should be a lot of scope for working with the grain of natural mental activity. One would be teaching mathematics because it is a human achievement which gives pleasure to people.

Then *how* should we teach mathematics? If one sees mathematics as a body of knowledge (about the world, or platonic ideals, or formal systems) then one will be led to a view of teaching in which knowledge is passed (poured?) from the knower to the ignorant. What is to be passed can be prescribed in a syllabus and tested in an exam. The most convincing argument against the value of this view, in my opinion, is that by and large people do not pass exams, and even those that do seem in a year or two to have been (cognitively) quite unaffected by their learning experience. If, however, one approaches mathematics as a set of human activities, one will try to introduce these activities to children. The aim will be to help them to experience them and enjoy them. The measure of success will be the extent to which learners subsequently think and act mathematically.

To be fair to Wittenberg, his penultimate paragraph recognises that mathematics is not "a defined entity in some logician's or philosopher's textbook", but an aspect of our experience, a living reality, and this is his resolution of the apparent conflict between different views of (the epistemology of) the subject. It seems to me that it would have been much better to start at this point, not to finish there.

Notes

[1] A. Wittenberg. An unusual course for future teachers of mathematics. *American Mathematical Monthly, 70* (1963a)

[2] David Bloor for instance ("Hamilton and Peacock on the essence of algebra", in: H. Bos, I. Schneider and H. Mehrtens (eds.) *Nineteenth century mathematics in context*. Basel and Boston: Birkhäuser. forthcoming) argues that Sir William Hamilton's opposition to the Cambridge School's development in the early 19th century of formal algebra, unrelated to number, was connected with his hierarchical view of society — you can't do just what you like with symbols, you must stick to the rules.

MINUTIAE

. . yet I cannot see any reason to suppose [Witt] gave a meaning to the quantity with its [decimal] separator inserted. I apprehend that if asked what his 123|456 was, he would have answered: It *gives* $123\frac{456}{1000}$ not it *is* $123\frac{456}{1000}$. This is a wire-drawn distinction: but what mathematician is there who does not know the great difference which so slight a change of idea has often led to? The person who first distinctly saw that the answer -7 always implies that the problem requires 7 things of the kind diametrically opposed to those which were assumed in the reasoning, made a great step in algebra. But some other stepped over his head, who first proposed to let -7 *stand for* 7 such diametrically opposed things.

Augustus de Morgan, *Arithmetic books from the invention of printing to the present time* (p. xxiv)

Communications

Educational mathematics

LARRY COPES

Even while issue number 3 is waiting in The Pile, I am reading number 2. I am reacting first to the Editorial, which makes the claim that writing about mathematics education should (but, frequently enough, doesn't) relate either to mathematics or to student learning. In the process of making that claim, it states

> Nevertheless, if it is appropriate, as it surely is, to demand of teaching that it works — i.e., that students learn mathematics — then it is equally fair and appropriate to demand of mathematics education that it works too — i.e. that mathematics teaching improves.

From this it appears that our aim as mathematics teachers is for students to learn mathematics. I believe that Bill Higginson is reiterating this apparently obvious goal on the next page when he writes

> The aim of a mathematics educator is to optimize, from both intellectual and emotional viewpoints, the mathematics learning experience of the student.

Now I don't disagree with the proposition that the success of mathematics education should be judged by its effect in the classroom Nor do I think we can be effective without students' "learning mathematics" (in any sense of the word "learning"). I do believe, however, that our goal might change in a significant way if we assume that mathematics education should help provide what students really need for creative survival in this complex world

But what do students "really need"? In a general sense, what I think we all need is as large and varied a collection of tools as we can get to help us deal with our experience By "tools," I mean much more than skills and habits — even complex skills like "problem solving." I think we need to have a variety of "lenses", through which we can view our world, a collection of metaphors to help us structure (understand?) and enrich our lives These include historical, aesthetic, scientific, and mathematical metaphors.

