SEEING THE CONSTRUCTION OF A MULTIPLICATIVE WORLD

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In a quest to make sense of first-year college students' understanding of mathematical concepts through individual interviews, I chose a series of tasks focused on the same concepts using different representations. The first of a series of four tasks involved my telling of a story related to what I considered to be a pictorial representation of the logarithmic function. One telling of the story from an interview transcript is provided here. [1]

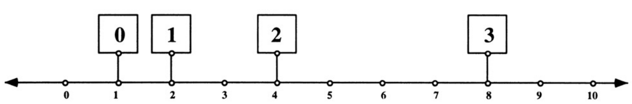

Figure 1: Representation used in task (adapted from Jacobs, 1970).

> I am going to start out by telling you a little story about this [Figure 1]:
>
> There is this guy and he gets in his car in the morning. He says, "I'm going to take a trip." And so he starts from his house and he drives 1 mile and he sees a giant sign with a 0 on it. He says, "Whew, what's this? This is very strange." So he drives 2 miles from his house and he sees another sign that has a giant 1 on it. He says, "Uh, what is going on here." So he drives from his house, he drives 4 miles and he sees a 2, he drives 8 miles, he sees a 3. He says, "What is going on?" But, then a light bulb comes on and he says, "OK, I know what's going on. I know where I'm going to see the next sign."

I asked students to predict numbers above the number line (signs) and below the number line:

- What sign do you think will be above the number 64 {256, $\frac{1}{2}$, $\sqrt{2}$, 3}?
- If a sign had the number **7** {**–7**, $\frac{1}{2}$, $\frac{3}{4}$, $\sqrt{2}$} on it, what number would be below the sign?

Following the completion of the task, I asked students to reflect on their activity in two ways. First, they were asked to draw a picture of a bulletin board to display findings from the investigation. Second, they were asked to "write down everything" they knew about the relationship between the sign numbers and the number line numbers.

To make sense of what I learned, I share my shift in attention from the logarithmic function, a structure known and defined in the mathematical world, to my model of students' construction of their mathematics. Steffe (1988) identified this shift in focus and cautioned against a focus on mathematical structures in the teaching of mathematics. "Our highly developed adult language may obscure our perception of the primitive operations of children" (p. 121). Steffe made a distinction between the *mathematics of children* and *mathematics for children*. He noted that much of the time children are subjected to mathematics for them, rather than provided with opportunities for the growth of their own mathematics. Also of importance is the perspective of the teacher and the practice of building models of the children's thinking (*mathematics of children*). These models can then be used as the basis for designing opportunities for the children to build mathematics.

Steffe's distinction, and the subsequent elaboration of the use of models in constructivist teaching (D'Ambrosio, 2004; Steffe and D'Ambrosio, 1995), applies equally well to my view of operations of first-year college algebra students. Although I attended to the students' actions, my observations were named in terms of the students' progress toward an understanding of the logarithmic function – the mathematical idea of the logarithmic function from my mathematical world. I described students' understandings of the logarithmic functions using my understanding of mathematics rather than my vision of their mathematics. One student understood the function in terms of exponents, while another understood the function in terms of the recursive action of multiplication by two. Using this way of looking at the students' work, I found very little. Students did not 'see' the logarithmic function in the problem, despite their recent success in class activities and assessments on the function. No student, of the six from three classes I observed and interviewed, used the logarithmic function to solve the task and only one used exponents. I was forced by my own view to assert that the students understood very little about the logarithmic function.

What I had overlooked was the *mathematics of the students* as described by Steffe (1988). The logarithmic function was neither an object nor an action for the students. It was not in the mathematical worlds they had constructed. This shift in attention allowed me to make conjectures about the mathematics of the students and use existing mathematics education frames to make sense of what I saw. Using this way of looking, I found quite a lot.

In this article, I discuss what I found as I revisited the students' activity and their sense-making actions. I argue that the students' actions are constructions of multiplicative worlds as described by Confrey and Smith (1995).

