Thinking Like a Mathematician: A Problematic Perspective

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"Thinking like a mathematician" is a phrase that has been impervious to several revolutions. It was a rallying cry for the "new math" movement of a generation and a half ago, and has withstood the transformation of that movement into the era of *The Standards*.

It has meant different things at different times In the case of the "new math," it implied an ability to appreciate and to operate in a deductive framework and it was a reaction to the popular perception of mathematics as operating on free-floating meaningless symbols supported by an arbitrary collection of rules and regulations—a view that was particularly popular (if overstated) with regard to algebra.

In the present era, it places greater emphasis on problem solving heuristics—focusing on students' ability to make use of a wide array of inductive and deductive skills as they operate on incomplete knowledge.

One of the most interesting ways of depicting the concept of "thinking like a mathematician" was proposed between the advent of the "new math" and its present mutation. In FLM in 1982, David Wheeler recalls and further develops some thoughts he had recorded earlier on the concept of *mathematizing*

Wheeler suggests that it is "more useful to know how to mathematize than to know a lot of mathematics," and he wonders why "the majority of teachers [do] not encourage their students to "function like a mathematician" "

Furthermore, he offers an insightful argument in defense of honoring the activity of mathematizing as an inherent skill or competence. He claims that anyone who can speak is already engaged in significant algebraic thought

> [T]he transformational requirements of a confident use of grammar are very complex and still defy complete analysis But the chief points at issue are that: (a) the child's mastery of grammar can only be adequately described in terms of mathematical operations, and (b) this mastery is not derived by imitation (p. 46).

In summarizing the elusive concept, Wheeler speaks of the following:

- the ability to perceive relationships
- the ability to idealize them into purely mental material
- the capacity to internalize actions and such, so as to ask "What would happen if?"
- the ability to transform along a number of dimensions such as from actions to perceptions and from images to concepts.

He views mathematization as a construct that "can be detected most easily in situations where something not obviously mathematical is being converted into something which obviously is." This leads him to suggest that mathematization is in fact something not so elusive to define—*putting a structure onto a structure*.

Eventually he expands the idea as he adds two other ways of detecting the presence of the concept: The triad of

- structuration (for which the feeling of "eureka" is an extreme indication).
- dependence of ideas upon each other
- infinity—connected with the search for generalizabilty

and the triad of:

- making distinctions
- extrapolating and iterating
- generating equivalence through transformation

I will not pursue the evolution of Wheeler's thinking about mathematization beyond the 1982 piece at this point. It is perhaps enough to point out that:

- he has an array of wonderful concepts that are not so much captured and clearly defined as they are appreciated and mulled over for a while—to be themselves the activity of transmutation over time New ones are added; some are discarded; some are incorporated into larger frameworks.
- an effort on Wheeler's part to make sense of this concept (over a lifetime ?) is in itself a self-referential act, a beautiful example of the very concept he is trying to illuminate That is, he is engaged himself in an act of

How do I finish the last bulleted sentence? Wheeler is mathematizing by virtue of his earlier insight that the ability to use language is itself an act of mathematization; he also is mathematizing by virtue of the fact that he seeks connectedness, generality, extrapolation, transformation—that is making use of the elements he is analyzing.

But his self-conscious use of language puts a different cast on what he is doing There are qualities not associated with mathematization. It is the tension expressed by this conflicting description of what it is he is doing in his analysis that provides the itch for what follows.

Thinking like a mathematician: A competing perspective

I am tempted to explore his analysis further, to integrate it with more recent thoughts of his and others on the concept of mathematizing and to further excavate the self-referential issue of whether Wheeler's *discussion* of the concept of mathematizing is itself an act of mathematizing, but, while I will mention other aspects of his thinking, I must move elsewhere if this is to contribute to the questions Wheeler has asked us to think about in this issue of FLM:

- Does mathematics education exist?
- Can we truly respect it as a science?

I would like to propose that the adoption of the slogan "teach so that they learn to think like a mathematician", while leading to a number of enticing questions, ignores some important educational issues. Such distortion makes it difficult for me to believe that we are on course in the field of mathematics education. My discussion here will be brief and sketchy. It is intended to suggest and to inspire debate rather than to be a persuasive argument.

It has been pointed out (and Wheeler mentions it in his essay as well) that mathematical genius is detectable early, in a way that is not observable in more complex fields such as literature and history.

