# AN INFERENTIALIST VIEW OF NOTIONS AND CONCEPTS

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Inferentialism, proposed by Robert Brandom (1994, 2000), is a nascent philosophy that has received increased attention in the philosophy of education (*e.g.,* Taylor, Noorloos & Bakker, 2017) and mathematics education research in particular (*e.g.,* Bakker & Derry, 2011). This article aims to demonstrate the potential of inferentialism to extend contemporary constructivist research, especially regarding conceptual development. Although current inferentialist research tends to criticize constructivism (Bakker & Derry, 2011), we do not take the same stance. This article attempts to reveal inconsistencies between implicit, constructivist views on conceptual development from the point of view of inferentialism and to view constructivist findings through a new lens.

First, we raise what we call *the referential problem of notions* in constructivism. Second, we introduce two key ideas of inferentialism for solving the proposed problem. Third, we highlight the general implications from an inferentialist perspective to conceptual learning. Fourth, we propose a solution to the referential problem based on inferentialist ideas and an additional advantage of this view over constructivism. Finally, we reinterpret two recent findings in research on conceptual learning.

## A problem in constructivist research

Conceptual development has been an issue of interest in mathematics education research since the very beginning (Vinner, 2014). Existing literature, especially in constructivism, has elaborated and conceptualized some essential theoretical constructs for research on conceptual development. Learners' idiosyncratic development of something conceptual (referred to as a 'conception') is distinguished from the corresponding concepts that researchers have (Simon, 2017). Although the operational nature of conceptual development that is theoretically assumed in constructivist research has recently been questioned (Scheiner, 2016), constructivist ideas have become common sense (Thompson, 2014). It is still an influential idea that, while moving toward richer development, learners continue to crystallize many aspects into a single concept (Tall, 2011). However, this line of research seems to implicitly suppose that we can treat conceptions as object-like entities, and we tend to understand the term 'conception' as being analogous to the idea of 'concept'.

Let us take an instance of articulating what the problem is, keeping in mind the distinction made by Vinner (2014):

> A *notion* is a lingual entity—a word, a word combination (written or pronounced); it can also be a symbol. A *concept* is the meaning associated in our mind with a notion. It is an idea in our mind. Thus, *a notion is a concept name.* (p. 92)

For example, an undergraduate student explains many things about the notion of real numbers by referring to infinitesimals, which belong to the nonstandard real numbers or *hyperreal* numbers (Ely, 2010). We then pose the following question: Which does this student possess—a conception of real numbers or of hyperreal numbers? As the student talks about what we call hyperreal numbers, we have the following two options to interpret what she conceptualizes: She idiosyncratically conceptualizes real numbers where we conceptualize hyperreal numbers; or she correctly conceptualizes hyperreal numbers, but calls them 'real numbers'. Generally, we cannot distinguish *wrong content Y with the correct name X* from *correct content Y with the wrong name X.*

The constructivist distinction between first- and second-order models (Steffe & Thompson, 2000) helps us to formulate this problem more precisely. For constructivists, any knowledge is a model built by the knowledge holder to interact with her experiential environment. Therefore, the observer's knowledge of the observed person's knowledge is the observer's model of the observed person's model. This model is called a *second-order model* and is distinguished from the general type of model, the so-called *first-order model.* Following Simon (2017), who proposed that the term 'conception' refers to second-order models and the term 'concept' refers to first-order models, the problem mentioned above can be explained as follows: If a learner possesses an idiosyncratic conception of real numbers similar to the concept of hyperreal numbers, then it means that *the observer's second-order model for what the learner calls real numbers is similar to the observer's first-order model for hyperreal numbers.* However, this way of labeling conceptions is logically problematic because a learner can never conceptualize anything before naming it. For example, in Japan, the term 'real number' is usually taught in high school, while repeating decimals are informally used even in elementary school. Although there may be Japanese students who conceptualize real numbers like Ely's student, we can never build our second-order model for their understanding of real numbers before they know the term 'real number'. While it is counterintuitive that a learner cannot conceptualize anything without its name, it is also strange that we cannot build a second-order model for a learner's anonymous conceptual entity.

The problem seems to stem from the fact that a notion (a signifier) and a conception (something signified) are separable.