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Creating a multiplicative unit

Confrey and Smith (1995) describe a multiplicative world that they conjecture may allow for a meaningful construction of growth functions such as the exponential and logarithmic functions. In the multiplicative world, objects are multiplicative units and the action is multiplication. As in the additive world, it is necessary to develop composite units that are simultaneously a whole and a combination of parts.

Using a continuous whole, for example 3, a learner can think of the units 1 and 2 within 3. He or she can also think of fractional units within the whole, for example,

$$3=\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\ .$$

Units within the whole are called parts of the whole and can be summed to make a whole. In a similar way, a unit can be thought of as belonging to a multiplicative world. If the multiplicative unit is 3, then parts of the unit could be $\sqrt[2]{3}, \sqrt[3]{3}, \sqrt[4]{3}$, for example. In the multiplicative world, fractional powers of the whole are the parts (Confrey and Smith, 1995). Using the action in the world, the parts can be composed into the whole, $3=\sqrt[4]{3}\cdot\sqrt[4]{3}\cdot\sqrt[4]{3}\cdot\sqrt[4]{3}$ or into other parts of the whole such as $\sqrt[4]{3}\cdot\sqrt[4]{3}=\sqrt[2]{3}$. The beauty of this idea is also its difficulty.

Confrey (1994) illustrated how a student might have difficulty constructing a part of a multiplicative unit using an example. The student was asked to "place a series of dates and events given in a tabular form in scientific notation (for the most part) on a number line" (p. 322). The student had difficulty placing values between 1×10^8 and 1×10^7, and initially "considered splitting the interval in half and labeling the middle 1.5" (p. 322). This example illustrates both the tension between and the difficulty coordinating "two competing frameworks" (p. 322): an additive world and a multiplicative world. The students' work that follows illustrates their possibilities and difficulties as they began to construct a multiplicative world.

Nora constructs a multiplicative unit

Nora was a fascinating young woman on a quest to be the first in her family to graduate from college. She saw her academic career as a challenge and her performance in mathematics classes as a competition. In her own words, "I want to be the best in the class." Nora excelled in her college algebra course. Her work on the problem in Figure 1 illustrates the development and use of a multiplicative unit. Nora was excited about the task and immediately began looking for patterns as I watched and listened. She examined the spaces and "dots" between numbers with signs.

> N: It skipped three [referring to 5, 6, 7 in Figure 1] and went to **3** [on the sign] and so it seems like whatever number is coming up [on a sign] is how many places [referring to integers] it is going to skip.
>
> S: So how would you know what the next number was going to be?
>
> N: I would probably skip four spaces or four dots.

Nora used her observation that there are three "dots" between 4 and 8 to predict that there should be four dots between 8 and the next number with a sign above it. To check her prediction, she extended the number line, but ran out of space and resumed the drawing on another page (see Figure 2).

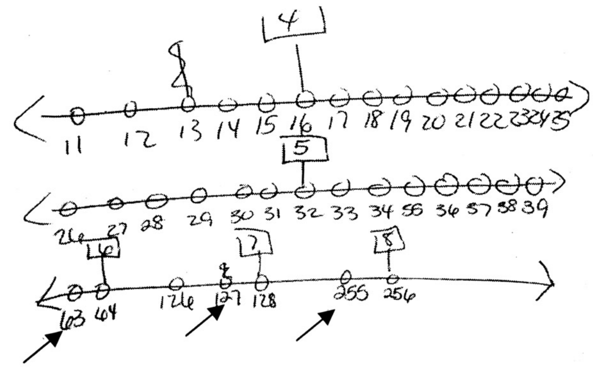

Figure 2: Nora's drawing of the number line task.

The line above 13 in the drawing illustrates Nora's prediction for the number below the sign with **4** on it. She counted four "dots" past 8 {9, 10, 11, 12} and drew a line over 13. However, she paused having observed another pattern.

