Some Unanswered Research Questions on the Teaching and Learning of Algebra

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> "Mathematics education research should be sensitive to context, and particularly to didactical intention, as a significant contextual element. It is unfortunate that the word didactics does not really exist in American educators' vocabularies—as if not using a word might make the thing it refers to disappear. M. Jourdain spoke prose without knowing it and American [1] educators follow didactical principles without admitting to them. Perhaps this is why educational research in America for the most part ignores the peculiarly effective power of the triadic teacher-learner-subject matter interactions that anyone would suppose must be at the very core of every educational endeavour." [Wheeler, 1989, p. 286]

The above quote occurs in the final paragraph of the final chapter of "Research Issues in the learning and teaching of Algebra" [Wagner & Kieran, 1989], and this paper began as a reflection on some of the issues raised in this book. What developed is an elaboration of my concerns about the predominant focus of algebra research in mathematics education.

Cognitive obstacles reassessed

Most of the earlier and well established research on children's understanding of algebra has been influenced by a Piagetian perspective [Booth, 1984; Collis, 1974; Herscovics, 1979; Küchemann, 1981]. The implicit assumption made in this research is that if pupils cannot perform satisfactorily on certain algebraic tasks then they have not yet reached the stage of formal operations. This is strongly reflected, for example, in the work of Herscovics who in discussing some of the research results on the teaching and learning of algebra focuses on the idea of a cognitive obstacle. "From the Piagetian perspective, the acquisition of knowledge is a process involving a constant interaction between the learning subject and his or her environment. This process of equilibration involves not only assimilation—the integration of the things to be known into some existing cognitive structure—but also accommodation—changes in the learner's cognitive structure necessitated by the acquisition of new knowledge. However, the learner's existing cognitive structures are difficult to change significantly, their very existence becoming cognitive obstacles in the construction of new structures." [Hersovics, 1989, p. 62]

In order to discuss my concern about this focus on cognitive obstacles I will describe two examples used by Herscovics [1989] in his chapter on "Cognitive obstacles encountered in the learning of algebra". The first is taken from the work of Collis [1974] who found that beginning algebra pupils view algebra expressions (e.g. $x + 7$) as incomplete. Collis explains this as being due to the pupils' inability to hold unevaluated operations in suspension, suggesting that it is not until the stage of formal operations that the pupil breaks away from the idea that algebraic expressions give unique results. At this point the pupil becomes able to accept these algebraic expressions as new algebraic objects. However results from our own studies of pupils programming in Logo have shown that pupils as young as 10 years old can accept and use "unclosed" algebraic expression in a Logo programming context (see for example Figure 1) and that moreover with appropriate experiences of Logo programming they can begin to accept these objects in a traditional algebra context. [Sutherland, 1989] Pupils also seem to accept without any difficulty these "unclosed" algebraic expressions in a spreadsheet environment. [Healy, Hoyles & Sutherland, 1990]

PANON 20

```
TO PANON :SIZE
FD : SIZE * 2
RT 90
FD (:SIZE / 3) * 2
RT 90
FD :SIZE
RT 90
FD (:SIZE / 3) * 2
RT 180
FD (:SIZE / 3) * 2
RT 90
FD :SIZE
RT 148
FD (:SIZE / 9) + :SIZE
BK (:SIZE / 9) + :SIZE
LT 148
BK SIZE
LT 90
BK (:SIZE / 3) * 2
RT 90
FD :SIZE
RT 45
END
```


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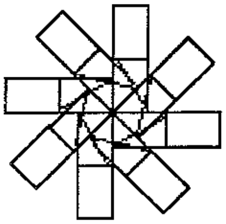

WINDMILL 20

```
TO WINDMILL :SIZE
REPEAT 20 (PANON :SIZE)
END
```

Figure 1
Operating on a variable in Logo

The second example of a cognitive obstacle is taken from the work of Wagner [1981] who found that students are not aware that the letters chosen to represent function variables in algebra are essentially arbitrary. Consequently they believe that a change in the name of the variable induces a change in the value of the function When pupils work with Logo functions they can see for themselves, as they experiment with changing the name of the function and the name of the variable, that changing these names does not effect the values generated by the function (see for example Figure 2). Consequently in the Logo context, in a situation in which the formalism is similar to the algebra formalism, pupils do not believe that a change in a literal symbol induces a change in the values of a Logo function. Tall questions this emphasis on cognitive obstacles suggesting that "Our curriculum, designed to present ideas in their logically simplest form, may actually cause cognitive obstacles." [Tall, 1989, p. 89]

a) TO FUNC :X
OUTPUT : $X + 4$
END

b)

