. Among their responses, we found three different types of dissimilarities.

First, they picked up examples of single asymmetric relations such as "people *[pointing, moving from one painting to the other]* no people" and "I see also another thing: fan or whatever it is [pointing to the projector lens shown in one of the paintings] is not within in that picture *[pointing to the other painting]*". Second, one of the girls noticed that "*[pointing to one of the paintings]* eyes *[moving to the other]* no eyes" and continued immediately after drawing two eyes on her paper "and I notice another difference: an eye without glasses and an eye with glasses *[pointing to two eyes in the same picture]*". Here she combines two asymmetric relations at different levels: a relation in the set of eyes in one painting and concerning wearing glasses, and a relation in the set of the paintings and concerning the presence of eyes in the paintings.

Third, a little later, the same girl had the following discussion with the teacher.

Girl	I see another thing in that painting. One *[pointing to a man in one of the paintings]* is holding his mouth sadly and one *[pointing to another person in the same painting]* is holding *[it]* up.
Teacher	One?
Girl	One holds down.
Teacher	A mouth down like a sad mouth?
Girl	hm *[agreeing, pause]* and one up like excited.

She combines a relation in the mouth positions of a set of people in one of the paintings and a more abstract relation which associates differences in mouth positions to differences in feelings. A transitive property in this reasoning

Figure 1.

Figure 2.

becomes more explicit when we summarise it as follows: Two people have different mouth positions. Different mouth positions indicate different feelings. These two people have different feelings.

The girls were also able to point out several pairs of things that are equal to one another with respect to some property. Their findings concerned, *e.g.,* the use of the same colour in both paintings: "*[pointing an area in one of the painting] Blue [moving to the other painting]* blue" and "Red chairs *[moving to the other painting]* red roof". The latter example shows that the objects themselves can be different, the connecting element is the colour. The girls also focused on finding similar things within one painting. Then they noticed, for example, people doing the same thing: "kissing [moving to the other painting] and kissing *[pointing to couples in both paintings]*".

Since the class had discussed the notion of patterns with the teacher earlier that year, their teacher started the following discussion.

Teacher	Say, do you see any patterns on those? Do you know what a pattern is?
Girl A	Yes.
Teacher	What is a pattern?
Girl A	A pattern is, when it consists of different shapes and colours.
Teacher	Hm, well described. So, do you see, is there a pattern?
Girl A	Let me think about it *[pause]* Yes. I think, I do *[Becomes silent]*
Girl B	*[Rises and goes to one of the paintings and points to the hair of a woman pictured]* One *[pause]* one like a stitch.
Teacher	*[You mean]* that there is a stripe in the hairdo?
Girl B	Hm *[agreeing].*
Girl A	*[Comes to the painting and points to other people in it]* And in the all hairdos here.
Girl B	And even here *[points to the forehead of a third person in the painting].*

In this episode, B shows a sense of both similarity and symmetry because the line pattern is repeated both as it is and mirror-symmetrically. When A joins the discussion again, she points to a pattern that is quite similar to the previous example but not exactly the same. The same applies to the pattern that B shows on the face of the third person. What is interesting here is that this pattern is not related anymore to hair but to a forehead, and its shape has also clearly changed. One may think that the pattern on the second face mediates the similarity from the first example to the third one. In this sense, it is a trace of transitivity.

Another example of transitive reasoning came up at the end of session.

Teacher	When do you think that this is happening *[referring to the painting where people are in the cinema].* Is it in the morning or *[pause]* at lunch time? When does this happen?
Girl	Eeh *[pause]* in *[pause]* in the evening!
Teacher	Why do you say so?
Girl	You know *[pause]* my mother has done a thing *[pause]* and my father *[pause]* when I was with my grandma and grandpa *[pause]*
Teacher	Hm?
Girl	*[pause]* then they were at the cinema and it was an evening.
Teacher	Hm! *[showing understanding].*

When teacher asked where the house in the other painting might be, the girls came up with the following ideas.

Girl A	I know, it is on a street!
Girl B	*[Starting at the same time and showing a great enthusiasm]* Eeh *[pauses while listening to the other girl]* you, you can think that *[pause]* that there, inside, there are all these persons!
Teacher	Yes, that the cinema is there! *[meaning the house in the other painting]*
Girl B	Hm *[Agreeing]*
Teacher	And here we have the outside *[view]* of the cinema and, in the other, the others in the cinema.

In other words, the girls construct a symmetry between the paintings that is conveyed by the themes of the paintings.

Discussion

In the domain of two paintings, there are only two alternatives for an asymmetric relation; the girls discovered both of them. The double relations discussed above could have other domains than those we mentioned. In the second example, a possible domain is the set of all eyes in the painting, not only the eyes that were picked up during the conversation. However, extending the domain does not affect the asymmetry property of the relations expressed by the girls.

A critical reader may question whether the girls really combined two relations in order to speak about the difference between the paintings or only noted another dissimilarity of the first type within one of the paintings. We cannot completely exclude this possibility, yet there are two reasons to believe that she simultaneously paid attention to both differences. Namely, the very short and meditative transition from the one point of view to the other and the fact that the girl was well aware that we now discuss the difference between the paintings. Further, the same girl definitely combined two different comparisons simultaneously in the double relation related to the third example. In this episode, it was clear to every participant that the discussion concerned only one of the paintings.

The examples expressed by the girls also contain many symmetries and transitivities, some of them quite advanced and abstract. The contextual symmetry between the paintings—an indoor picture of people in the cinema and the street view of the house where the cinema is—is especially interesting. There is an element in the latter painting representing the house that may explain this symmetry. The girl who came up with this symmetry also pointed to an upper part of the painting and said that the colour of the sky is same as it is in the evening. By combining this with her memory of her parents going to the cinema in an evening, she seems to have constructed the evening=evening similarity between the paintings. This may have led her to think that also the cinema can be a theme that relates the paintings to one another.

Interestingly, the girls did not mention any examples based on showing that an object is similar to itself. Actually, there are studies showing that reflexivity is an obscure notion for learners at any age. Blanton *et al.* (2018) noticed that some children relate the equal sign automatically to performing an arithmetic operation, *i.e.,* they interpret $5 = 5$ as $5 + 5$ and not as a relation. Similarly, Tossavainen, Attorps, and Väisänen (2011) noticed that many student teachers claim that $x = x$ is not an equation because "it is weird to relate a variable to itself in this way". In their data, some student teachers explicitly related the equal sign to a command to solve an equation. They claimed that this expression cannot be an equation because it would "collapse" if one tried to solve it.

Van den Heuvel-Panhuizen and van den Boogaard (2008) studied the cognitive engagement occurring when a non-mathematical picture book is read to young children. They noticed that the children, finding themselves in an inspiring environment with elements that can be mathematized, spontaneously came up with mathematics-related thinking. Our experiment clearly supports their conclusion in the context of visual arts. Already five-year-olds can have versatile mental templates for advanced discussions on mathematical relations. An especially interesting observation revealed by the above quotes is that the girls developed their mathematical thoughts both from the paintings and their interaction related to the artworks. This reflects what we also see in the formal contexts for learning mathematics. Both a personal engagement with mathematical text and social interaction are needed for developing deep understanding about a new mathematical notion.

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Readers interested in looking at paintings mathematically may want to read 'Looking at a painting with a mathematical eye' by Marion Walter (1928-2021), in issue **21**(2). In that article she explores the painting 'Arithmetic composition' by Theo van Doesburg (See Figure).

Marion had a lifelong interest in connections between mathematics and art, was a member of the FLM Advisory Board from 1980 to 1997 and contributed much to the visual 'look' of FLM in its first seventeen years.