> N: Now my numbers are running together. So that would be [Nora counts 9, 10, 11, 12] one, two, three, four … [long pause]
>
> S: What?
>
> N: [In this quotation Nora is referring to a pattern in the numbers] I noticed that if you do like 1 plus 2 is 2, there is the dot 3. 4 plus 3 is 7 and there is the dot for 7 and then it [referring to sign **3**] is on the next one [number 8]. I want to say that it is going to be more like 2 plus 1 is 3, there is the dot for 3. There is no sign, but then if you go to 4 there is a sign. But then, if you add 4 to 3, which would be seven, there is the dot for 7. Then on the next number [8] is a little line [referring to line for sign **3**].

Nora identified a pattern for which she had two examples in the given data. She summarized her findings noting that the sum of 1 and 2 is 3, which precedes 4, a number with a sign over it. Similarly, the sum of numbers 3 and 4 is 7, which precedes 8, the next number with a sign. Nora used this method to predict the numbers that would correspond to signs **4**, **5**, **6**, **7**, and **8**. She carefully identified numbers that preceded numbers with signs above them in her diagram (see arrows on Figure 2).

Nora's prediction procedure, based on her observation of patterns in the numbers on the number line, was efficient and effective for her. So much so that she did not modify her thinking until she faced a problem to which her procedure did not apply: predicting the sign above the number **½**.

Nora quickly estimated that the sign above **½** would be "point five" [**0.5**]. Nora tried several times to reverse her prediction procedure using subtraction. The approach that appeared to provoke her creation of a multiplicative unit began with Nora finding successive differences between numbers with signs:

> If I did 8 minus 4, I would get 4. And minus two… And 2 minus 1, I would get 1.

She summarized her thinking:

> If you subtract what you've got, then you get what you get.

This statement confused me, so I asked for clarification:

> If you divided [a number] by two you would get the same one [result you would get using subtraction]. And then that would probably, more than likely, be right.

Nora tested her conjecture by dividing:

> If you divided 8 by two you get 4. And if you divided two into 4 you get 2. And if you divided two into 2 you get 1.

This statement marked Nora's first use of a multiplicative unit, perhaps created to explain her subtraction process to me.

Nora verified her conjecture by dividing each number with a sign by two to predict a predecessor. Using this method she quickly predicted signs for ½, ¼, and negative integer powers of two. Nora's activity while making these predictions prompted her to revise her procedure for finding numbers below signs:

> You could multiply by two, because 1 times two is 2, 2 times two is 4, two times 4 is 8.

I view this simplification as the completion of Nora's construction of the multiplicative unit for this task. The unit can be used to generate integer powers by repeated multiplication and division.

Nora's path to the use of the multiplicative unit began with counting. She quickly identified an interesting pattern in the pictorial representation, spaces and dots, which allowed her to develop a prediction procedure built on addition. The multiplicative unit was only generated and used when Nora had difficulty reversing her procedure to predict the sign above ½.

Ken's multiplicative unit

Ken was a polite young man who did not like mathematics very much. Although Ken was successful in elementary school mathematics, he identified ninth grade (14-15 years old) as the origin of his difficulty with the subject. Ken struggled in college algebra and hoped to pass the course, which he did. His goal was to obtain the degree his father insisted he have before he joined the family construction business.

Ken's initial focus was on the numbers with signs and finding a relationship between them.

> If you go from 0 to 1 [on the number line], that is … you just go one. And that's where they [the signs] start, **0**. Then you go from 1 to 2 … you double 1, one plus one. And then you know on the numbers at the top [signs] it goes up one …. You double 2 and then your next sign on 4 is **2**, because if you double 4 you get 8. The sign above that is **3**, so the next one has got to be double 8, which is 16, then 32 ….

