Action Proofs in Primary Mathematics Teaching and in Teacher Training

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The first draft of this paper was presented (under the title "The role of definitions and proofs in primary teaching") at an international conference on theoretical problems of teaching mathematics in primary schools held in Eger (Hungary), 18-21 June 1973, and published in Semadeni [1973]. Another version (entitled "The concept of premathematics as a theoretical background for primary mathematics teaching") circulated in 1976 as a preprint of the Institute of Mathematics of the Polish Academy of Sciences. The approach proposed there was developed by Kirsch [1979] and by Turnau [1978], pp. 45-64.

The term "premathematical proof" used in all these papers has proved to be inadequate for many reasons. Therefore I now borrow the term "action proof", once used by Arthur Morley [1967], p. 23, in a similar situation. The concept is somewhat close to "operativer Beweis des Satzes" (cf. Müller and Wittmann [1977], p. 230).

The present text owes much to the discussions I have had with many people, in Europe and North America, over the last decade.

Introduction

The terms "definition" and "proof" have well-established (though different) meanings in the foundations of mathematics, in papers by working mathematicians, and in upper secondary school mathematics teaching

As a rule, proofs are not used in primary mathematics teaching. The reason for this is obvious: the concrete-operational child is not capable of hypothetical reasoning, of deduction expressed in words and symbols

On the other hand, many believe that no mathematical fact should be taught in school unless an average child can somehow satisfy oneself that it is valid (there are a few exceptions to this rule, e.g. the formulas involving π).

The purpose of this paper is to propose a general scheme for devising primary-school proofs, called action proofs. They are not intended to be ready-made proofs for the classroom, but rather should serve as guiding ideas for teachers. One may think of an action proof as an idealized, simplified version of a recommended way in which children can convince themselves of the validity of a statement; in practice, an action proof will require some preliminary or additional exploration.

The use of action proofs to replace formal proofs should be particularly significant in educating elementary teachers (cf. Semadeni [1983]). Showing them proofs in normal mathematical language is usually pointless: even if student teachers master them, it is mathematics imposed upon them rather than actively learnt. Worse, formal proofs often give wrong hints of how to teach children. If the tradition of narrow, "practical" training of preservice and inservice elementary teachers is to be abandoned, an action proof (discussed and analysed by student teachers) offers the right level of sophistication, falling intermediate between an intuitive explanation and a deductive argument.

Furthermore, action proofs may help curriculum designers and texbook authors in working out a global organization of primary teaching.

General description of an action proof

An action proof of a statement S should consist of the following steps:

1° Choose a special case of S. The case should be generic (that is without special features), not too complicated, and not too simple (trivial examples may later be particularly hard to generalize). Choose an enactive and/or iconic representation of this case or a paradigmatic example (in the sense of Freudenthal [1980]).

Perform certain concrete, physical actions (manipulating objects, drawing pictures, moving the body etc.) so as to verify the statement in the given case.

2° Choose other examples, keeping the general schema permanent but varying the constants involved. In each case verify the statement, trying to use the same method as in 1°.

3° When you no longer need physical actions, continue performing them mentally until you are convinced that you know how to do the same for many other examples.

4° Try to determine the class of cases for which this method works

Thus, an action proof is a result of internalizing an action rather than a logical inference from given premises. In didactics of mathematics one often speaks of abstracting concepts from reality. The basic idea of an action proof is that we abstract not only single concepts involved in the statement but also certain proofs. Of course, the concrete actions performed in steps 1° and 2° cannot be separated from some form of reasoning; this reasoning is therefore part of the action proof.

The physical and mental activities included in an action proof should satisfy the following conditions:

1° In substance, all arguments must be semantically correct.

2° The concrete actions must be concretizations of a formal mathematical proof of the given statement.

3° The method must be valid in all cases specified in the statement.

Thus, it requires a competent mathematician to judge whether a given action proof is acceptable

It may happen that in devising an action proof we have to distinguish various special cases. Then each case must be dealt with separately, that is, we are to find an action proof for each case, and then to combine the results by some straightforward argument.

When a concept is to be formed in a student's mind, exploring several representations of it is usually indispensable In an action proof, however, a single representation may suffice, since it is tacitly assumed that the concept has already been formed. Nevertheless, it is preferable to reinforce an action proof by transferring it to different representations.

The concept of an action proof fits the naive Platonic approach discussed by S. Vinner (1975); one may say that activities with concrete objects are to be replaced later by activities in the world of abstract objects

Example 1. Commutativity of multiplication of natural numbers. The statement to be proved is that the product does not depend on the order of factors (the phrasing may be different).

We first choose a pair of numbers, e.g. 3 and 5. We are to show that $3 \times 5 = 5 \times 3$. As the concretization of $n \times m$ we choose n rows with m counters in each. Thus, we begin the action by arranging the counters as in Figure 1a.

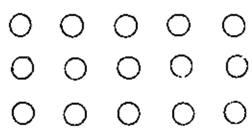

Figure 1a

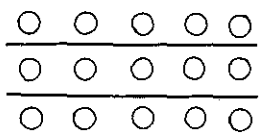

Figure 1b

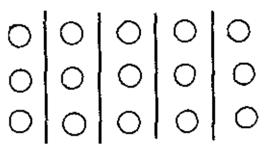

Figure 1c

We separate them horizontally (Figure 1b) and infer that the number of counters is 3×5. Then we separate them vertically (Figure 1c) and get 5×3. The number of counters in Figure 1a must be independent of the way of counting Hence $3 \times 5 = 5 \times 3$. (This argument presupposes, in particular, the conservation of number.)