As one example, imagine a walk through the woods in the spring I can enjoy such an experience because of the sounds and smells and sights But I can appreciate it much more if I bring various metaphors into play in interpreting (not replacing) those sensory perceptions I can appreciate this wildflower more if I know something of the theories about the biological origin of it and related plants. My knowledge of a particular poem in which the plant was used as a symbol, and my awareness of its medicinal use in other cultures, also deepen my appreciation And my understanding of mathematical ideas of symmetry and group theory lend another dimension to my observation

A middle-management position in a corporation would provide a more "practical" example. Such managers find themselves having to integrate their organizational skills and knowledge of the company with the perhaps differing viewpoints of those around them. An awareness of the social and economic contexts in which the corporation is embedded, as well as various psychological metaphors, are very valuable, if not indispensable, for effective management Perhaps also essential is some knowledge of mathematical models for, say, amortization and sinking funds; but isn't it just as just as important for them to accept as legitimate different views of the same phenomenon, just as amortization calculations can be thought of as geometric series rather than merely as substituting into a formula? Or, when faced with several equally valid alternatives, shouldn't they recognize that final decisions must be made on aesthetic or personal grounds, just as choices among equally valid algorithms, proofs, or even axiom systems must be made?

I should make several disclaimers at this point. First, this is not to say that mathematics is not an exciting field of study in itself. Indeed, the interrelationships and elegance that make mathematics interesting, as well as the processes of research (as so vividly illustrated by David Tall's article), can probably be the most valuable aspects of the mathematical lenses people can use Second, I don't mean to be saying that students are aware of these less obvious goals. Perhaps the chance to help students become more aware of their own needs can be seen as a major reason for teaching anything. Finally, in no way am I implying that the structuring of our worlds is entirely an "intellectual" process The use of mathematical metaphors, like others, involves ethical and emotional dimensions as well.

It does seem, though, that taking as our goal "students learning mathematics" is too narrow, and may actually detract from the objective that they learn to view the world through the lenses of mathematics Even granted that familiarity with mathematical results and processes is needed for using mathematical metaphors, shouldn't we focus more on the excitement of "educational mathematics" than on more efficient or effective acquisition of that familiarity?

Teacher-student interactions

DAVID STURGESS

I very much enjoyed the article "Ye shall be known by your generations" by Stephen Brown (Vol 1, No 3). My first reaction on reading it was that here was someone expressing what I believed but had not succeeded in formulating for myself This I find a rare event but very rewarding when it happens In my work with teachers I have been concerned with getting them to the stage of asking questions and generating problems, often, I have to confess, without a lot of success Recently, however, I have been trying a slightly different approach which is to help them to become aware of the nature of their interaction with the children that they teach, and to reflect upon this. If one couples this with experience of problem solving for themselves there seems to be a "double" awareness each of which reinforces the other The nature of the experience of personal problem solving activities put beside the actions in the classroom, underlines the need for allowing children to experience problem solving, and thence on to problem generation

One technique that I have used for this is to ask teachers to make a tape recording of a normal interaction between themselves and children to reflect on what takes place This is essentially a task that is not done in isolation, because that makes the reflection very difficult, but one that is undertaken by all the members of a group who can then provide mutual criticism and support, or undertaken in some other situation that is supportive

On a recent course that I ran for secondary teachers I introduced the group to a series of common investigations and discussions of mathematical problems, toward the end of which several of them expressed a wish to try the ideas out with small groups of children and analyse the process for themselves In each case I suggested that all the sessions should be recorded so that they could afterwards check on what had taken place One of these teachers, John Lloyd, after his first session with children, produced a list of "criticisms" of his own performance which seem to me to be a blueprint of the actions of teachers that most often prevent the generation of problems I asked John if I could show this list in the hope that others may be tempted to look at their own classroom behaviour and see how they feel about it. (My own "performance" with the group was also analysed!)

