# Early Roots

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*Small children have many more perceptions than they have terms to translate them; their vision is at any moment much richer, their apprehension even constantly stronger, than their prompt, their all too producible vocabulary* [1]

The fact that many famous mathematicians began their careers early in life is well known. It is often dealt with by speaking of their extraordinary talent or even genius, but as though it had nothing to do with the nature of the discipline itself, in comparison for example with historiography or jurisprudence or state craft. By its nature mathematics offers some special modes of access, and sometimes of invitation, to the young. I know of no census as to the precocity of the many who could be called just good at this trade, but suspect the story would be similar. In any case early talent cannot be estimated retrospectively, mostly lost among the competing concerns of growing up. One must observe it directly. But the record of such observations is sparse, and we have not much beyond conjecture - in which I intend happily to engage.

I believe that the study of children's mathematical insights can bring one to a view of mathematics itself that might at first seem close to Platonism, which for that reason if no other, one should treat with some special interest. I made that suggestion in an earlier review article [2]. But the philosophy of Plato, read two millennia later, exists in more versions than one. I shall slant a special version. This version, debated by Plato scholars, allows divergence from the metaphysics of an ideal world of Forms.

The competing line of interpretation leads first to Aristotle, who was for many years Plato's student, almost always neglected in modern discussions of mathematics. By a short-cut I shall bring it to some recent philosophy, that of Charles Saunders Peirce (1839-1914). Peirce was the acknowledged founder of the general philosophical outlook he called Pragmaticism, or Pragmatism, that claimed the allegiance (with variations) of William James (1842-1910), John Dewey (1859-1952), George Herbert Mead (1863-1931) and many others. Contemporary discussions in the philosophy of mathematics have generally overlooked the latter three, who generally followed Peirce in their discussions of the subject.

But Plato, first. In the minds of the young, inborn mathematical Forms or Ideas are the least obscured, he seems to say, but all those shadows on the cave wall that at best suggests some *deja vu*, some reminiscence of the true Ideas. For children the access to these Ideas is least hindered. In his discussion of the Meno, David Wheeler gives a detailed account of Socrates' impeccable art in evoking the slave boy's latent geometrical understanding [3]. Wheeler leaves aside the context of Platonic metaphysics. In my story the metaphysics needs to be considered but only as a way of opening doors to inquiry about the nature of mathematical ideas, whether children's or adults'.

By all reports - not just those of his best student - the historical Socrates was indeed a great teacher, as such both loved and feared. His art appeared, however - as still it seems to many - to violate all sensible beliefs about teaching. Teaching must - must it not? - be a process by which admonition and knowledge are conveyed from teacher to student. That is common sense, what everyone has experienced and takes for granted! How, then, can one be a great teacher who only asks questions? Aha! The questions are sneaky, loaded, sophistical. That was a common reaction, as Plato's dialogues frequently reveal, from those discomfited by the Socratic art. Plato puts forward the only possible defense: what such a teacher may teach indeed cannot convey knowledge; that knowledge must be already latent, somehow, in the minds of those taught. The Socratic art, he says, is a kind of midwifery; what is taught is there ready to be born, needing only that patient questioning to assist in parturition.

Plato's earlier dialogues were devoted to recording and dramatizing his master's art, even sometimes leaving an examination half-finished, where questioning reached some pause. In these dialogues Plato vividly evokes the intent and style of the master. Along the way, however, the messenger is sometimes tempted to extend the message, digging more deeply under the paradox of the teacher who did not seem to instruct. I believe it was this hermeneutic temptation that led in the end to the full-blown metaphysical doctrine we call Platonism.

I spoke of doors that Plato opened but his metaphysics passed by. Opening them, one can begin to develop a view of the roots of mathematics in childhood, a view that suggests, I think almost compels, a more general view, of the nature of mathematics itself, of its history, of the network of converging and branching tracks evolved from such beginnings. In the observation of childhood one can recognize a kind of learning-by-abstraction that is vital, but perhaps less easily recognized, in adult experience.

