LEARNING THE = SIGN: A SIGNIFICANT COGNITIVE OBSTACLE

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Many first grade children consider the = sign as a prompt to perform an arithmetic operation. They see this sign as an operator and not as an indicator of a relation of equality. This conception leads to two main types of difficulties: these children do not accept unconventional equalities and they have significant difficulties in completing equations that do not correspond to a $a + b =$ __ structure.

In this article, I present the results of research whose main objective was to describe the process of understanding the = sign with first grade students. I taught this symbol in a six-week teaching experiment. In a case study, I describe the extent to which the conception of the = sign as an operator is strongly anchored in first grade children. I show that it is possible to move children from the beginning of primary school towards a conception of the = sign as an indicator of a relation. However, this transition is not evident; it constitutes an important cognitive obstacle [1].

Poor curriculum or cognitive limitations?

Why is the introduction of algebraic notation, which is traditionally done at the beginning of high school, so difficult for a significant number of students? Several authors suggest that it is not because the concepts and structures covered are conceptually beyond their reach that many adolescents have difficulty with the introduction to algebra. Rather, it is the teaching they received in elementary school that prevents them from succeeding in algebra. In a recent article in this journal, Brizuela and Schliemann argue that students' difficulties are the result of a curriculum that simply does not allow them to develop algebraic reasoning:

> Perhaps it is not that students are not prepared or ready for learning algebra, but that the teaching or curriculum to which the students have been exposed has been preventing them from developing mathematical ideas and representations they would otherwise be capable of developing. (2004, p. 33)

Their research suggests that fourth graders are able to solve linear equations with one or more unknowns on both sides of the = sign, if appropriate teaching activities are implemented. Dickinson and Eade (2004) reach similar conclusions after working with fifth grade children on linear equations using number lines. Research by Carpenter, Franke, and Levi (2003) also shows that it is possible for children in first and second grade to work on algebraic tasks such as generalising some properties of basic operations.

In fact, for several years now, there has been a tendency to develop algebraic reasoning from the beginning of elementary school. This is the case, for example, in the Ontario Curriculum (Ministry of Education and Training, 1997) and the National Council of Teachers of Mathematics *Standards* (2000). Of course, the underlying idea is not to confront children with formal algebraic symbols at an early age, but rather to offer them a different and deeper reflection on arithmetic (Carpenter, Franke & Levi, 2003). In this context, Brizuela and Schliemann argue for a reduction of the boundaries between arithmetic and algebra.

To achieve this, Kaput and Blanton (2000) suggest *algebraizing* teaching materials and developing teaching practices that are conducive to the development of algebraic thinking.

One of the most important difficulties that children who have not been exposed to algebraic activities in elementary school experience when they begin working with algebra is the recognition of the = sign as an indicator of a relation of equivalence, not an operator (Bodin & Capponi, 1996; Smith, 2002). This conception is already strongly anchored, even in early elementary school children (Carpenter, Franke & Levi, 2003; Vance, 1998; Falkner, Levi & Carpenter, 1999), preventing them from working with equalities that have a structure other than $a + b = c$. Sáenz-Ludlow and Walgamuth (1998) observed that third-grade children who conceive of the = sign as an operator transform equations like $2 + 4 =$ __ $+ 2$ into $2 + 4 = 6 + 2$.

Shoecraft (1989) reported that his research participants separated equations such as $a + b = c + d$ into two different equalities. For example, one of his students transformed $2 + 7 = 4 + 5$ into $2 + 7 = 9$ and $4 + 5 = 9$, to maintain the *question-answer* pattern.

Considering the importance of this obstacle, would it not be appropriate to put in place measures that allow children to develop an understanding of the = sign as an indicator of a relation from the beginning of primary school? Would such a measure prevent children from constructing limited conceptions about this symbol and from experiencing difficulties later on, when they are confronted with algebraic notation?