IN		OUT
3	→	7
-2	→	2
1.5	→	5.5

c) FUNC (X) = $X + 4$
or $X \rightarrow X + 4$

Figure 2
Logo and algebraic representations of a mathematical function

There are at least two ways in which the computer environment could change the conceptions developed by pupils in the classroom situation. Firstly, as Tall has suggested, they can allow pupils to work with richer and more complex mathematical ideas than is usually the case at the beginning stages of school algebra. Secondly, in computer environments pupils are able to refine their own constructions by interacting with the computer, as opposed to having to be told that their constructions are incorrect by a teacher. In the traditional algebra context, for example, pupils can only know that changing the name of a function does not induce a change in the values generated by accepting the word of the teacher. Of course, the computer environment may also cause new misconceptions to develop. It is because these misconceptions are so tied to practice that it is difficult to view them as being linked to cognitive development.

Although I have discussed only two of the cognitive obstacles described by Herscovics [1989], I suggest that the idea of a cognitive obstacle needs serious re-questioning. We need to analyse in which ways the practice in the classroom may be contributing to the development of such obstacles. Herscovics suggests that we can rarely account for the errors which pupils make by studying the classroom situation. Teaching is viewed as a "problem of communication" and errors due to instruction are written off as being "the result of a formalistic presentation of the new subject matter." [ibid, p. 82] "for any mathematical concept that is new to learners, the best we can do is create conditions likely to enable them to complete the difficult process of accommodation." [ibid, p. 83] This view of teaching, inextricably linked to the idea of overcoming cognitive obstacles needs to be reassessed. It is in sharp contrast to the more Vygotskian idea that knowledge is socially constructed by pupils within the classroom. From a Piagetian perspective the role of language and symbolism is one of a translation process of an already understood and internalised method. Whereas from a Vygotskian perspective language plays an important role in the construction of knowledge. Thus the theoretical position adopted will strongly influence the view of the contribution of language in general and the contribution of algebraic language in particular to pupils' developing understanding. In much of the established research on pupils' errors in algebra we find the dominant and Piagetian view that pupils' obstacles are a result of their cognitive development, linked to the role of the teacher as someone who helps pupils overcome these inevitable cognitive obstacles. In this view, the teaching situation is not analysed as a possible source of the pupils' constructions and thus a possible source of the errors developed by pupils

Computer-based environments

Both Thomas & Tall [1876] and Thompson [1989] have found that in certain computer environments pupils view a literal symbol as representing a general number and do not succumb to many of the reported misconceptions developed in algebra (as reported for example in Küchemann [1981]) This is consistent with the findings of Logo research [Noss, 1986; Sutherland, 1989] and I suggest that the implications of these emerging results on both the theory and practice of mathematics education needs to be addressed by the mathematics education community.

At this point I shall describe in detail some of the results from our Logo research which suggests that pupil's developing algebraic ideas in Logo are related to the practices in which they have engaged. [Sutherland, 1989] The Logo Maths Project [Hoyles & Sutherland, 1989] was a study of eight pupils (aged 11-14) programming in Logo during their normal mathematics lessons. Within this activity pairs of pupils were able to take turns at working at two computers which were place in the classroom. The researcher acted as a participant observer and video data was collected for the four pairs of pupils throughout the three years of the project. The eight pupils represented a spread of mathematical attainment within the class. One of the aims of the Logo Maths Project was to probe in a longitudinal case study pupils' use and understanding of algebra-related ideas in a Logo context and relate this to their use and understanding of these same ideas in a "paper and pencil" environment. The following categories were used as a framework:

- Understanding that a variable name represents a range of numbers.
- Understanding that different variable names can represent the same value.
- Understanding the idea of "lack of closure" in a variable-dependent expression.
- Understanding the nature of the second order relationship between two variable-dependent expressions.
- Ability to use variables to express a general method.

Evidence for the longitudinal case study of pupils' understanding in Logo was derived from a detailed analysis of all the transcripts in which the pupils had used variables over the three-year period of research, together with the results of individually administered Logo tasks and an individually-administered structured interview. Evidence for pupils' understanding in an algebra context was provided by an individually-administered structured interview [2] designed to probe pupils' understanding from the perspective of the above categories (for a detailed description of these items see Sutherland [1986]). Many of the structured items were derived from the CSMS study. [Küchemann, 1989] So, for example, to probe whether or not the pupils understood that different variable names can represent the same value the following Logo and algebra items were given (not consecutively):

a) When is the following true?