In contrast with official Platonism, Aristotle's philosophy was essentially empirical. But his understanding of experience was very different from what we usually call "empiricism". The young Aristotle was for many years a student of Plato, as Plato had been of Socrates. He rejected Plato's view of a world of Ideas known independently of the world of nature. The Ideas were, rather, essential forms of things we find in nature, some of which we earliest learn. He rejected Plato's metaphysics rather casually, however, as though it had never seemed to him to be crucial to Plato's thought.

But to bring Aristotle's views of mathematics adequately into the context of present-day discussion is too large a job for this essay. I quote only one remark, appropriate to the present text. Speaking of practical wisdom, which as he says depends on mature experience of complex affairs, he contrasts it with mathematical understanding; the objects of mathematics do not evolve in that way, they underlie quite ordinary experience and are special distillations from it. They "exist by abstraction." Concerning the complexities of practical life the young, he says, lack adequate understanding, they "merely use the proper language, while the essence of mathematical objects is plain enough to them."[4]

Here is one child's version of the abstract world of number. After she had heard a name for "the biggest number" from another child - a trillion-trillion-trillion - and a waggish physicist friend, $-10^{69}$- she interpreted charitably: "They just meant it was the biggest numbers they had a name for." Why did she say that? "Well, you could always add one more." Many of us remember taking the comparable step to a spatial infinity, and the later trouble about a world that might be finite but unbounded. Could the numbers loop around like that? Never! One knew that.

Quite rigorously, I believe, one can show that any formal elucidation of the natural numbers must - not as a sufficient condition, but as one irreplaceably necessary - involve some literal and reproducible instantiation of them; of the first few, at least, and their sequence. In one of many stories one can invent, an early-times hunter-gatherer fisherman tied knots in a rawhide string - remote ancestor perhaps of the Inca quipu - for each member of his small band. He then tried to catch at least one fish for each knot. The first crucial step is that knotted string, or some equivalent standardization.

As part of his general semiotic theory, Peirce gives special attention to two such kinds of mathematical signs. One kind is primary, that he calls iconic. A sign that first represents any basic mathematical abstraction must itself possess the very structure which it represents. Thus the icon of three is always some standardized triple. That of any $n$ is an $n$-tuple. Such icons are part and basis of any arithmetical code. By themselves they give us a monary code. An iconic sign is typically portable or reproducible from memory. Painters' color charts are also icons. The charts that present primary-color mixtures, additive or subtractive, are icons of a mathematical kind.

But the use of iconic signs is not a sufficient condition for conceptual understanding; they are necessary, and in a way that deserves remark. The necessary possession and use of icons can be understood, of course, as a means only, a practical necessity. But what has previously been assimilated as means later becomes an object of reflective investigation. That brings about a reversal of ends and means characteristic of reflective thought.

But there are other mathematical signs that Peirce calls indexical, they just point to what is signified. Some are like demonstrative pronouns, they have meaning only when the thing meant is literally present in the discourse. If a sentence contains a demonstrative "this" or "that", and an apple is displayed, or a quintuple, or a mountain pointed to, then the apple or quintuple or mountain is its own icon, itself part of the sentence. Another kind of index points intelligibly to something that need not then and there be exhibited, a name or descriptive expression: thus the "3" serves to indicate some icon (*, *, *), while the indices 1, 2, 3 can in turn be elements of the icon (1, 2, 3), giving us back the cardinal icon composed of the ordinal indices. In such ways we have invented number codes other than the purely iconic. Thus the definition of a number in the decimal code is an even more complex organization of indices and icons; the sequence of digits is an icon for the powers 0, 1, 2,... of ten, the digits indices of the corresponding multiples. In any such code, happily, large numbers no longer need to be signified by icons, but by formulas which serves, *inter alia*, for constructing or identifying an icon (knots, pebbles) that would or does exemplify it.

In that way, defined first by the monary code, large numbers need no longer be exemplified. But always, be hypothesis, they could be. They are well-defined, waiting if you wish to be exemplified, but in any case already inhabiting a world of potentiality, standing arrayed there by themselves. They have mathematical existence, independently of the contingencies of nature.