In this article, we call this problem 'the referential problem of notions'. Following Brandom's (1994, 2000) terminology, this is a representationalist standpoint, which assumes a notion can embody atomistic conceptual content. In the next section, we draw upon the inferentialism proposed by Brandom to solve this theoretical problem and build a new relationship between notions and concepts. Using the inferentialist angle suits our purpose because it criticizes the representationalist view.

## Two key ideas of inferentialism

First, we review two main ideas of inferentialism—*conceptual pragmatism* and *aboutness*—which provide a non-traditional perspective for learning. We argue that learning does not occur by inputting a piece of information but rather by outputting one's thoughts in an observable manner; learning is involved in *when* and *how* one makes one's own thoughts explicit.

### Conceptual pragmatism

We assert that inferentialism's most unique feature is its use of conceptual pragmatism, which reverses the order of explanation regarding conceptual understanding:

> An account of the conceptual might explain the *use* of concepts in terms of a prior understanding of conceptual *content*. Or it might pursue a complementary explanatory strategy, beginning with a story about the practice or activity of applying concepts, and elaborating on that basis an understanding of conceptual content. The first can be called a *platonist* strategy, and the second a *pragmatist* (in this usage, a species of functionalist) strategy. […] [Inferentialism] is a kind of conceptual pragmatism (broadly, a form of functionalism) in this sense. It offers an account of knowing (or believing, or saying) *that* such and such is the case in terms of knowing *how* (being able) to *do* something. (Brandom, 2000, p. 4)

Adopting this order of explanation, the role of expression changes from a traditionally assumed role. Expression is not an act that takes something out from inside one's head but rather actualizes something's potential.

> We might think of the process of expression in the more complex and interesting cases as a matter not of transforming what is inner into what is outer but of making *explicit* what is *implicit*. This can be understood in a pragmatist sense of turning something we can initially only *do* into something we can *say*: codifying some sort of knowing *how* in the form of a knowing *that*. (p. 8)

Due to this new role of expression, conceptualization is defined in a new way:

> The process of explicitation is to be the process of applying concepts: conceptualizing some subject matter. (p. 8)

Brandom (2000) uses the term 'explicitation' to explain the process of making explicit through the form of *a knowing that* what has been implicit through the form of *a knowing how*. Although inferentialism, as a pure philosophy, does not explicitly explain how conceptualization proceeds in detail, the process of explicitation should be considered a key process of conceptualization based on the above claim. As mathematics education researchers, we can interpret the above claim as follows: *Conceptualizing something occurs by expressing it* (Uegatani & Otani, 2021).

This new definition of conceptualization is innovative because it has profound implications for the referential problem of notions. The definition denies the possibility of conceptualization without notions. That means a learner cannot conceptualize anything without its name from an inferentialist point of view, though it may be counterintuitive from a constructivist point of view, as we mentioned in the previous section. The conceptualization of concept *A* occurs if, and only if, the notion *A* appears in an expression.

From this point of view, even if individuals implicitly use many concepts in their heads, it is not sufficient to conceptualize them. A concept's conceptualization only proceeds after making it explicit within an expression. Whenever a concept is made explicit in a fresh context, we can interpret its novel aspect as conceptualized. Thus, understanding when and how to express concepts is part of conceptual development (see Sfard & Lavie, 2005).

### Aboutness

Another special inferentialist idea is *aboutness*, though it is less emphasized in existing inferentialist research on mathematics education. In inferentialism, communication with concept use is understood as *a game of giving and asking for reasons* (a GoGAR), asserting with reasons, and teleologically assessing others' assertions to judge whether they could become the grounds for one's own statements.

> The context within which concern with what is thought and talked *about* arises is the assessment of how the judgments of one individual can serve as reasons for another. The representational content of claims and the beliefs they express reflect the social dimension of the game of giving and asking for reasons. (Brandom 2000, p. 159)

For each communicator, a claim's veracity is not the only important thing; what the true claim is about is also significant. For example, the declaration that "two is a prime number" can be about two or a prime number. Depending on what one talks about, the direction of the conversation may vary. Our locutions "make the words 'of' and 'about' express the intentional directedness of thought and talk" (p. 169).

