Two Letters

To the Editor:

I enjoyed "On the Aesthetics of Mathematical Thought" (FLM 6, 1, February 1986) and wholeheartedly agree with the authors' final conclusion that the failure in schools to promote the development of aesthetic appreciation of mathematics is mistaken.

However, there seems to be a potentially disastrous thread of oversimplification running through their argument. It appears when commenting on Papert where they ask, "Isn't metricizing a structure necesarily one of the main components in assessing aesthetic value?" Do they really think that they can effectively attach numbers to mathematical phenomena so that each is less, as or more beautiful than another specified phenomenon?

Apparently they do, since they write of Birkhoff, "Probing deeper, (he) looked at the problem of quantifying aesthetics ... In this classic paper ..." which strongly suggests their approval of his quantitative approach, while they repeat their use of "metric" in such phrases as "the placement of a personal internalized metric" and "The metric used by experts for such assessments ..."

They do quote Hofstadter's opinion against Birkhoff but their response is that while "(aesthetic) assessments are very personal; nevertheless, there is far-reaching agreement among scholars as to what arguments are beautiful and elegant"

This claim raises a host of problems, many of them prompted by the fate of aesthetic theories in literature and the arts

How complete is this "far-reaching agreement"? If less than complete it will not support a metric, though it may support the view that in particular periods, even in particular places, there is often broad agreement on aesthetic judgements, as there was among 18th century classicists *against* Shakespeare, and about the same time, against Leonardo da Vinci.

Here, their own evidence is against them. They discuss five proofs of the irrationality of root 2, and assert that a sample of mathematicians consistently preferred proofs A and C; this is only consistent with a general metric if all the sample placed A and C equal first, which the authors do not assert.

The validity of a "metric" can also be tested in the experience of individual mathematicians. Here I can only speak for myself; there are many proofs, and structures, which I enjoy and appreciate, but regard as different as chalk and cheese. I cannot order them, neither does it make any sense to say that they are "equally" enjoyable, when the enjoyment is so different.

When mathematical beauty is broken down into specific features, further problems arise, again paralleled elsewhere. Discussing solutions, the authors refer to level of prerequisite knowledge, clarity, simplicity, structure, cleverness, surprise and so on; they mention possible relationships between them, such as greater prerequisite knowledge implying lesser elegance, and they once again refer to quantification.

What evidence have they that mathematicians consistently agree in the manner in which they combine their judgements of these factors into an overall judgement? Do they, incidentally, suppose that mathematicians have metrics for these judgements individually?

What evidence do they have that mathematicians will agree in ranking solutions, or structures, on even one of these criteria, say, simplicity? Roger Penrose [1] discussing the beauty of simplicity takes the example of a simple square lattice and remarks,

> There is no doubt that it is something simple. But I do not really feel that there is much beauty about it. As a pattern it is just boring. It may just be a question of familiarity, of course ...

I pondered this, and decided that for me the lattice is beautiful, but that the reason is probably much as he suggests; I am not as familiar with such lattices as he is, I am not a professional mathematician, and so its beauty is not for me exhausted. Familiarity, however, is applicable to professionals and amateurs. Does an individual's aesthetic appreciation of a phenomenon change with time, due to increased familiarity (or possibly decreased familiarity, rediscover, and renewed appreciation) or some other reason? If so, then what of the metric?

Penrose then turns to his non-repeating tiling of the plane, and admits incidentally that he chose the topic of his lecture in order to show this pattern! Penrose's delight in his own discovery is itself delightful and natural, and a similar delight may be observed in children in their own discoveries, but it also raises interesting problems. What evidence is there that mathematicians are not aesthetically biased, as many artists seem to be, towards their own fields and their own works?

The concept of learning aesthetic appreciation raises its own problems. The authors refer to researchers who have claimed that "It is possible to learn how to appreciate art, music and poetry by developing an inner metric against which to measure them." I would not dispute that, as a first approximation, this is possible, and has been achieved in recent years by numerous teachers and students from I.A. Richards downwards, though if they really believe that "individuals learn to appreciate music, art and poetry, by understanding their underlying structures", I do wonder where other aspects of meaning have escaped to. However the failure of the "best" critics and authors to agree in their own judgements contradicts any simplistic claim that this process can be complete.