In discussing such matters Peirce emphasizes what Quine later recognized as "metaphysical commitment", to the reality of numbers as objects, entities in their own right, no longer mere attributes of things in nature. Peirce makes light of it "Honey is sweet" can be transformed into "Honey possesses sweetness." What was a property (of honey) has become a kind of (fictitious?) thing, and the statement that of a relation between two things, one "possessing" the other. Such transformations he calls hypostatic abstractions, turning properties into things. Detached from adjectival use, hypostatized, the sweetness is still not fictitious, it is just as real as the taste! And there are different kinds of sweetness, as there are of honey. The numbers, similarly, now can be qualified by their own kinds or properties and the study of them transforms arithmetic into number theory.

So my central topic must be abstraction, meant in just the way Peirce (and Aristotle) intended it. For a first look, I go back to a fortunate two-year-old who, sitting on the kitchen floor with a few pots and pans and spoons, has dropped a spoon in a pot, then removed it, repeating this pair of operations over and over again: interatively, each time pleasurably. Adults may wonder how this monotonous repetition, 30 or 60 times, can be so absorbing. Could it be, to catch gravity napping?

"You would think a few times would be enough!" But should you? For a second look, observe a pre-school four-year-old building a long roadway across the block room, from one "city" to another. Iteration again, but now additive as well. A kind of entertainment often repeated in different contexts.

Along such developmental tracks the roots of two mathematical ideas are laid down: iteration and sequence among others, all underlined by repetition and accessible to abstraction. Those of number can come next, depending of course on the child and the human ambience: sequential ordering, then the one-one correspondence. In some such way young children may evolve the kind of meaning my fisherman knew, with portable or reproducible standards. Thus far, I think, abstraction of a distinctively mathematical kind need not have occurred, although an instrument has been found that can be taken from one kind of situation and dappled in others.

That abstractive step does take place, however, when the sequence of counters is acknowledged in detachment from practical use and considered, reflectively, as an object in its own right. Young children, morally supported in their investigations, are able to find and enjoy the essence of this abstraction. Whether or by how much they choose to explore beyond it, the door has been opened

Peirce rejected, as emphatically as anyone else, John Stuart Mill's notion that the propositions of arithmetic are simply well-tested empirical generalizations. Indeed, they do have verisimilitude; they are reflections of generic patterns of the world around us as we experience it, yet detached from it. When mathematical generalizations are "applied" in the sciences, they are describing the very kinds of experience that first gave birth to them. But such applications have wholly hypothetical status; the mathematics generalizes, by its art of definitions, beyond that empirical world. If certain empirical assumptions were correct, then certain empirical conclusions would follow. Thus: when considered *in abstracto*, every number has a successor, the sequence is infinite. It is imaginable that of every enumerable set of things in nature, there is only a finite number. That would not affect, say, the infinity of primes. Held apart from such questions, the numerical infinity neither accords with nor contradicts empirical fact. The hypothetical "if $n$, then $n$+1" is a Peano axiom. A child has laid it down: you could always add one more.

This seems a proper place to pause, to remark that the arithmetical apprehension of some small children may be "much stronger than their prompt, their all too producible vocabulary." Much of their thought and expression is enactive, iconic; the acquisition of speech is second language, not first. I want to say that the essential mathematical ideas of childhood belong to the first language, then only derivatively to the second.

How, next, to describe young children's available powers of geometrical iconography? Sometimes it seems more related to those of Picasso than of Euclid; they can become intelligible to adults as the latter learn (or relearn) the language of play - play with materials found eolithically or introduced by observant adults; sometimes also in painting or sculpture.