I was unsure what Ken meant by the term "double." The initial term in Ken's sequence and his statements "you just go one" and "you double 1, one plus one," indicated to me that he might have been using addition to make sense of the data presented in Figure 1. To clarify, I asked Ken what he meant. He noted that, "you can't do four times four. You have to do four plus four." He then quickly added, "you multiply them all by two is what it is." Based on this statement, Ken may have only used the phrase "one plus one" to clarify his thinking for either himself or me. If addition was used to make sense of the task, it was only a transitional approach, quickly abandoned for the more explanatory, multiplicative unit two. There is no evidence that Ken was counting "spaces or dots" as Nora had, instead he appeared to be using each number with a sign to predict the next. Ken focused on the terms in the sequence 1, 2, 4, 8, …. This focus and use of the multiplicative unit was articulated in Ken's written explanation of the relationship he saw between the numbers below the number line and signs.

> On the # line, from 1 through infinity, you multip. by 2 and increase the sign above your answer everytime by one.

Ken used his unit as an intensifier, repeatedly multiplying and extending the sequence as he went: 1, 2, 4, 8, 16, 32, 64, 128, 256, …. His approach to my request that he predict the sign above the number ½, illustrates his continued use of the multiplicative unit in this way. Ken extended the number line to –2 (see Figure 3).

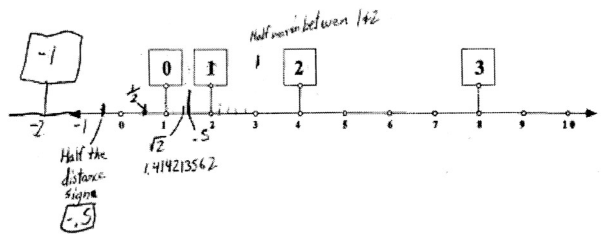

Figure 3: Ken's extension of the number line.

Since Ken did not explain his actions as he drew, I was unsure how to interpret them. I asked him to explain.

> I just continued my line on out and wrote [numbers] –1 and –2. For –2 the sign would be **–1**. And then half of that would be … your last sign is at 1 and your sign is **0**. And then you have to come back three [units] to –2 to your next sign … Half of three is one point seven five. No, one point five I believe. And, you know, this is half way right here [writes *half the distance* on his diagram]. And so, you know, if this is half and you have **–1** here [corresponding with number –2], so it [sign corresponding to *half the distance*] would be negative point five [**–0.5**] right here [writes *sign –.5*]. And then this [sign corresponding with ½] looks like about negative point one five [**–0.15**]. Maybe negative point two [**–0.2**].

Two ideas immediately struck me. First, that Ken was extending the number line to the left in an unusual way. In retrospect, I wondered what his approach meant in terms of his multiplicative unit. If we focus on the sequence Ken generated, then a plausible reason for his extension emerges. Ken was extending the sequence of numbers 1, 2, 4, 8, … whose signs were, in sequence, **0**, **1**, **2**, **3**, …. His prediction method may have been based on his observation about the numbers in the sequences. Ken noticed the sequence of signs, **0**, **1**, **2**, **3**, … decreased by one to the left, thus Ken predicted that the next sign to the left should be **–1**. To use his multiplicative unit to extend the number sequence 1, 2, 4, 8, … to the left, Ken developed a novel approach. He multiplied 1 by negative two and thereafter applied the multiplicative unit two. This action created a reflection of the positive portion of the number line. For Ken, the growing distances between signs were either not observed or not relevant. His focus was on the number sequence he was creating.

The second interesting idea Ken explained was his use of linear interpolation to predict the sign for ½. Ken associated the sign, **–0.5**, above the number –0.5, the midpoint of the three unit segment between –2 and 1, with the midpoint of the unit length between signs **–1** and **0**. He used the same method to predict the sign above number 3 was **1.5**.

Several explanations for approaches like this one have been shared in the literature (Stavy and Tirosh, 2000), including the focus on proportional reasoning and linear relationships in school mathematics. This method of sense making – used with so much success in school and cultural settings, such as in purchasing candy: If one piece of candy cost 5 cents, then how much will five pieces of candy cost? – may be an obstacle when students attempt to model situations seen as non-linear by experts. For me, the central question was how to create cognitive dissonance that might lead to the elimination of the obstacle and the construction of parts of multiplicative units. My investigation of Demetrius' work suggested one possibility.