I have queried the authors' assertions at some length because the question of a metric for aesthetic judgement is so important, not least to readers of *For the Learning of Mathematics*.

A major plank of child-centred teaching and learning is that differences between individual pupils should be acknowledged and taken into account, as the authors themselves assert. If teachers, many of whom will have little or no experience of the aesthetic aspects of mathematics, get hold of the idea that aesthetics *is* important in the classroom and if in addition they believe that there is a metric of aesthetic judgement, then heaven help the pupils who do not appreciate the beauty which the teacher "knows" is there! Heaven help pupils who have their own ideas of what is beautiful. Heaven help—even—pupils for whom geometrical ways of thinking are consistently more attractive than algebraic, or conversely, Heaven help pupils who prove—too late—that Dreyfus and Eisenberg are oversimplifying an extraordinarily complex issue.

What we should be doing, I suggest, is to investigate with minds as open as possible the variety of mathematicians' and students' appreciation of mathematics, the differences as well as the similarities that appear and the reasons why preferences and differences exist, and whether these relate to psychological factors, or cognitive factors, and so on. At best we might be able to construct interpretations of particular subjects' aesthetic appreciation which would allow us to partially predict their future preferences [2] and adapt our teaching and their learning towards their own perceptions and ways of thinking and feeling, rather than towards some claimed uniformity of appreciation interpreted in a metric. At worst we would have far more and better information for the construction and testing of effective theories.

Should it turn out, amazingly, that all this potential complexity can be interpreted in terms of a concept as simple as that of an aesthetic metric, in contrast to the lack of such a metric in the arts generally, all well and good. But do not make that assumption initially. That would be the worst possible scientific method and the most dangerous possible course for mathematics teaching

David Wells

[1] R. Penrose The role of aesthetics in pure and applied mathematical research In *Bulletin of the Institute of Mathematics and its Applications*. 1974, Vol. 10, p 266

[2] D.G. Wells. A theory of one individual's responses In *Studies of Meaning Language & Change* 1985 No 16 p. 16

To the Editor:

David Wells' reaction to our paper is thoughtful and we are grateful to him for giving us a chance to clarify a number of points which we may not have made clear enough in the paper. Our underlying contention is that it is possible to metricize the aesthetic appeal of mathematical arguments, structures, etc. Wells over-interprets this contention. The problem, perhaps, lies in our use of the word metric, for we do not mean to attach absolute numbers per se to mathematical arguments. But there is, nevertheless, far-reaching agreement in the mathematics and the mathetmatics education community that certain standards and criteria lend themselves as benchmarks in assessing mathematical structures and problem solutions from the point of view of their elegance and their aesthetic appeal. Furthermore, within broad and general guidelines, these criteria can form the basis of what we call, for lack of a better word, an aesthetic metric The point to remember here is that this is not and cannot be an absolute metric, but most definitely a relative one, encompassing the complex interactions between the components of the chain we discussed in the paper: prerequisite knowledge, simplicity, surprise, ...

The fact that there was no complete agreement between the mathematicians queried, as to which proof of the irrationality of $\sqrt{2}$ they preferred, is evidence that they are assessing the arguments against internal, probably non-explicit criteria. However, there was quite far-reaching agreement; for instance, it is noteworthy that none of them chose proofs B, D or E. Why? We thus conclude that there were some common criteria at work. Doesn't this suggest that, for whatever reason, there exists a general framework, within which judgements are made?