In this article, I want to show that a conception of the = sign as an operator is strongly anchored, since it can already be found in children of the first grade of primary school. I will also show (through an example) that there are activities that can help students overcome such a conception. Finally, I will show that it is not easy for such young children to *fully* understand the meaning of the = sign. I will also discuss the appropriateness of introducing this symbol in the first grade of elementary school.

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The = sign conceived as an operator

In my research, I conducted interviews with eleven first-grade children from a low socioeconomic urban area in the city of Luxembourg (Grand Duchy of Luxembourg). Then, I conducted a didactical experiment with six of them to teach a conception of the = sign as an indicator of a relation. In this article, I analyse more precisely the case of Mathieu, considered a high-achieving student by his teacher. The process of this child is particularly interesting because, despite his good academic performance, understanding the = sign as an indicator of a relation was as difficult for him as for less successful students. Furthermore, the strategies he used and the errors he made are representative of those observed with the other children I worked with.

In order to learn about Mathieu's initial conceptions of the = sign, I first asked him to evaluate several equalities, including 4 + 5 = 9 and 7 = 3 + 4. When looking at the first one, which was very familiar to Mathieu since it corresponds to the type of *calculations* he performs at school, his conception of this symbol seemed adequate.

Teacher Can you tell me if what is written here *[4 + 5 = 9]* is correct?

Mathieu Yes, its correct, because 4 + 5 is 9 *[Mathieu counts on his hands, shows 4 on one hand and 5 on the other hand, then counts all fingers shown]*

Teacher And this sign *[shows the = sign]*, what does it mean?

Mathieu It's the same thing *[…]* It's the same here *[5 + 4]* and here *[9].*

As soon as I offer him an equality that no longer corresponds to an expression of *calculations* (7 = 3 + 4), Mathieu's answer changes:

Teacher Can you read out loud what it says here?

Mathieu It's backwards *[Mathieu tries to turn the sheet of paper on which the equality is written]*

Teacher Is what is written right?

Mathieu No, it's backwards. It should be 4 + 3 = 7, and it's 7 = 3 + 4. It's backwards. It's not done right.

Teacher If we had put 4 + 3 = 7, would that be right?

Mathieu Yes, but, like that *[7 = 3 + 4]*, it's wrong.

This excerpt clearly shows that Mathieu does not consider the = sign as an indicator of a relation. Although he does not say so explicitly, he believes that the *answer* (the sum of 4 and 3) must come after the equality sign. By using a right-to-left reading of the equality, Mathieu is able to return to an equality where the answer immediately follows the equality sign. This conception of Mathieu is confirmed when I ask him to complete the expression 6 + 2 = __ + 3.

Teacher What number do you have to put in the space, to make it right?

Mathieu If it goes up to here *[immediately after the empty space, which Mathieu has to fill]*, I know the calculation. [Mathieu then crosses out '+ 3' and writes '8'].

Teacher And if we had said *six plus two equals something plus eight*, would that have been right?

Mathieu No, it would be wrong.

Here again, Mathieu returns to an equality in which the result of the addition that is on the left of the = sign immediately follows that symbol. He therefore considers once again the = sign as a prompt to write an answer.

Mathieu is not alone in experiencing these difficulties. They have been noted by other authors (*e.g.,* Vance, 1998; Sáenz-Ludlow & Walgamuth, 1998; Falkner, Levi & Carpenter, 1999; Kieran, 1992) and not only in first grade. For example, the students interviewed by Sáenz-Ludlow and Walgamuth were in third grade. Falkner, Levi, and Carpenter even found that this conception appears to be widespread among all elementary school children. When they presented the equality 8 + 4 =__ + 5 to 752 children of different primary grades, the success rate remained lower than 10% in both first and sixth grade.