Afterwards it may be desirable to perform analogous actions for some other pairs of numbers. In the case of larger numbers, e.g., when we want to show that $157 \times 29 = 29 \times 157$, complete manipulation is not feasible and we have to imagine 157 rows with 29 counters in each, and to perform the action in the mind.

Special care is needed with the extreme case $0 \times n = n \times 0$ (if one wants to include it)

The action proof outlined above is validated by making sure that it is a concretization of some formal mathematical proof: If a set A has m elements and a set B has n elements, then $A \times B$ has $m \times n$ elements, whereas $B \times A$ has $n \times m$ elements. Switching the variables

$$(a,b) \mapsto (b,a)$$

yields a bijection from $A \times B$ onto $B \times A$. Hence $m \times n = n \times m$. Changing the way of counting from horizontal to vertical corresponds to rearranging the elements in each ordered pair (a,b).

An action proof should not be confused with the didactically wrong) procedure of verifying the statement for just a few concrete cases and then asserting that it is true in general. Some teachers argue as follows: "3×5 is 15 and 5×3 is 15; therefore $3 \times 5 = 5 \times 3$. The same is true for all positive integers." Such an argument is not acceptable. Note that in the action proof above we need not know that $3 \times 5 = 15$; similarly we can be convinced that $157 \times 29 = 29 \times 157$ without multiplying out.

Action proofs are particularly useful in arithmetic and in combinatorics (see Kirsch [1977], Nowicki [1978], [1981]) that is, for proving statements involving finite sets Many nice examples can be found in textbooks and in articles for teachers.

In combinatorics one finds many "recurrent" action proofs, which are action proofs applied to one or another recurrence procedure. One has to learn systematically (in the process of exploring suitable representations) how to pass from each given number 1, 2, 3 etc. to the next, until the way of iterating the action and making the step $n \mapsto n + 1$ is grasped. It is the idea of recurrence rather than proofs by induction which should be taught to primary-school teachers

Example 2. The number of permutations of n different objects. Our recurrence will start at $n = 2$ The case $n = 1$ may be considered afterwards as special, extreme or singular; the case $n = 0$ is beyond the scope of action proofs. In Figure 2 a strategy of adding a new bead is shown. For each of the two possible permutations of two beads (white, black) we can insert the third (striped) in three different places; hence there are 2×3 permutations of 3 beads in total. The fourth bead can be put in four different places, so there are altogether $2 \times 3 \times 4$ permutations of 4 beads, etc.

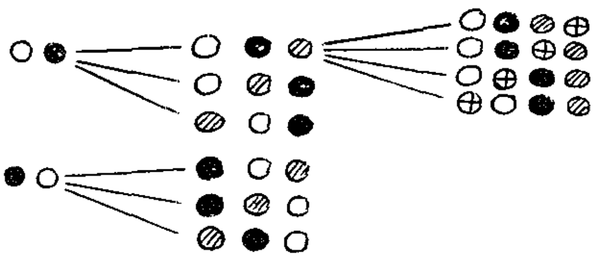

Figure 2

This example shows that a recurrent action proof can be used when the statement S involves a recursive way of computation rather than a single formula. In the case of permutation, the passage to the formula $1 \times 2 \times 3 = \ldots = n$ is immediate. In the case of the number of k-element subsets of an n-element set, however, an action proof leads only to the recurrence relationship expressed by the Pascal triangle; the proof of the global formual $n!/k!(n-k)!$ is more involved.

In geometry the role of action proofs is more delicate. The easiest instances can be obtained in geoboard geometry, which may be regarded as a discrete approximation to Euclid geometry; in fact, if a polygon has vertices at points with rational coordinates, it can be thought of as a polygon of integer coordinates at a suitable square net. The main advantage of square-net geometry is that it does not require deduction in Euclid's style and the arguments can often be reduced to counting squares, points etc.

There are also examples of action proofs of theorems about specific finite groups and finite probability spaces. Of course, action proofs do not work for general statements about groups or probability spaces, where formal deduction from axioms is indispensable.

Heuristic, plausible arguments, visual proofs, etc. should not be confused with action proofs, which are valid mathematical proofs, that is, provide absolute certainty (unless, of course, somebody makes an error, which is—for human reasons—as inevitable here as in a standard mathematical proof).

Some visual proofs (like the celebrated "Behold!" in a proof of the Pythagorean theorem) are, in fact, action proofs. To understand such a proof requires concrete actions (e.g., cutting the given square and moving the pieces) performed in the mind. Many people are unable to do this without help—they "cannot see".

One may criticize the concept of an action proof by arguing that it involves a psychological question: how can one know whether the child is convinced of the validity of the proof by inner understanding and not just by being prompted by the authority of the teacher? Without dismissing this criticism we note that it applies to any proof in a textbook: if the author finds his proof correct and complete, this does not automatically imply that students understand it.

Empirical work is needed to determine which action proofs really work in the classroom and, in particular, to clarify the role of recurrent action proofs. (Significant work on the latter is described in Greco *et al* [1963]).

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