From this standpoint, contexts always exist within texts in an inseparable manner, and both texts and contexts should be synthetically examined in mathematics education research on concept use (Bakker & Derry, 2011). In focusing on aboutness during a moment in the GoGAR, we, as researchers, are expected to pursue how conceptualization proceeds.

## General implications for conceptual learning

There are two general implications of the inferentialist ideas mentioned above. First, as Bransen (2002) argues that the key to understanding inferentialists' objectivity is normativity, research on conceptual learning becomes research on locally emerging normativity regarding using terms in GoGARs. Simply speaking, we assert that *notions appear first, followed by concepts,* in GoGARs. Brandom (1994) argues that "the inferential role, which is the conceptual role, *is* the content" (p. 618). "Inferentialism is basically the claim that meaning (*i.e.* conceptual content) should not be analysed in terms of *reference* but in terms of *inference*" (Bransen, 2002, p. 374). Therefore, inferentialists do not view a mathematical notion (or term) as representing a mathematical concept or a mathematical object. Rather, if one explicitly uses a mathematical notion (or term) to generate the next sentence in a GoGAR, then the notion plays some inferential role, and its role is viewed as the conceptual content.

Second, conceptualization is *only* considered to be making explicit what is implicit. Thus, conceptual learning only occurs when learners make an observable decision. Although learning may often imply inputting, we interpret learning as outputting. The radical constructivist account on re-presentation—which "is always the replay, or *re*-construction from memory, of a past experience and not a picture of something else, let alone a picture of the real world" (von Glasersfeld, 1995, p. 59)—suggests from the very beginning that a human's output never matches one's original input. If learners never portray their knowledge in its unchanged form at the moment when they construct it, we will never know for certain what they subjectively input into their heads. Thus, it is unfruitful to focus on what learners input into their heads; rather, we should only stress learners' outputs. Each time learners output something, its conceptualization progresses for them.

This heavy emphasis on outputs alters our understanding of the significance of social interactions in conceptual learning. This is critical not because social interactions influence what learners can input into their heads but because social interactions with others define potentially acceptable forms of outputs in discourse.

Constructivism denies the stance that learning implies internalizing socially constructed meanings (Thompson, 2014). The implication we draw from inferentialist ideas is consistent with this constructivist view. Learning does not involve internalizing socially constructed meanings but rather externalizing personally invented ones (*i.e.,* making personal, implicit ideas explicit). In this sense, every learner has the potential to challenge socially pre-existing ideas.

Note that our educational interpretation of inferentialism differs from its existing interpretations in mathematics education. For example, Hußmann, Schacht & Schindler (2019) built an inferentialist epistemological theory of conceptual development. However, we focus on conceptual development in a public social domain rather than that in a private mental domain. As another example, Taylor, Noorloos & Bakker, (2017) propose the metaphor of mastering for learning, which complementarily resolves both deficiencies of Sfard's (1998) metaphors of learning as acquisition and participation. However, we do not emphasize the necessity of exploring students' processes of mastering socially existing conceptual practices. We empathize with the constructivist perspective and are interested in the socially emergent process of learners' new conceptual practices.

## A solution to the referential problem of notions

Based on inferentialist ideas, we propose reversing the order of explanation about the relationship between notions and concepts, going against Vinner's perspective (2014) and also against a constructivist way of labeling first- and second-order models.

First of all, we redraw Vinner's articulation of the relationship: "*a notion is a concept name*" (p. 92). In this quote, Vinner treats a concept as an atomistic entity. However, from our inferentialist point of view, many concepts are related to each other. The concepts are semantically interwoven and cannot be procedurally decomposed into some basic separable components. Since mathematical and everyday concepts can be interwoven, the construction of mathematical concepts cannot be reduced to the hierarchy of concept definitions (Bakker & Derry, 2011). Rich meanings are lost immediately after we detangle the interwoven relationship between concepts, although they retain some core significance. Although mathematics, as an academic discipline, is only interested in fundamental meanings, our general mathematics education research on concept development should capture all meanings, including peripheral ones within multiple uses.

In this regard, Vinner's interpretation of a notion as a concept name might lose its peripheral value. The problem here is that we implicitly assume that a concept pre-exists its name. We often give a mathematical object its name in a way analogous to giving a physical object its name. However, an actual conceptual learning process is the opposite of this. Conceptualizing a concept in a context is explicitly expressing its notion within that context.