Then, what are the origins of this conception and why is it already so strongly anchored in children who have just started school? A first hypothesis concerns the teaching that the children have received. In the Luxembourgish textbooks that were used in the classroom of the children I worked with, the meaning of the = sign is treated only in the context of comparisons of collections, and in conjunction with the signs: *smaller than*, <, and *larger than*, >. Similarly, the vast majority of the equations that children have to solve correspond to a structure $a + b =$. The conception that one must write an answer after the = sign can therefore be easily maintained in this type of activity. Finally, in the teacher's guide, no mention is made regarding how the meaning of the = sign should be worked through with children. In fact, the teacher of the classroom confided to me that she did not specifically intervene on the meaning of this symbol.

If the teaching received by the children influences the construction of a conception of the = sign as an operator, other elements, more related to the nature of mathematical learning of young children, can also play an important role. Thus, during the first spontaneous contacts with addition, the child first considers the parts to be added, and then finds the sum, which is in a way an answer.

Activities aimed at challenging students' initial conception

In order to determine whether children in the first grade of primary school are able to develop an understanding of the = sign as an indicator of a relation of equality, I conducted a teaching experiment. This allowed me to follow the evolution of the students throughout a teaching sequence of 7 half-hour sessions during which I worked with them individually. Different tasks prompted students to evaluate equalities or to complete equations. During my teaching sequence, I presented two types of additive structures to children. The first was $a + b = c$ or $a = b + c$ structures and the second was $a + b = c + d$ structures. At the beginning, equalities were always accompanied by a concrete representation made with plastic tokens of the same size and shape. I distinguished two types of tasks.

First type of task. Students were asked to *evaluate* equalities, with no unknowns. I asked the children to *determine* whether a statement was correct and, if not, to transform the *false equality* into an equality. For example, when I asked children to evaluate the equality 3 + 4 = 7 + 1, they had to evaluate whether this equality was correct, justify their answer and change one or more of the numbers involved to obtain a *true equality*.

Second type of task. Here children are asked to *complete* equalities that include an unknown. The concrete representation of the unknown can then take two different forms: first, that of a transparent plastic bag, into which the children must add a certain number of tokens, and also that of a non-transparent cardboard box containing tokens, the number of which the children must determine. This latter representation is similar to that used by Radford and Grenier (1996) who also represented the unknown as a hidden quantity in a teaching sequence on high school algebra.

The degree of support provided by the concrete representation gradually decreased, and towards the end of the work on a structure, children were led to complete equations working with written forms only.

Significant progress

Towards the end of my teaching sequence, Mathieu had made significant progress and was now able to conceive of the = sign as an indicator of a relation of equivalence in different situations. As an example, let us take a situation from the last session with Mathieu, where I asked him to determine whether the = sign can be inserted between 3 + 4 and 7 + 1. To complete the task, Mathieu could, if he wished, use objects that had been made available to him.

Teacher — Can you tell me if we can put a = sign here?

Mathieu — Let me see, I'm not sure *[he puts two subsets of 3 tokens and 4 tokens on the left sheet and two subsets of 7 tokens and 1 token on the right sheet]*. No. Because here *[indicates left]* there are 7, and here 8 *[indicates right]*.

Teacher — What would you have to change to put it in?

Mathieu — You would have to do + 1 *[replace 7 + 1 with 6 + 1]*

Teacher — Can you read what it says?

Mathieu — 3 + 4 = 6 + 1.

Teacher — Why is this correct now?

Mathieu — Because we have changed something now. Instead of 8, we have 7 on this side *[on the right]*.

In this situation, Mathieu understands the = sign as an indicator of a relation of equality. Contrary to the answers he gave in the initial interview, for him 3 + 4 = 6 + 1 is a correct mathematical expression. However, the obvious progress Mathieu made in the four weeks I worked with him does not suggest that this learning is easy for a child of this age.