After his first session with children John felt satisfied with what had taken place until he played back the tape! He then made these notes, some of which he has amplified:

> "I rushed on and did not stay with problems for long enough
>
> I gave too many cues, e g "Ah!"
>
> I frequently asked questions and made it obvious I had an answer in mind
>
> I had problems in generating group discussion My presence seemed to inhibit them
>
> I often said: "We will come back to that later" (so that I could make the point that I had in mind?)
>
> I often stopped the whole group to draw their attention to an interesting point made by a pupil (in an attempt to direct their work?)
>
> I avoided using the words: "Can you explain that?"
>
> I said I understood their ideas when I didn't
>
> I often asked the whole group a question rather than individuals
>
> I frequently cut off pupils' explanations (because they were not the ones I had in mind?)
>
> My role should have been as a "consultant", however I often found myself "directing" the children For example, I often stopped the group working and asked questions or talked to the whole group at the same time I found it very difficult to listen to pupils' explaining their ideas without interrupting them and putting them on the "right track" (my idea of the "right track")
>
> The group were very quiet and little pupil-pupil interaction took place When it did I intervened with a comment of my own Most teacher-pupil interaction was initiated by me in the way mentioned above. However, pupil-pupil interaction did take place once I had left the room Then the discussion involved all group members and was concerned with the mathematical work Once I returned to the room, discussion ceased
>
> On the one hand, when listening to pupils I gave a cue such as "Ah!" making it obvious that I had an explanation or "right answer" in mind On the other hand, I frequently "cut off" pupils' explanations or failed to try hard enough or stay long enough in order to understand their reasoning
>
> I experienced difficulty in knowing when pupils had reached frustration level and therefore when to intervene When intervening, I was not sure how to give the pupils the impetus needed to involve them in the work without giving them the answer."

Having become aware of these behaviours John was able to avoid many of them in later sessions and was delighted at the way in which the children developed as problem solvers and generators when these particular verbal patterns were avoided

Teaching and learning algorithms

ARTHUR MORLEY

Jere Confrey's article on concepts in the July 1981 issue was most stimulating, but it was her first three sentences which produced this response on a different but related topic!

She wrote, "In mathematics education, the term "concept" repeatedly surfaces, often in contrast to the term "skill". Yet the teaching of mathematics as skills still predominates in our schools, partially because advocates of conceptual learning often assume the value of concepts without explicitly defending it by defining precisely what they are. Until an adequate response to this question is given, the question of how to teach concepts will remain unanswered and the techniques of skill teaching will continue to dominate mathematics teaching"

I do not agree with this conclusion The emphasis on concepts was born out of reaction to (revulsion at!) traditional skill teaching, but this has prevented us taking a cool look at the central role of algorithms in mathematics, and facing up to the problem of teaching their construction better. (I remain uncomfortable with Skemp's term "instrumental understanding" because it seems to me to arise from leaving algorithms out of mathematics and then finding they won't go away, though I would be happy to accept Schwarzenberger's interpretation [1] of the term as "the ability to follow a set of computational procedures correctly".)

My experience of in-service courses in the past twenty years is that if we cannot show different approaches in what teachers still regard as the core of their jobs, teaching algorithms, and show the relevance of concepts in that context, then there is no change in classroom style in any part of the teaching of mathematics (and who can blame them given all the pressures?). Succeed in this, and "all else is added unto you" Fail here, and the rest is looked on as a fringe activity

Every branch of mathematics throws up its characteristic algorithms We can marvel at the sheer cleverness and subtlety of, say, integration by parts, coset-enumeration, the Simplex method, iterative solution of equation, Euler paths, and, yes, the algorithms of arithmetic Yet in school we seem to have lost any vision of the magnitude of the achievement of their construction, and so fail to convey it to pupils Why is the challenge to *construct* the algorithms in the course of solving problems so rarely put in the classroom, before the showing and the explaining begin? This way children might capture more of the excitement by participating in the construction of the algorithms and *seeing* how powerful they are. Once constructed they become routines to be followed, and dull to that extent; but that in a way is what they are meant to be. They free us to think about other things Even so we should not play down the personal feelings of power and satisfaction that possession of an algorithm gives by enabling us to do things we couldn't (or only with great difficulty) do before

During the 1960's only in some of the work of Edith Biggs in the U K. [2] and in Madeleine Goutard's "Mathematics and Children" [3], especially in the chapter "How Numeration Could Come About", did I feel the challenge to construct put strongly, and the response from the children reflected the strength of the challenge. That tradition was continued in the 1970's by the Dutch IOWO team in the Wiskobas project using their method of "progressive schematisation". The use of the loop-abacus for addition and subtraction, the grid-model for multiplication, and the "cups for sharing" scheme for division, make it possible for problems with larger numbers to be put, but leaving the level of operation under the children's control [4,5]. Though it contains much useful advice, the NCTM 1978 Yearbook "Developing Computational Skills" has a totally different pedagogic flavour to the work mentioned above, with showing and explaining and understanding well to the fore.