$L + M + N = L + P + N$

Always Never Sometimes, when ..

b) When do these Logo commands draw the same length line?

```
TO LINES "L "M "N "P
FD :L FD :M FD :N
RT 90
FD :L FD :P FD :N
END
```

Always Never Sometimes, when ...

Figure 3
Example of algebraic and related Logo structured interview item

This methodology allowed us to begin to relate pupils' developing understandings of algebraic ideas to their Logo experiences. The first priority in this work was to refine the Logo classroom activities so that pupils began to use and manipulate appropriate algebra-related ideas within Logo. This took time and initially we were handicapped by the resistance of the teachers with whom we were working to the idea of introducing pupils to variables in Logo. They reasoned that because pupils have difficulties with algebra they are also likely to have difficulties with variables in Logo. For this reason we were initially cautious about how we introduced variable to pupils and attempted to introduce the idea as a means of generalising within the pupils' own projects. Analysis of our transcripts showed that this was often inappropriate. Pupils usually rejected the teacher interventions because they had not conceived the goal as one of generalising. However, if we presented pupils with the idea of variable in a teacher-directed task then they eventually began to take up this idea within their own projects. This need for relatively "heavy" teacher intervention, in order to provoke pupils to use the idea of variable in Logo, suggests that there may be a gap between pupil's arithmetical and algebraic thinking. This relates to the findings of Filloy and Rojano who suggest that "Theoretical and historical considerations seem to indicate that there is a didactical cut in the evolution line that goes from an arithmetical to algebraic thought." [Filloy & Rojano, 1984, p. 51] The most important result from this study was that the links which pupils made between variable in Logo and variable in algebra depended very much on the nature and extent of their Logo experiences. So, for example, it turned out that those pupils who could correctly answer the question about whether or not different variable names can represent the same value had used this idea whilst programming in Logo.

More recently we have been working with a group of 12-13 year old pupils. [Healy, Hoyles & Sutherland, 1990] Although not the main focus of this project, it has been possible to investigate the link between these pupils' developing understanding of algebraic ideas and their Logo experiences. We introduced the pupils to a number of variable letter tasks (see for example Figure 4) after 3-4 sessions of Logo in which the pupils had already become confident with the use of the beginning Logo commands and the idea of defining and editing a procedure.

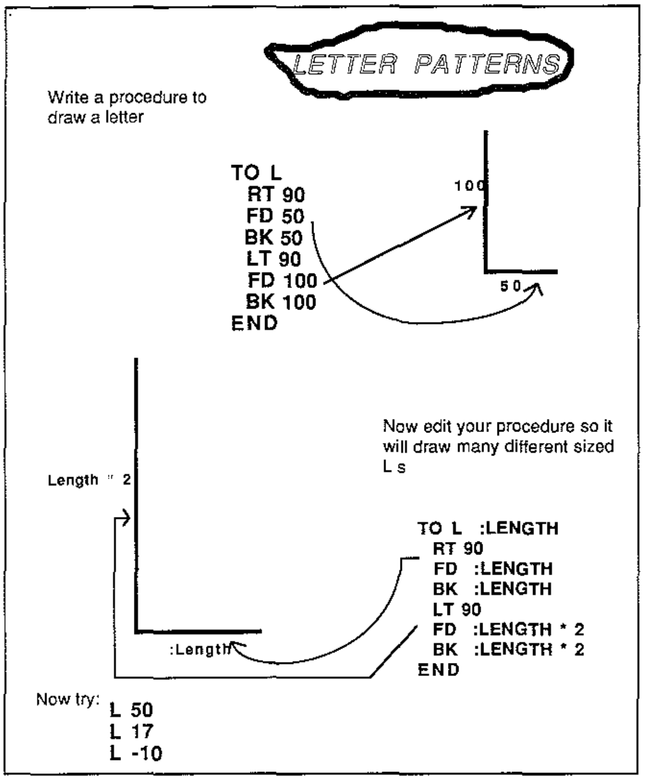

Figure 4
Variable letter task

The pupils were then encouraged to work on projects of their own choice (see for example Figure 1). We found that after between 4 and 10 hours of Logo, 12 out of 17 of these pupils could confidently operate on a variable whilst programming at the computer, and just over half (10 out of 17) could operate on a variable in order to solve a task away from the computer (Figure 5).