A significant cognitive obstacle

Even though Mathieu managed to make progress throughout the teaching sequence, understanding the = sign as an indicator of a relation of equivalence remains a powerful cognitive obstacle for him. Three types of observations allow me to come to this conclusion. First, after the explanation of the = sign during the first session, Mathieu is reluctant to modify his initial conception of this sign. Second, understanding the = sign as an indicator of a relation of equivalence in a certain type of situation does not guarantee that the meaning attributed to this sign will be the same in other situations. Thirdly, only ten days after the end of the teaching sequence, during a post-test, I was already able to observe Mathieu's tendency to return to a conception of the = sign as an operator in certain situations. I give below some details concerning these three observations.

First observation: difficult construction of a new meaning for the = sign

In the first session, I placed two sets of 5 and 3 tokens of the same size on a sheet of paper on the left, and a set of 8 tokens on a sheet of paper on the right (see Figure 1). Mathieu had to determine if there was the same quantity on both sides and if it was possible to put the = sign.

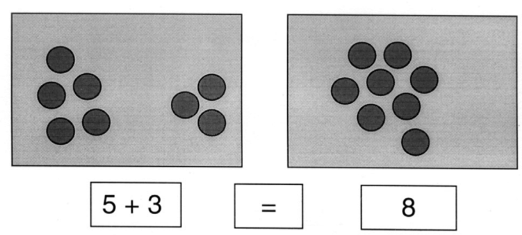

Figure 1. Representation of 5 + 3 = 8.

Teacher — Can we put the = sign between the two sheets? [*between 5 + 3 and 8]*

Mathieu — Yes, here *[shows the two sets of 5 and 3]* is a calculation and here is the result *[shows the set of 8 chips]*.

Teacher — What does the = sign mean here?

Mathieu — It's the same thing. Here, there are 5 and 3 *[shows the objects on the left]* and it's the same thing as 8 *[shows the 8 objects on the right]*.

Afterwards, the two cards were switched, and Mathieu had to decide again whether the same quantity of objects was present on each of the two sheets and whether the = sign could be inserted between the numbers 8 and 5 + 3 (see Figure 2).

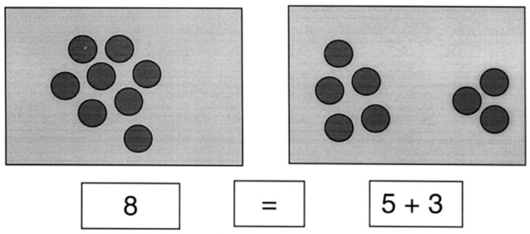

Figure 2. *Concrete representation of 8 = 5 + 3.*

Mathieu It doesn't work. You can't put the = sign now.

Teacher Why not?

Mathieu It only works if you do it the other way around *[5 + 3 = 8]*. But like this *[8 = 5 + 3]*, it doesn't work. We can't do the calculation.

Teacher Are there as many objects on this sheet as on this sheet?

Mathieu Yes

Teacher The sign = means that there is the same amount on one side as on the other side. If this sign means that there are as many on both sides, can we put the sign here?

Mathieu No. Here, there are 8, that's yours, and here there are these, that's mine. Does it look the same? I'm doing this *[joining 5 + 3 into a set of 8]*, because then we have as many, you and I.

Teacher So, can we put the equal sign?

Mathieu Yes, it works.

Teacher Can you read me what it says?

Mathieu It would be better to read like this *[from right to left]*. Because you can read the calculation, and after =, see if it's the same, but 8 = 5 + 3, it doesn't work.

If Mathieu accepts my explanation here, he nevertheless insists on a right-to-left reading of the equality. In this way, he succeeds in reducing it to a structure that remains consistent with a conception of the = sign as an operator. The conception of the = sign as a prompt to write an answer is sufficiently well anchored that all my participants tried, in one way or another, to maintain it.

Second observation: back to a conception of the = sign as an operator

Throughout the teaching sequence, Mathieu tended to regress in his understanding of the = sign by attributing the meaning of operator to it. This phenomenon appeared especially when Mathieu was confronted with situations that were new for him. In some situations, Mathieu *transformed the equality* in such a way that the = sign can be seen as an operator. In other situations, he insisted on a *right-to-left reading.* Here is an example of each situation.