When we move to fractions and decimals the position seems to me worse. I do not know of a single textbook which does not assume that no problem of meaning is created for children by inserting a "×" (reinforced by use of the word "multiplication") between two fractions, say 2/3 × 5/7. They then go on either to define this "multiplication" in terms of the usual rule, or to state that the usual interpretation of "×" in the situation is "of", perhaps accompanied by a diagram with an explanation of compositions.

Yet up to that point in school the "×" sign has been used for multiplication of whole numbers (or fractions multiplied by a whole number) where the operation can be interpreted as a repeated addition. (Fractions of whole numbers are dealt with by the slight of hand described above in that the "of" is just replaced by a "×"). But in what sense can 2/3 be added to itself 5/7 of a time? We should face up to the fact that the pedagogic problem is one of constructing meaning for a quite *new* operation and it should not be slurred over by the use of "×" and "multiply" as if these carried their earlier meanings.

If children are asked to find the areas of a 2½ unit square, a 1¾ unit square, and a 2½ by 1¼ unit rectangle, using counting of squares and bits of squares, and the dimensions and answers are then tabulated:

$$\frac{5}{2} \quad \frac{5}{2} \quad \frac{25}{4}$$
$$\frac{7}{4} \quad \frac{7}{4} \quad \frac{49}{16}$$
$$\frac{5}{2} \quad \frac{5}{4} \quad \frac{25}{8}$$

we have a basis for the induction of a rule which will replace counting on a diagram to find the area. However, the crucial point is that when you ask children if they can spot a rule, they *say* things like "five times five" and "two times two", and when you ask them to write this down they offer

$$\frac{5 \times 5}{2 \times 2}$$

never the single "×" sign. It is then an easy step to say that we agree to put the one "×" sign as a shorthand for our special rule of "multiply the top, multiply the bottom". So we *finish* with the single "×" sign and its *new* meaning, we do not start with it [6]. If we start with an "of" situation applied to area or numbers and pursue it honestly with children, an exactly similar line of development appears.

There are difficult pedagogic questions to consider about justifying to pupils why algorithms work. Bell and Beeby [7] found that while counting up the decimal places was the most efficient rule for inserting the decimal point after multiplying two decimal numbers, reference back to the rule for multiplying fractions was unconvincing, partly because that had been learnt by rote and partly because the required comparisons moved attention away from the given decimal problem. A group of primary teachers who discussed this last winter preferred to express 2.5 × 3.06 as

$$\frac{2.5 \times 10 \times 3.06 \times 100}{10 \times 100} = \frac{25 \times 306}{1000}$$

which they felt relied on an understanding of decimal notation and an analogy to multiplication with whole numbers.

More difficult is using the division procedure for obtaining 3/7 in decimal form. It is not too difficult to make the step from 3/7 to 3 ÷ 7, but why should pupils believe the answer which appears on the calculator? A sequence with easier fractions where the decimal form can be obtained by use of equivalent fractions and the results compared with those from the division process at any rate gives an inductive basis for generalising to its use with 3/7, but I do not think we know how far this is convincing to pupils.

The algorithms discussed so far might be classified into those for counting and those where the rule arises out of giving meaning to an operation. To these we can add, more explicitly, approximating algorithms which take on importance with the ready availability of calculators and computers, geometric constructions and game strategies [8].

A series of questions which I have found helpful for student-teachers to consider in planning lesson sequences on algorithms is the following:

(i) What problem can I choose which shows the *need* for the algorithm? (Preferably one which can also be solved without the algorithm, using what the pupils already know.)
(ii) What apparatus or diagrams may be helpful?
(iii) How can the procedure be justified to the children?
(iv) What areas of application of the procedure do I want the children to tackle?

Here is another much-needed research programme which takes the role of algorithms seriously, but with teaching approaches chosen in the light of our overall aims in the teaching of mathematics.

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