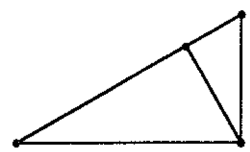
A town is sketched out on the ground in an area of a few square meters, complete with roads and road signs, an airport and a river. The river is made of thin small blocks. Waves on the water, deemed necessary, are made by the way the blocks slightly overlap. It is all an icon, or what Peirce sometimes calls a schema, a system of icons, proto-architecture, proto-geometry, with much attention to direction, to symmetry and dissymmetry. One of the planners brings for the zoo a too-large giraffe from the schoolroom. Despite its off-scale size his partner accepts it, but with hesitant politeness [5]

None of this iconic play needs as yet to be judged mathematical, reflectively abstracted from its early representative intent. We need suppose no sharp transitions, hard to catch even when they occur. Yet as with the early stages of arithmetical construction, geometry is on the way. Slice a raw potato once, for a plane; twice for a line; three times for a point. Shrink the potato to small size, and consider its location in relation to two wall and a ceiling [6] Invariance to scale is one inescapable facet of the learning, scaling the big world down. We all remember the picture of the child holding the container, printed on the container. A five-year old friend grasped immediately the iterative implication of the suggestion of a doll's house among the furniture of her doll's house: "And in the little doll's house there could be another doll's house... !"

Mathematics, Peirce repeatedly asserts, is an experimental science. Since mathematical signs are iconic abstractions from empirical reality, they can be manipulated much as scientists manipulate natural processes in the laboratory. I believe what Peirce had in mind were such "experiments" as we can make, for example, in searching for prime numbers, combining algebraic equations, inventing new geometrical designs. Conjectures arrived at in this way are not in the end established by empirical tests; but they can lay out the way to formal definitions and proof, often strongly suggestive of some idea of proof.

It makes matters clearer here to invoke a distinction of Peirce's between two kinds of deductive inference, which - with Euclid in mind - he calls corollarial and theorematic, and to which he attaches great importance. Corollaries are immediate inferences, special cases of more general propositions. Theorems, on the other hand, are found as consequences of some experimentally discovered combination of different and previously unrelated icons. He might have been thinking of the Pythagorean theorem.

There are a good many proofs of that theorem, some more immediate than Euclid's pons asinorum, but all depend on subsidiary constructions that either embed the right triangle in some larger structure already understood, or else subdivide it, as in the elegantly equivalent theorem which divides the triangle into two smaller similars



in which a single added perpendicular serves the "theorematic" need most simply. By dividing the triangle into two that are similar, it reminds us that what is true of the triangles will be true of their embedding squares

More generally: we all know the experience that by some analogy, hunch or sheer chance we "bring to mind" together two facts previously but separately known, then discover something new, something that they only jointly entail. Granted that $p \cdot q$ entails $r$, it does not follow from "A knows $p$" and "A knows $q$" that "A knows $r$". Such two facts may well have been stored on different occasions and in different parts of the mind's library; yet must be brought out together, attended to, before cross-indexing and discovery are possible. This I believe is the basis of Peirce's account of mathematics as an experimental science. One does not know in advance which propositions, previously isolated, will combine to form new conclusions. Deduction is the end result, but not the method.

Learning a few geometrical constructions, fourth-grade students were challenged to find ways of subdividing a given figure into congruent parts. Equilateral triangles were readily subdivided into 2, 3, 6 such parts, then finally into 4. With that construction, one student announced that she could also divide it into 9, or 16, or 25, on and on. She had previously discovered, in playing with plastic equilateral triangles, that when you put them together you can make larger triangles. With some urging from her teacher she had found, to her great surprise, that the numbers were not the triangular numbers that she and others had previously found. They were instead the numbers they had gotten when poker chips were packed in squares, now expressed in triangular shapes. She saw that you get the sequence by adding successive odd numbers (the Pythagorean gnomons). Instead of subdividing the big triangle into smaller ones, she reversed the process, and built it up from them. Theorem? Corollary?

In any case there was found what Peirce calls a schema, a generative procedure that can produce square numbers of triangles to make larger triangles, or squares to make squares, just by adding each time the successive odds. The actual procedure with the tiles might eventually fail, cumulatively, from slight irregularities in the plastic shapes, if not from boredom. But that would no longer matter; the schema itself does not fail. This is mathematics! Yet it derives, palpably, from children's experience, from geometrical play.