Transforming equality: Toward the end of the teaching sequence, Mathieu needed to determine the value of the unknown in the expression __ + 4 = 2 + 8. To help him, the situation was represented with a non-transparent cardboard box containing six tokens and a set of four tokens on a sheet to his left, and, to his right, a sheet with sets of two and eight tokens. Mathieu had to determine how many chips were in the box if the = sign can be placed between __ + 4 and 2 + 8 (see Figure 3).

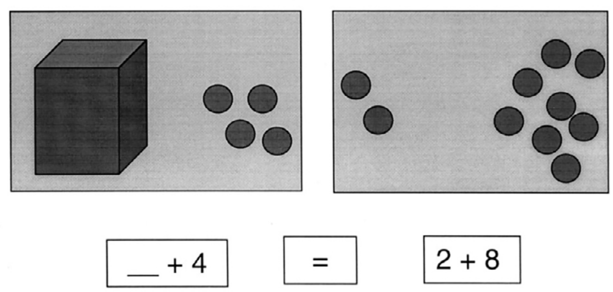

Figure 3. *Concrete representation of __ + 4 and 2 + 8.*

Teacher If I put an = here, how many are in the box?

Mathieu There are 4, because 4+4 is 8.

Teacher Can you read what it says?

Mathieu 4 + 4 = 2 + 8

In this situation, Mathieu chooses a number that allows him to have a sum corresponding to the last number of the equality (8). Even with the concrete representation in front of him, he ignores the set of two tokens, which leads him to transform the equality into a structure $a + b = c$ and allows him to maintain a conception of the = sign as an operator.

Right-to-left reading: Occasionally, especially in the early sessions, Mathieu uses right-to-left reading to transform an equality into an $a + b = c$ structure. For example, when solving the equation 7 = 2 + __ from the symbolic writing only, Mathieu first wants to write 9. Then, reading backwards seems to be the most appropriate strategy. "Can I also read the other way around *[from right to left]*?" When asked why he reverses the direction of reading, Mathieu explicitly refers to a transformation into a structure $a + b = c$, "The calculation is always put at the beginning, and the result after". This shows a real lack of understanding of the situation that prompts Mathieu to read the equality from right to left.

Third observation: low retention in the post-test

A third factor which illustrates the power of the cognitive obstacle associated with the learning of the = sign is Mathieu's limited retention observed during the post-test, only ten days after the end of the teaching sequence. After a rather short period, I observed the beginning of a return to a conception of the = sign as an operator.

While Mathieu generally used the = sign as an indicator of a relation of equivalence in the post-test, his learning remained rather fragile since, in certain situations, he began to return to a conception of the = sign as an operator. At the very beginning of the post-test session, Mathieu insisted on reading, from right to left, the equality 8 = 4 + 4.

Teacher Can you tell me if what is written here *[8 = 4 + 4]* is right.

Mathieu Yes, it is right, because 4 + 4 is 8.

Teacher Can you read what it says?

Mathieu 8 + ... No, that doesn't work. But you always have to start from the window side. 4 + 4 = 8.

Teacher Why are you reading this way now?

Mathieu Because you always read this way. You always have to start on the window side.

In the classroom, the windows are indeed located on the left side and their location coincides with the side of the beginning of the reading. For the post-test session, however, I had taken Mathieu to a different room, where the windows were located on his right.

This change of location was enough to encourage Mathieu to reverse the direction of reading the equality.

Mathieu's learning is thus far from being definitive, even if, during the teaching sequence, his explanations seemed convincing. The regression phenomenon thus provides a further indication that the learning of the = sign is a powerful cognitive obstacle. The representation of the = sign as an operator is strongly anchored in children and it is very difficult for the participants in my research to definitively overcome this conception.