This view resolves the referential problem of notions. Conceptual understanding of something without its name is impossible in principle. This means that, for example, *two* and *an even prime number* are seen as completely different concepts. In this context, 'two', 'even', 'prime', and 'number' are words. 'Two' and 'even prime numbers' are notions for different concepts. When one uses the two terms 'two' and 'an even prime number' interchangeably, this indicates that one understands the two concepts in this way.

From this point of view, the student in the study by Ely (2010), who has a nonstandard conception of real numbers, can be described slightly differently. Ely does not report her use of the term 'real number', but rather her use of the symbol '…' (ellipsis). Hence, we describe her neither as one who has a conception of hyperreal numbers without knowing the correct name nor as one who has a nonstandard conception of real numbers. We describe her simply as one who conceptualizes the concept /…/ [1] in a nonstandard way. Using the symbol '…' is seen as conceptualizing the concept /…/. We do not need to care about the correct correspondence between a term (name) and a concept (content). Therefore, the referential problem of notions theoretically disappears.

This view also has two advantages over a constructivist view. First, a constructivist framework for second-order models usually only includes the information on *how* but not on *when*. Considering both *how* and *when* helps us make overgeneralized models of students' thinking unviable and allows us to use second-order models with more precise timing. As a second-order model is a representation of a way of understanding how students think, *when to use the second-order model* should also be a part of our knowledge about students' thinking, though a great deal of effort may be required to make explicit when to use the second-order model. For example, Ely also reported that the student used '…' in the exam in a standard way. This example shows that she conceptualizes a single concept /…/, which plays two different inferential roles in GoGARs of the exam and the interview, not that she has both standard and nonstandard conceptions of real numbers.

## Reinterpreting recent findings on conceptual learning

Our proposed view provides different interpretations of recent findings on conceptual learning in mathematics education research. Let us discuss two examples: Tsamir and Tirosh (2022) and Scheiner and Pinto (2019).

First, Tsamir and Tirosh extend Tall and Vinner's (1981) idea of concept images to *mis-in* and *mis-out* concept images. The former mistakenly includes a non-example as an example, while the latter mistakenly excludes a correct example. Tsamir and Tirosh reported that university students have a mis-in concept image (*e.g.,* 8π is even) and a mis-out concept image (*e.g.,* zero is not even). Because some students only gave idiosyncratic reasons (*e.g.,* an even number is a multiple of 2, zero is a special number) without referring to the formal definition, Tsamir and Tirosh also suggest that correct conceptualization needs a mathematical task that addresses both critical attributes of even numbers, *being an integer* and *being divisible by 2*. However, a different practical implication is drawn from our perspective. Both a concept image and a concept definition of even numbers are parts of the conceptualization of even numbers. In GoGARs between the students and the researchers, the students made explicit what is an even number as forms of personal definitions. The concept *even* did not play an inferential role in activating its formal definition. Hence, a necessary mathematical task for the students is developing a disposition to use the formal definition whenever they use the term 'even'. Rather than students having a misconception of even numbers, they had insufficient experience in using its formal definition.

Our explanation is simpler than Tsamir and Tirosh's classic constructivist one. This simplicity comes from our theoretical choice of *notion* followed by *concept*. For capturing not only the core but also peripheral meanings of a concept, we do not identify understanding the concept with understanding its formal definition. As part of conceptualizing a concept, students should learn the necessity to activate its formal definition from its name.

Second, Scheiner and Pinto extend Scheiner's (2016) discussion and propose three processes of ascribing meanings to objects: contextualizing, complementizing, and complexifying. They defined as follows:

> contextualizing has an *epistemological* function (viz., directing one's thinking to particular senses$_F$), complementizing has a *conceptual* function (viz., coordinating diverse senses$_F$ that forms conceptual unity), and complexifying has a *cognitive* function (viz., blending ideas$_F$ that emerges new dynamics and structure). (Scheiner & Pinto, 2019, p. 370)

Here, the subscript F indicates the term comes from Frege's terminology. Scheiner and Pinto elaborate on these three processes based on a reanalysis of a university student who reconstructed and remembered the definition of the limit of a sequence from his concept image.