I use that story as a reminder that at the beginning of Greek mathematics there was one rather grand "theorematic" development, in which two whole clusters of prior knowledge, stored separately within the practical culture of ancient Greece, were brought together. Quite a bit of geometry had early been learned and reduced to practice by builders and surveyors, and of arithmetic, by merchants and other keepers of books. But there was no name for the union of these two arts that the Pythagoreans and others discovered, it could only be passed on and developed in teaching. Hence it was called *mathematica*, a word originally meaning, merely, that which can only be learned from teachers. I think "theory" was being invented.

A long tradition has robbed school children of the richness of this Pythagorean discipline, well within their reach. Leaving out any but the most trivial elementary school geometry, as we generally do, we concentrate many hundreds of childhood hours on algorithms of computation, calling this mathematics. The early Greeks knew better.

We know some of those early discoveries that came from this union of number and form, each complementing the other. The fourth-grade class of my story was "right on" to one of them. The discovery that most numbers can be classified under many kinds of shapes - triangular, rectangular, square, cube, and "prime", led not only to numerology but also to the beginnings of number theory and of measure theory, depending as it does jointly, on multiplicative and additive properties. The Pythagorean theorem and the discovery of the irrationals may have been conjectured first from such identities as $3^2 + 4^2 = 5^2$; or the reverse. Some basic theorems about areas and volumes could be derived from number patterns - for example, the volume of the pyramid, the tetrahedron, and other solids.

It stretches Peirce's term only slightly to say that this early junction of arithmetic and geometry was "theorematic" on a grand scale. In the subsequent history quite other areas of practical and scientific investigation have given rise to new orders of what Peirce called hypostatic abstraction. In confrontation with the older mathematics these became themselves mathematized.

A beautiful illustration of this extension is Archimedes' discovery that his theory of the unequal-arm balance could be used for "weighing" the volume of the sphere. The law of moments enabled him to extend the long-known properties of the circle to those of the sphere. This was no ordinary balance, but one he had mathematized: it weighed no solid weights, but abstract measures, areas and volumes. Metaphorically, also, he added a third dimension to the two that the Pythagoreans had brought together.

It does not denigrate Archimedes' genius to mention that this new abstraction can be very close to the enactive understanding, when they have been given ample opportunity to explore it, of young children. One should mention also that the theory of the balance gave Galileo a mathematical imagery for investigations in mechanics. And his geometry of motion represented the beginning of another major synthesis, later developed by Fermat, Newton and Leibniz: analysis.

A fourth domain was brought under mathematical scrutiny by Bernoulli and others. His theorem pointed toward new syntheses far beyond the simple mathematics of gaming; that was another beginning that led to "theorematic" confrontations, those between areas of experience previously isolated, and with results as radical as those the Pythagorean had initiated long before. When abstractions from such diverse areas of experience converge, new branches of mathematics emerge, and new levels of abstraction.