That is, it is possible to present challenging tasks to highly talented youngsters. The history of the task may be ignored; the necessary machinery is minimal; and the manner in which such youngsters express their insights does not require elaboration in order to generate mathematical inquiry. Young Gauss' solution to finding the sum of the numbers from 1 to 100 (perhaps an apocryphal story) by cleverly observing an important Gestalt is a case in point

The life of Ramanujan supports the view that innocence may be an asset in much of mathematical thinking. As a matter of fact, Ramanujan, who had received minimal formal mathematical training, came upon the most remarkable connections, and many of his arguments defied accepted canons of proof. That many of his conclusions were wrong is beside the point, for given his untutored notion of proof and his lack of formal education, it is noteworthy that he was able to come upon so many discoveries and in fact to create so many new fields

In Krutetskii's [1980] painstaking analysis of mathematical genius, he explores unusual talent by having his students use "think aloud" techniques. He points out that seeking clarity, simplicity, and elegance in solving problems distinguishes the youngsters who excelled from others. Furthermore, aside from problem solving *per se*, information is both acquired and retained in an economical manner so that the pupil "does not load the brain with surplus information".

Though well documented, none of this is surprising. It does however, remind us that there is something special about mathematical thinking that may distinguish it from most other fields.

But such an orientation towards a field of inquiry generates problems of an educational nature. If to be educated is to increase our awarenesses of the integrative potential of disparate experiences, should we not view with skepticism a call to isolate such activity from other parts of our thinking and feeling lives?

Wheeler tells us that we may not (perhaps cannot) observe mathematization directly but only in action and perhaps in hindsight. This is a provocative thought and it surely has its pedagogical implications.

But do we not wish to encourage a kind of mathematical reflection that far outstrips anything that has been depicted as "thinking like a mathematician" or of mathematizing in particular? We are generally committed to "looking back" as an important ingredient in problem solving. It is an effort to try to understand what one has done; what strategies may have been used implicitly as well as explicitly; what ones may be generalizable; what new territory may be opened by the inquiry and so forth.

But even "looking back" has a limited educational perspective. It does not encourage us to ask some very important questions like the following:

- What do I find out about how I think and what I prize when I consider my analysis of this situation, problem or set of problems?
- What motivated me to even take this activity seriously?
- How does the thinking I experienced in this situation relate to thinking in other areas of my life?
- What made this thing I explored a problem in the first place? What would have to be done so that instead of solving it, I might look at it so that it was not a problem at all? Are there other people who would not see this as a problem? Why?
- How does this problem or situation fit into the scope of mathematical things I have thought about? of other things?
- What emotions did I experience as I worked on the task? How do those emotions relate to cognitions I held about this problem?

I am not sure these are good questions, but they are examples of both meta-mathematical and educational questions that require a kind of thinking that is different from "thinking like a mathematician".

They are different not only because the subject matter is different, but because they make different use of language and thought. In particular, these are *not* the kinds of questions that can be explored with the same sense of awareness as in *mathematizing*. I am reminded of the claim by Papert [1980] that the computerized turtle Logo environment enables the child to learn mathematics in the same way that s/he learns a first language, but at the same time exemplifies what is meant by thinking like an epistemologist. It is not hard to see the exaggerated nature of the claim once we realize that to be an epistemologist is not only to use language, but to *reflect upon* how language is used. It is *not* an activity that is either acquired or expressed in the same way that anyone learns a language naturally.

Similarly the act of understanding ourselves and becoming educated is fundamentally at odds with the qualities we associate with mathematization. While I am not suggesting an either/or mentality, it is worth appreciating that an educational perspective frequently strives for uniqueness rather than generality, for understanding of one's emotions and not just *experiencing* them, for thinking in dialogical and linguistic rather than in monological ways; for seeking confusion rather than eliminating "irrelevancies".

How to define what I am calling *an educational perspective* in a clearer way than I have done so far, and how to orchestrate and integrate what may be two quite different orientations towards knowing and experiencing the world, are matters that require our attention if mathematics education is to be thought of as a field that takes both mathematics and education seriously

Like Fermat, I have developed a clear analysis of the concept of *an educational perspective*, but "this margin is too narrow to contain it " *

* Actually it would have been a major contribution had I been able to do for the concept of *an educational perspective* what Wheeler has done for the concept of *mathematizing*

References

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