Discussion of results

My work with first graders has shown that it is possible to make students of that age progress in their understanding of the = sign. This is not only the case for Mathieu. The other children I worked with also showed significant progress. These results are consistent with those of Falkner, Levi and Carpenter (1999), Carpenter and Levi (2000) and Behr, Erlwanger and Nichols (1980) who concluded that their work on the = sign in the first grade enabled children to progress towards a more adequate understanding of this sign.

Progress is not made, however, without specifically addressing children's conceptions of the = sign. The widespread conception of this sign as an operator in the pre-test and the difficulties associated with it show that, when the = sign is not explicitly taught, children will not understand it as an indicator of a relation between two numbers. Carpenter, Franke, and Levi (2003) came to the same conclusion: according to these authors, learning the = sign is not simply a maturation effect, but children's conceptions must be confronted directly.

Moreover, progress, visible in all the participants of my research, is not easy to achieve. I found, not only with Mathieu, but also with the other children I worked with, that the learning the = sign is a powerful cognitive obstacle for first graders.

With respect to children's resistance to changing their conception of the = sign, my results are consistent with those obtained by Sáenz-Ludlow and Walgamuth (1998) who found with third grade children that they have great difficulty detaching themselves from their initial conception of the = sign as an operator, even assiduously defending their point of view:

> The dialogues and the arithmetical tasks on equality indicate these children's intellectual commitment, logical coherence, and persistence to defend their thinking unless they were convinced otherwise. (p. 185)

As for the low retention of learning, my results go in the sense of those of Falkner, Levi and Carpenter (1999). After teaching the meaning of the = sign using $a = b + c$ structured equalities to preschool children, they found that this learning does not generalize to structures previously unknown to the children. When these same children are confronted with an equality of structure $a = a$, they admit that it is the same quantity, while refusing to write such an expression.

This fragility of learning that I have noted is also found in the research of Carpenter and Levi (2000), aimed at teaching the = sign to first grade children. The learning of the = sign by these children does not seem to be stable over time. Children who were initially able to use the = sign as an indicator of a relation of equivalence in different situations were unable to retain some of this learning. A few months after the explicit teaching of the = sign, some of them had returned to a conception of the = sign as an operator when evaluating an $a + b = c + d$ structured equality. Carpenter, Franke and Levi (2003) therefore recommend that children continue to be exposed to equalities that do not correspond to an $a + b = c$ structure.

The depth to which a conception of the = sign as an operator is already anchored in children in the first year of primary school, as well as the difficulty of getting them to overcome this conception raises the question of the relevance of introducing this symbol at the very beginning of primary school. Indeed, as Dougherty (2004) points out, an early introduction of the = sign as an operator means that children's conceptions will have to be undone later on:

> In order, then, for older children to solve equations with meaning, we have to first 'undo' their idea about the equals sign before any approach to solving equations makes sense. (p. 29)

Dougherty then suggests having children work on equality relations at the concrete level, in a measurement context, before addressing them in a numerical context:

> Showing that students can solve equations with different methods at an earlier age is encouraging. If they are capable of using these methods, even after coming from an almost strictly numerical perspective in their early beginnings in mathematics, what would be possible if students started with a focus on the structure of mathematics within a measurement context? (p. 29)

In relation to this observation, one may wonder whether it would not be more appropriate to delay the introduction of this symbol for a few months in order to ensure that the children are more ready to understand it as an indicator of a relation of equality.

However, it should be remembered that many children already arrive at school with a conception of the = sign as an operator. This sign is present in many children's books designed to teach counting, which necessarily leads children to construct a representation of it even before they enter school. I believe that it is important to take this factor into account when introducing the = sign at the beginning of primary school. Another approach would be to replace the = sign with another symbol (for example, an arrow) during the very first arithmetic operations on numbers.

Would such a measure help to avoid the development of conceptions that are difficult to challenge later on? There are many questions that call for further research on this subject.

Note

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