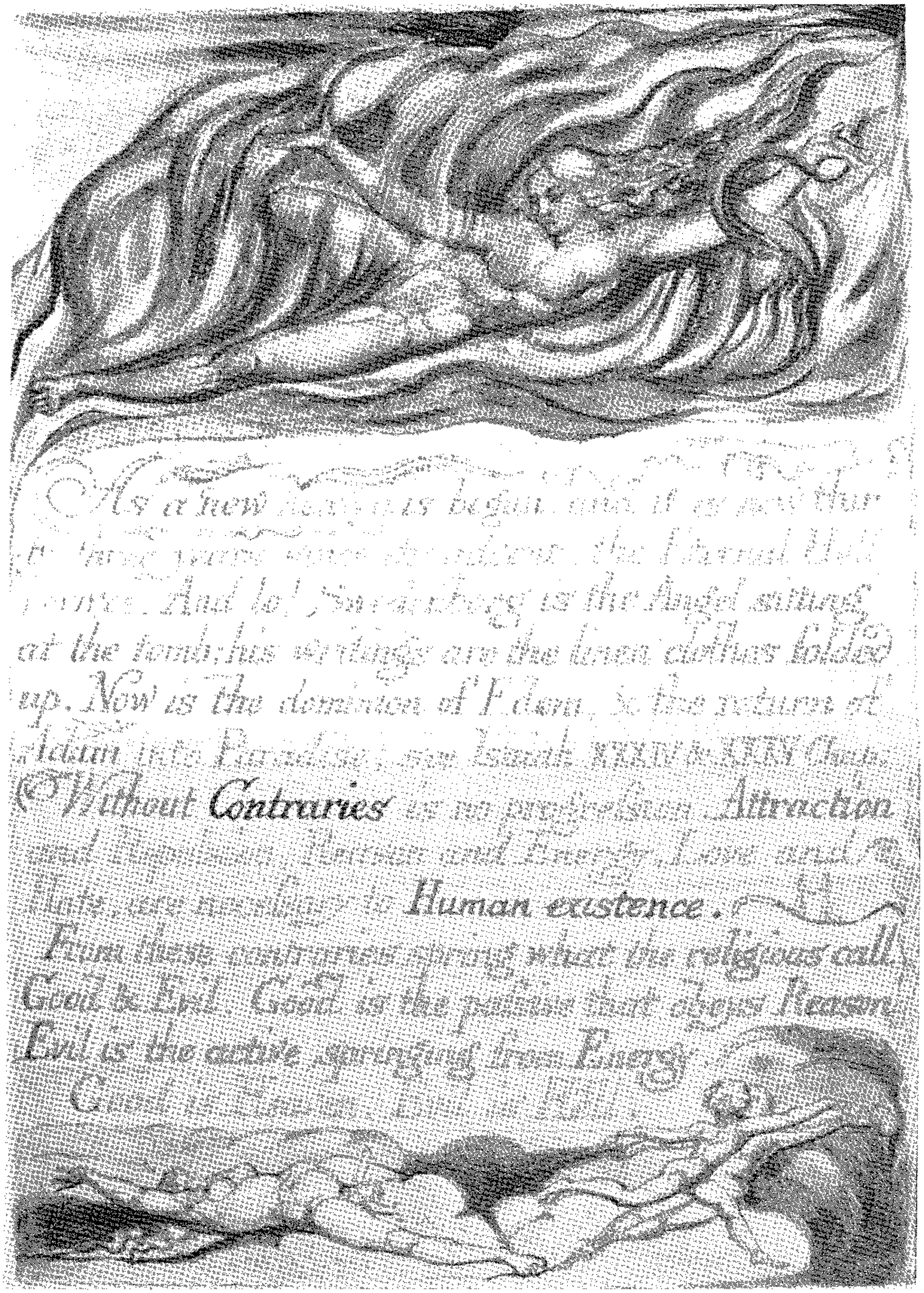
When we begin to open the doors that Plato first dramatized but then passed by, and begin to explore the experiential sources of this rich quasi-world of abstraction, we can begin to understand, I believe, how mathematics can at once help to extend our experience and reduce it to order; stand apart from the empirical sciences and yet, at the same time, map cities in the sand.

## Notes

[1] Henry James, *What Masie knew* - borrowed with permission, from a manuscript of Frances Hawkins. For her, as a teacher, it is a well-formed reminder of some thing essential about the estate of childhood. For Henry James, it circumscribed the lives of his great novel. Here I wish to take it, modestly, as a guide to the stuff of young-age mathematics, for which I believe it is singularly apt.

[2] D. Hawkins, "The edge of platonism", *for the learning of mathematics*, 5.2 (1985)
[3] D. Wheeler, "Teaching for discovery", *Outlook*, 14 (1974)
[4] Aristotle, *Niocomachean ethics*, Bk VI, 1142a, 12-22
[5] F. Hawkins, "Turn here", *Outlook*, 25 (1977)
[6] M. Vicentini, personal communication

# Without Contraries is no Progression…




William Blake, page from *The Marriage of Heaven and Hell*, 1790