It is interesting here to compare their and our views on conceptual learning. From their view, a single mathematical concept can become richer through processes of complementizing multiple senses$_F$ and can be contextualized in multiple situations. However, we do not theoretically assume the existence of such a conceptual unity.

When the student was asked how he remembered the formal definition, he responded:

> I think of it [the formal definition of limit of a sequence] … graphically … I think of it … so like you've got the graph there … and you've got like the function there, and I think that … … it's got the limit there [fixing the middle line position in the vertical axis] … and then epsilon [marking the two side positions], once like that … and you can draw along [drawing the three lines] and then … all the … points after $N$ there … are inside of those bounds [the space between the two lines each side of the middle line]. … … It's just … err when I first … thought of this, it was hard to understand, so I thought of it like this, like … that's the $n$ going across there and that's $a_n$ … (Scheiner & Pinto, 2019, pp. 364–365, ellipses in original)

We agree that his way of remembering the definition is a mathematically good strategy for remembering a complex definition. However, we think he might not use this strategy to remember other simple definitions, such as of even numbers. In his response, he used many mathematical terms, such as 'function', 'graph', 'epsilon', 'point', 'bound', '*N*', '*n*' and so on. He also used many non-mathematical terms 'graphically', 'got', 'draw', 'hard', and so on. Hence, from our view, this response demonstrates that many concepts play inferential roles in remembering the definition of the limit of a sequence in the GoGAR of the interview rather than demonstrating how he understood the concept of the limit of a sequence. Although this suggestion may be counterintuitive, it suggests that the definition of a limit of a sequence has not yet been strongly connected with his conceptualization of the limit of a sequence and that a new research question can be posed, for example: *Why and what types of definitions of concepts are weakly connected with their conceptualization?*

Scheiner and Pinto's approach is suitable for researching ascribing meanings to a target mathematical object, such as the limit of a sequence. However, our approach is suitable for investigating how various concepts play inferential roles in GoGARs. Since mathematics education researchers tend to focus on a particular mathematical concept they are interested in, they may overlook the roles of relatively minor mathematical concepts, such as *ellipsis* and *epsilon*, and everyday concepts, such as *graphically* and *hard* in conceptual learning. For example, in the above student's response, the terms 'epsilon' and 'hard' have peculiar inferential roles.

He nuanced 'epsilon' for expressing a positive variable approaching zero and used his remembering strategy because "it was hard to understand". Our inferentialist view helps us research conceptual learning more holistically.

## Conclusion

To rebuild the relationship between notions and concepts, we introduced the philosophy of inferentialism. From this angle, we posited a reversed order of explanation: Notions first, concepts second. Whenever one uses a notion of a concept for generating the next sentence in a GoGAR, one conceptualizes the concept. In this sense, a concept is a piece of knowledge-how that allows an individual to infer from what to what in a certain context; it is not an object-like, abstract entity.

From our proposed view, we have succeeded in solving the theoretical problem of Vinner's (2014) distinction between notions and concepts and the referential problem of notions. Our view restricts our way of describing conceptualization and slightly shifts the focus of what is really conceptualized. For example, we describe the student reported by Ely (2010) as conceptualizing '…' in a nonstandard way rather than having a nonstandard conception of a real number. Thus, the referential problem theoretically disappears. Although we problematized a constructivist view on the relationships between notions and concepts, our proposal is still consistent with a constructivist core motivation and characterizes conceptual learning as creative development in terms of word use in GoGARs. However, at least two issues remain. First, we only indicated a new view on conceptual learning, and a methodological elaboration for analyzing empirical data from our inferentialist perspective is needed. Second, we did not discuss how objectivity emerges (see Bransen, 2002). Identifying the origin of objectivity in a mathematics classroom is an important task for future mathematics education research and can be explored from an inferentialist perspective.

## Acknowledgments

We would like to thank the reviewers for their thoughtful comments and efforts toward improving our manuscript. This work was supported by the Japan Society for the Promotion of Science (JSPS) KAKENHI Grant Number 21K13587 and 20K14013. We would also like to thank Editage (www.editage.jp) for English language editing. Any opinions, findings, and conclusions expressed in this article are those of the authors and do not necessarily reflect the views of JSPS or Editage.

## Note

[1] Here we use /…/ to denote the concept and '…' to denote the symbol or notion.

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*Wall painting from the Matthias Church, Budapest.*