Write **one** procedure to draw this letter in different sizes.

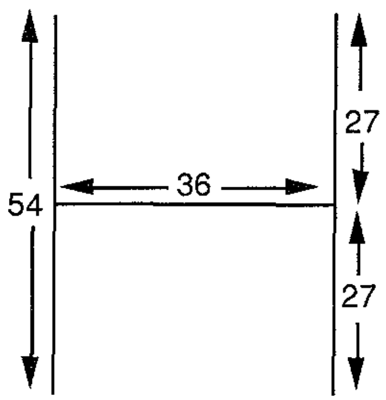

Figure 5
The "H" task: structured interview question

Table 1 presents a summary of the results of the structured algebra interview (identical to the one used in the Logo Maths Project) administered to the pupils at the end of the study. The responses to the test were categorised using the categories developed in the Logo Maths Project.

It can be seen from the table that those pupils who could operate on a variable to solve the "H" task performed better on the algebra interview than those who could not. The majority of these pupils who could operate on a variable understood that a variable represents a range of numbers (7/10) and accepted lack of closure in an algebraic expression (8/10). Even the overall proportion (3/17) of these 12-13 year olds who understood that a second order relationship exists between variable dependent expressions compares favourably with the figure of 6% for 14-15 year olds deriving from the CSMS study.

		Proportion of pupils who exhibit understanding of variable out of those who:		
		operated on a variable in "H" Task	used unrelated variables in "H" Task	did not use variables in "H" Task
CATEGORIES OF VARIABLE	Variable name represents a range of numbers	7/10	0/1	2/6
	Different variable names can represent the same value	6/10	0/1	2/6
	Acceptance of "lack of closure"	8/10	0/1	1/6
	Understanding the second order relationship between two expressions	3/10	0/1	0/6
	Using variable to express a general method	8/10	0/1	1/6

Table 1
Classification of secondary school pupils (n=17) responses to algebra structured interview categorised according to their solution to the Logo "H" task

More recently the algebra structured interview has been given to a group of seven [3] 10-11 year olds who have received no formal algebra instruction. These pupils have all been taught Logo by their class teacher. Although the number of pupils is small the results (Table 2) again indicate that it is the way the variable has been used in Logo which seems to effect the developing understanding of algebraic ideas. So, for example, if pupils have used "unclosed" algebraic expressions in Logo then they seem to be able to accept these ideas in algebra, even when they have had no previous formal algebra experience.

CATEGORIES OF VARIABLE	Proportion of pupils who exhibit understanding of variable out of those who:	
	operated on a vari-able in "H" Task	used unrelated vari-ables in "H" Task
Variable name represents a range of numbers	3/3	1/4
Different variable names can represent the same value	3/3	0/4
Acceptance of "lack of closure"	3/3	0/4
Understanding the second order relationship between two expressions	2/3	0/4
Using variable to express a general method	2/3	0/4

Table 2
Classification of primary school pupils (n-7) responses to algebra structured interview categorised according to their solution to the Logo "H" task

Again, the number who understand that a second order relationship exists between variable-dependent expressions (2/7) is higher than would be expected for this age group of pupils with no previous experience of algebra.

What emerges from analysing the results of the previous studies is that it is the experience of operating a variable in Logo which is most likely to influence pupils' performance on the algebra items. Additionally, the ways in which pupils use variable in Logo seems to be linked to the ways in which they are first introduced to the idea Without appropriate intervention pupils can develop the practice of adding unrelated variable to their Logo procedures without making any relationship between these variables explicit. So what can we conclude from these results? With instruction it seems that pupils can write Logo procedures in which they operate on a variable in order to make a geometric relationship explicit. Secondly, there is emerging evidence that pupils with these types of Logo experiences can correctly answer algebra items taken from the CSMS study, items which pupils with a traditional algebra background normally find very difficult. The studies we have carried out so far have involved a relatively small number of pupils but in an on-going project [Sutherland, 1990b], we shall be investigating the effects of computer-based experiences on pupils' development with larger groups of pupils.

The result of these Logo studies cause me to question the excessive attribution of pupils' misconceptions in algebra to their cognitive development. It seems to me that it is more likely to be the "algebra" classroom practices which are contributing to many of these misconceptions: how else can we explain that with different practices different conceptions develop?