Penrose's comment on the square lattice shows that he was relating to the simplicity of the structure, one of the links in the chain mentioned above. His comment that simplicity alone was not enough to make the lattice beautiful to him, again attests to the interaction with other criteria, which may be those of our chain or others, perhaps familiarity, as Wells suggests. But here Wells is falling victim to his own criticism, for when he studied the square lattice, simplicity and familiarity, or other constructs not yet identified, interacted in a complex manner; this interaction may be dependent on the individual as, to some extent, "Beauty is in the eye of the beholder." Note, however, that the effect of the individual in assessing beauty may be less strong than one might think (see "Beauty is in the Genes of the Beholder" by Harel, Unger and Sussman) Wells reacted to certain aspects of the lattice structure—and all we have tried to do was to identify these aspects. Perhaps the constructs involved in aesthetic judgements are hierarchically organized and internalized only through experience. But the point is, once again, that Penrose, and Wells, assessed a particular structure against some specific criteria.

Wells is correct in his observation that one can often appreciate and enjoy proofs and structures which may be as different as "chalk and cheese"—and that their aesthetic value cannot and should not be compared. They, as Wells so ably states, are different and beautiful each one in its own way—and nothing more need be said about them. But consider now two arguments designed for the same goal. For a simple example let us take the formulae for the sum and product of the roots to a quadratic equation:

> *If* r and s are the roots to the equation $ax^2 + bx + c = 0$ $(a,b,c \in R,\ a \neq 0)$ then $r + s = -b/a$ and $rs = c/a$. Let us say that Solution A consists of algebraic manipulation of the two branches of the quadratic formula,
> $r = (-b + \sqrt{(b^2 - 4ac)})/2a$, and
> $s = (-b - \sqrt{(b^2 - 4ac)})/2a$.
> Let Solution B entail a modelling of the equation $(x - r)(x - s) = 0$ against the original $x^2 + (b/a)x + c/a = 0$, and observing that the desired formulae follow from comparing the coefficients of the two equations.

This is a "cheese and cheese" case, and we venture that the arguments can be assessed, compared and contrasted against a metric which encompasses many of the links in the previously recalled chain. Incidentally, the problem of developing these formulae was given to a group of preservice teachers; 80% of them solved the problem via Solution A, whereas only 10% used Solution B. Nevertheless, they considered Solution A clumsy after having been presented with the second possibility and its merits.

We fully agree with Wells' comments that teaching should be child-oriented with individual thinking patterns and personal judgement being respected and encouraged. Seeing this requirement within the general framework alluded to earlier, there are criteria against which the individual himself must assess his belief system, and among these criteria there are some standard measures. It is not that criteria should be foisted on the children by fiat but rather that they develop their own belief system and aesthetic standards together with the ability to rationally discuss them. Since this development takes place within a given cultural environment, the relative uniformity of the emerging standards comes about rather naturally.

Wells seems to be accusing us, and vice-versa, of oversimplifying and generalizing a complex educational goal. Nevertheless, there seems to be quite a large measure of agreement between us as to what should and what should not be done. Wells, as usual, has raised serious issues and concerns; he addresses more than a simple debate on the moot point of the existence or dearth of aesthetic appeal in school mathematics. What is really being addressed is the role of the curriculum and teacher in developing, in the student, elements from the aesthetic domain. Here, we believe, discussing problems in the spirit we have described in our paper will be a boon and not a bane to school mathematics.

Tommy Dreyfus, Ted Eisenberg

Harel, D., Unger, R. and Sussman, J. *Trends in the biological sciences* 1986

Suppose that a very young (perhaps two-and-a-half-year-old) child wants something—for example, to occupy her mother's role. She wants it at once. If she cannot have it, she may throw a temper tantrum, but she can usually be sidetracked and pacified so that she forgets her desire. Toward the beginning of preschool age, when desires that cannot be immediately gratified or forgotten make their appearance and the tendency to immediate fulfillment of desires, characteristic of the preceding stage, is retained, the child's behavior changes. To resolve this tension, the preschool child enters an imaginary, illusory world in which unrealizable desires can be realized, and this world is what we call play. Imagination is a new psychological process for the child; it is not present in the consciousness of the very young child, is totally absent in animals, and represents a specifically human form of conscious activity. Like all functions of consciousness, it originally arises from action. The old adage that child's play is imagination in action must be reversed: we can say that imagination in adolescents and school children is play without action.

L. S. Vygotsky