Demetrius and a part of the multiplicative unit

Demetrius was a pleasant young man who dreamed of becoming a special education teacher. While he had always had trouble with mathematics, Demetrius continued to value the subject. He noted that mathematics was important and he wished he could understand it. Like Ken, Demetrius struggled with, but passed, college algebra – his last required mathematics course.

Like Nora, Demetrius's identification and use of a multiplicative unit evolved from his counting acts. He, too, used a multiplicative unit two to make predictions for numbers corresponding to integer signs:

$$\left\{\frac{1}{256},\frac{1}{2},1,2,4,8,16,32,64,128,256\right\}.$$

For Demetrius, it was the prediction of the sign corresponding to numbers between those in the previous set that caused difficulty. I asked him to reflect on his activity and identify sources of confusion:

> D: I would say the thing that really flunks [confuses] me the most was when the distance messed with me, more than anything.
>
> S: Like when you were trying to find the sign over 3.
>
> D: Mainly when I was trying to find the sign of 1.4 [approximation for √2] or a half [½] or something right in between. That is way shorter than going from 4 to 8.

Demetrius noticed the change in distance between signs. The distance from 4 to 8 was larger than the preceding distances between numbers with signs. Demtrius's predictions of $1\frac{1}{4}$ and $1\frac{1}{8}$ for the sign above 3 suggest that he tried to account for the changing distance. He selected numbers less than **1.5**, the prediction that other students made for the sign above 3. Demetrius appeared to struggle with "two competing frameworks" as had the student in Confrey's (1994) study. He seemed to realize that his multiplicative unit two was insufficient for the task of predicting signs for some numbers. As he tried to find the sign associated with 3 he expressed his struggle,

> See, what we have been working with all day is multiplying by two, but I don't have no kind of way of knowing what to multiply [2] by to get that [number 3].

Demetrius seemed to realize something new was needed. He suggested intermediate numbers for signs that were consistent with an interpretation of his actions as an attempt to coordinate a part of his multiplicative unit 2 with some part of his additive unit 1 to make predictions for √2 or 3.

Reflecting on my role in students' constructions

These three stories illustrate different paths adults may take to and from a multiplicative unit. My focus on and analysis of the actions of the students, rather than on how their activity could be connected to the logarithmic function, allowed me to begin to build models of their multiplicative worlds. This analysis and my investigations have raised additional questions for me.

The pictorial representation I selected was loosely based on the idea that number lines on which terms from geometric and arithmetic sequences were represented might support an investigation of students' thinking about the logarithmic function. The associated story was created to support and ease students into the exploration. However, the story may have encouraged students to use an additive or counting approach to the task. While the distances driven, rather than distances between signs, were reported, experience driving a car may have provided support for additive reasoning. Nora commuted to school approximately twenty miles each day. Ken and Demetrius both lived on campus, but drove home each weekend to visit their parents. These experiences may have made counting and additive approaches more likely. A story that suggested that the signs were the result of multiplicative action may have eliminated such initial counting acts.

While Demetrius's work suggests that constructions of parts of a multiplicative unit were within his reach, how could I stimulate this growth? As the interviewer, I provided very little stimulation. The students struggled with the task and in some cases left feeling discouraged and frustrated. One factor that may have contributed to the students' feelings was my unflinching resolve to observe students' understanding of the logarithmic function and not to teach or assist. My ability to generate questions that might result in a construction of a multiplicative unit was also impaired by my rationale for posing the task. I was determined to explore links between students' thinking on this task and their thinking about the logarithmic function. Seeing and hearing the student work with these filters, in addition to the limitations of the story context, made the crafting of questions to test conjectures about their constructions of a multiplicative world impossible.

It is possible that asking Demetrius to tell me more about the distances he saw could have helped. This reflection might easily have led to further conjectures about the growth of the distances between signs or may have allowed me to gather evidence that I could use to pose better questions. His approximations for the sign above 3 illustrate that questions about distances may have been appropriate.