The role of algebraic symbolism

As I have begun discussing in this paper, the predominant view of the role of algebraic symbolism seems to be that pupils first have to "understand" a mathematical problem and then have to "translate" this understanding into symbolic language. "The essential feature of algebraic representation and symbolic manipulation, then, is that it should proceed from an understanding of the semantics of referential meaning that underlie it." [Booth, 1984, p. 58] Kieran suggests that the difficulties pupils have with translating are still likely to persist even within computer-based environments: "I believe that the result of much of the research discussed in this chapter would be applicable to computer intensive algebra learning situations for the following reason: In this hypothetical algebra programme, there would probably still be the need to represent formal mathematical methods, that is, to formalise procedures and to symbolise them." [Kieran, 1989, p. 53] The implication is that the formalisation process is in some way an add-on and final stage of the algebraic process, a position which I have already suggested is influenced by the Piagetian view that language is grafted on to understanding. In contrast, Vygotsky views language as a crucial mediator of inter-psychological functioning and an essential agent in intra-psychological functioning. Vygotsky was predominantly concerned with the role of natural language, but he also suggested that "The new higher concepts in turn transform the meaning of the lower. The adolescent who has mastered algebraic concepts has gained a vantage point from which he sees arithmetic concepts in a broader perspective." [Vygotsky, 1934, p. 115] If algebra is a language which can structure thinking then we might predict that methods which present the algebraic language as a final translation of an already understood process will restrict pupils in their development.

When we consider certain interactive computer-based languages (e.g. Logo and spreadsheet packages), then pupils' learning of these languages seems to be much more related to how they learn natural language than to the way they are traditionally expected to learn algebra. For many pupils the computer-based symbolism is an essential tool in their negotiation of a generalisation, and the way they come to progressively formalise informal methods. The process is dialectical. Writing a general Logo procedure involves both identifying the mathematical relationships within a problem and making these relationships explicit with a formal language. There is evidence that naming and declaring the variables in the title line of a Logo procedure helps pupils come to terms with the algebraic relationships within the problem. The language is used as a means of expressing and exploring mathematical ideas, and the learning of syntax and semantics are inextricably linked.

The naming of the variables seems to provide a structure to the generalising process. In Logo pupils sometimes start by naming "too many" variables. As they proceed through the procedure definition process they often decide to remove "named" variables from the title line as they begin to make relationships between variables explicit within their procedure. As I have already said, pupils' use of the symbolic Logo language is not grafted on to an existing problem solution: the discussion in natural language and intermediate interactions with the computer play crucially interacting roles in the evolution of a formal Logo procedure. The following discussion taken from a transcript of Sally and Janet working on the arrowhead task illustrates this point:

Janet: But I don't know how we're going to do it .. we can get rid of JILL.
Sally: I don't know. We never done it, did we? .. JACK divided by 2 ... is it divided first? .. or what...
Janet: FD divide JACK ... divide by JILL
Sally: No.... Right. So we want it to go forward by half of JACK .. so would that be JILL?
Janet: Yeah I know .. but just forget about JILL for the moment .. how do we do it?

Although this process of negotiation by referring to variable names is not identical within a spreadsheet environment, there is evidence that the syntax of the spreadsheet environment provides important scaffolding which serves as a bridge between a pattern seen with respect to a specific case and its generalisation in the spreadsheet language. In the spreadsheet environment it is the use of pointing (to different cells on the screen) which seems to be an important intermediate language in the generalising process. As with Logo the spreadsheet language places certain restrictions on the problem solving process: so, for example, pupils are likely to generate a sequence using an iterative rule in a spreadsheet environment although the same pupils may use a universal rule in the traditional algebra context. [Healy, Hoyles & Sutherland, 1990] Thus the pupils' thinking and problem solving processes are moulded by the tools available in the language and by the ways in which they have first been introduced to working with these tools. If, for example, in the beginning stages of working with Logo pupils were asked to plan a problem solution away from the computer, then they might be more likely to use the Logo language as a final stage in the problem solving process. It seems that it is only by directly interacting with the language whilst working at the computer that pupils develop a way of using the language to express their mathematical ideas. Similarly the ways pupils work with the spreadsheet language will be related to the ways in which they have been introduced to this practice.

Kaput believes that the current mathematics curriculum is misdirected, having an undue focus on syntax as opposed to semantics. He believes, along with others, that if we increase the referential teaching of algebra most of the difficulties pupils have with learning algebra will disappear. Meaning for Kaput derives from linking representation systems: "Ours is a relational semantics. That is we do not assume the existence of absolute meanings or absolute sources of meaning. Rather, meanings are developed within or relative to particular representations or ensembles of such." [Kaput, 1989, p. 168] Kaput has been involved with the development of a number of computer programs which reflect these ideas. The key element in all these environments is the possibility of dynamically linking notation systems. One such environment is the Function Analyser, in which it is possible to make linear transformations to a graph (by using the arrow keys) and observe corresponding changes in the algebra representation. Or, vice-versa, a modification of the algebra representation produces a corresponding change in the graph. "Actions are supported in any of the several notations, with consequences inspectable in any of the others. In this type of environment the computations required to translate actions across representations are done by the computer, leaving the student free to perform the actions and monitor their consequences across representations." [ibid, p. 179]

Kirshner strongly disagrees with Kaput's perspective. "I believe that the human mind is uniquely fashioned to learn syntax as syntax and that current syntactic instruction fails not because it is syntactic, but because research has not begun to fathom the depth of complexity and intricacy required of syntactic performance in algebra. I believe that the natural predisposition of the mind is to approach new, structured domains syntactically. Regretfully, I will not be surprised to find that the computer environments designed to promote meaningful conceptual development result first and foremost in learning about button pushing on computer keyboards." [Kirshner, 1989, p. 197] Our experience with Logo and spreadsheet packages is that if syntax is introduced within an accessible, motivating, and interactive problem solving situation then syntax is learned with surprising ease. If on the other hand pupils are allowed to explore the syntax with no clear goal (either devised by themselves or provided by the teacher) then a button-pushing phenomenon does develop. We have described this as the "whizzy effects" phenomenon [Hoyles & Sutherland, 1989] and it could involve, for example, unreflective use of the REPEAT command. So it seems to me that it is the welding together of syntax and semantics within a motivating problem situation which is critical to the learning of the computer programming language Logo.

Conclusions

At a time when most mathematics educators are still clinging strongly to a Piagetian perspective, psychologists are focusing on the role of the social context as a critical and fundamental factor in cognitive development (see for example Light [1986]. Theories which stress the social construction of meaning seem particularly relevant for an understanding of the classroom situation. For some [Walkerdine, 1982] knowledge is developed within practices and from this point of view into algebra is not initiation into decontextualised knowledge but initiation into another social practice. This position predicts that pupils will not easily make links between one practice and another, which fits well with the findings that pupils make very few connections between their arithmetical and their algebraic practices. [Lee & Wheeler, 1989]

As Light has pointed out, "The focus of attention is thus shifted away from the abstract "epistemic subject" of Piaget's structuralist approach, towards the real child's experience in specific social contexts. Paradoxically the achievement of abstract thought is seen as context dependent and context driven." [Light, 1988, p. 235] Light goes on to suggest that a pragmatic approach [Goodnow, 1972] to understanding might offer as much as the universalistic cognitive development approach: "The child is apprenticed to a language and culture which are grounded in practical human purposes." [Light, p. 236] He also points out that the mathematical tools developed by a particular culture are very related to the social practices of that culture (see for example Gallardo's discussion of the development of negative numbers [1990]).

We know that pupils can and do solve mathematical problems without using algebraic symbolism, as did Diophantus. Can we develop a school algebra culture in which pupils find a need for algebraic symbolism to express and explore their mathematical ideas? The technology is now available to design many new mathematical environments. I suspect that these new environments will not replace the need for school algebra, but quite what the nature of this algebra should be is still an open question. My main concern is that a research community which firmly believes in the absolute nature of pupils' algebraic errors and their inevitable relationship to cognitive development is likely to reject the real potential of these computer environments and will not be able to constructively inform the creation of such environments. I do not wish to suggest that cognitive development is not a factor influencing pupils' developing understanding but merely that we need to know more about the influence of both context and language on pupils' development.

Notes

[1] This paper was written during a study visit to Mexico and I cannot help notice the over-generalised use of the word America I assume that what is meant is USA and Canada

[2] The structured interview consisted of questions taken from the CSMS algebra interview [Küchemann, 1981]; algebra questions asking pupils to generalise and formalise a geometrical pattern; Logo questions probing similar ideas to those used in the algebra questions; Logo questions asking pupils to write general procedures.

[3] These pupils came from a class in which Logo was part of the normal classroom activity The pupils chosen for the interview were the ones who had been introduced to the idea of variable

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