Some mathematics educators might suggest that I could encourage students to use graphs and tables to represent their data. Would these other representations allow for exploration of or a different view of the data? Perhaps they would, however this is a fine line and brings us back to Steffe's discussion of the mathematics of children. Further investigation or questions should be developed and framed using the mathematics of the students. Even as I attempted to stimulate the use of other representations, I avoided directing the students to use a particular representation in this interview. I asked them to represent their findings on a bulletin board, but did not direct them to use a particular representation.

None of the students represented their data using either a table or a graph. There are several possible interpretations for this. First, the pictorial representation in the task may have limited the approaches students took to the problem. Students seemed to think in terms of the picture. Evidence of this interpretation can be seen in the representation drawn by Nora and Ken – extensions of the original number line. Another interpretation might be that the students did not see graphs and tables as useful tools. In their classes, these students saw and made graphs and tables by hand and with their calculators every day. Yet faced with quite a bit of data, they did not use these tools. I speculate that these tools, so useful to mathematics educators, were not seen as useful to students in their problem solving. Rather than suggest or encourage the use of tools from my mathematics, I hypothesize about the objects and actions used by the students and design tools based on those actions.

Using the lens of the multiplicative unit as a tool has allowed me to explore how a new lens can change my view of students' thinking. Returning to this data has also forced me to consider how my tasks and perceived role as an interviewer or as a teacher may impede students' actions and ultimately growth of their mathematics. This growth may not always forge a path with which I am familiar or that I immediately understand, but it is certainly the one most meaningful and useful for the students – and it is this meaning that I wish to stimulate.

Notes

[1] This, and other, excerpts from tape-transcripts of interviews included in this article have been lightly edited for readability. In addition, the following conventions have been adopted to limit the need for explanatory notes in the transcripts:

- When reference was made to a numeral below the number line, the numeral is used.
- When reference was made to a numeral on a sign, the numeral in bold is used.

References

Confrey, J. (1994) 'Splitting, similarity, and rate of change: a new approach to multiplication and exponential functions', in Harel, G. and Confrey, J. (eds), *The development of multiplicative reasoning in the learning of mathematics*, Albany, NY, State University of New York Press, pp. 291-330.

Confrey, J. and Smith, E. (1995) 'Splitting, covariation, and their role in the development of exponential functions', *Journal for Research in Mathematics Education* **26**, 66-86.

D'Ambrosio, B. (2004) 'Preparing teachers to teach mathematics within a constructivist framework: the importance of listening to children', in Watanabe, T. and Thompson, D. (eds), *The work of mathematics teacher educators: exchanging ideas for effective practice*, San Diego, CA, Association of Mathematics Teacher Educators, **1**, pp. 135-150.

Jacobs, H. (1970) *Mathematics, a human endeavor; a textbook for those who think they don't like the subject*, San Francisco, CA, W. H. Freeman.

Menninger, K. (1969; original work published, 1958; Broneer, P., translator) *Number words and number symbols; a cultural history of numbers,* Cambridge, MA, MIT Press (see below).

Stavy, R. and Tirosh, D. (2000) *How students (mis-) understand science and mathematics: intuitive rules*, New York, NY, Teachers College Press.

Steffe, L. and D'Ambrosio, B. (1995) 'Toward a working model of constructivist teaching: a reaction to Simon', *Journal for Research in Mathematics Education* **26**, 146-159.

Steffe, L. (1988) 'Children's construction of number sequences and multiplying schemes', in Hiebert, J. and Behr, M. (eds), *Number concepts and operations in the middle grades*, Reston, VA, National Council of Teachers of Mathematics with Lawrence Erlbaum Associates, pp. 119-140.

Logarithms were a magic tool: how, for instance, could anyone previously manage to extract the seventh root of 5?

(Menninger, 1958/1969, p. 443, see references above)