A Linguistic Basis for Student Difficulties with Algebra

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Student difficulties with algebra are common, not only when students are first introduced to the subject in secondary school, but also when they are presumed to have mastered the subject and are beginning to study calculus One need only utter the phrase "word problems" to elicit a sympathetic response, though this is certainly not the only trouble calculus students have with algebra It is the thesis of this paper that a major component of student difficulty with algebra is the inability to make sense of the algebraic symbol system as a language, and accordingly that remedies should be sought by considering algebra in a linguistic sense.

Let us assume that the usual system of algebraic variable notions which we expect beginning calculus students to have learned to manipulate is a symbolic language of some kind. Then why should so many beginners have difficulty reading and writing in the algebraic language? Do students find difficulty in processing it simply because it is a symbolic language? If this is so, then how do we explain their fluency in English, whose expressions are also symbols?

Then is it reasonable to describe the algebraic symbol system as a language at all? And if it is a language, what is it used to convey, what do its sentences look like, and how is it acquired? Is it really distinct from the student's own spoken language?

Languages have both grammars and meanings — syntactic and semantic components It is difficult to watch beginning students for long without concluding that they slight the semantic component of their algebraic language Whatever their syntactic facility in manipulating the expressions of the algebraic language, many students cannot attach meaning to an algebraic expression.

From the point of view of understanding and possibly even addressing the mathematical problems of beginning students, we need to consider the linguistic aspects of their difficulty. Are there ways to restore meaning to a language students are trying to use in a semantic-free way? Are there ways to help younger students as they first acquire the algebra language, so that they can properly associate algebraic expressions with their meanings?

The students whose activities have inspired these speculations are first year calculus students at the University of New Hampshire, particularly those who visit the Mathematics Center, a Mathematics Department facility in support of the first year calculus program. It is staffed with a director, a graduate teaching assistant and a group of undergraduate assistants. Several hundred first semester calculus students are required to complete individualized Mathematics Center programs of remediation in algebra and trigonometry while continuing in the calculus course. Other calculus students visit the Mathematics Center on their own to use calculus review materials prepared for them, or to ask questions about the course content. In addition, each of about 400 second semester calculus students completes a laboratory style project in numerical integration in the Mathematics Center.

A consequence of this organization of the first year calculus program is that the director of the Mathematics Center has a privileged view of the ways in which large numbers of students address problems in mathematics. Because so many students are seen, mathematical behaviors that at first seem isolated aberrations finally obtrude themselves on the observer's notice by their very repetitiveness

Let us begin with some examples, in the hope of illustraing the linguistic aspects of student difficulties

An example: generating algebraic sentences

It is only fitting to begin with word problems here, as in fact some early efforts in the Mathematics Center began with attention to maximum-minimum word problems in calculus. It has become de rigeur recently for calculus texts to incude algorithms, some better than others, for solving the various types of word problems as they come up. But the real difficulty resides at a lower level, for the central feature of word problems is that they are presented in natural language but must be resolved in algebraic language Trouble with word problems has its origins in language use.

An example word problem occurs in the remedial module on solving inequalities:

> *The Cornhusker Combine Company has manufacturing facilities in Omaha and Irkutsk. Management finds that for every 5 combines made at the Omaha plant. 3 are produced at the Irkutsk plant. Each year 75 combines from Omaha, and 325 combines from Irkutsk, are found to be defective as they come off the assembly line Management wishes to market in all, at least 8000 combines annually. How many should be produced in Omaha?*

Students almost never initiate a solution for this problem by writing a descriptive inequality for it, though they are aware that this is expected. Some students offer solutions in a numerical spirit, and often their explanations would be reasonable approaches to a problem that could not be solved in closed form Here are the solutions proposed by

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three students doing this module in their algebra remediation during the Spring 1987 semester

> Solution 1. We need to produce 8000 combines from the two plants, and their production levels are in the ratio 5:3. So as a first approximation, let us say Omaha will produce 5000 combines and Irkutsk 3000 (making up the required 8000). However, the Omaha plant will discard 75 combines Then to complete its quota, it must produce 5075 combines.
>
> Solution 2. As above, the ratio requirement suggests the first approximation that Omaha produce 5000 combines and Irkutsk 3000. But 5000 less 75 is 4925 marketable combines from Omaha; and 3000 less 325 is 3675 (sic) marketable combines from Irkutsk. The total of 4925 and 3675 is 8600 combines, which is more than enough: so presumably the 5000 combines produced in Omaha were sufficient.
>
> Solution 3. The 3:5 ratio again suggests the first estimate — 5000 combines from Ohama, and 3000 from Irkutsk. Yet the ratio of "discards," 325:75, is not the same as the production ratio. Attempting to "correct for" this, the student multiplied $5 \times 75 = 375$ and decided that Omaha should produce not 5000 combines, but 5000+ combines The student was aware that this method was faulty, and offered the sum 5375 as a rough approximation only

The first student presented the answer hoping that it was correct The second student was aware only that something must be wrong because too many combines resulted. The third student was aware that the method, not just the answer, was unsatisfying in that it depended on the ratios of production levels and discards being identical, which wasn't the case.

Many algorithms for solving word problems of various kinds begin with an instruction to "read the problem carefully." But note that it was not failure to read carefully that hampered these three students. All three evidently understood the problem clearly enough in its natural language setting. Each student took as the starting point for its solution the 5:3 production ratio between the two plants, and proceeded in an entirely numerical spirit Two of the students even made reasonable numerical guesses as to the solution. Yet none of them was able to describe the problem requirement in an algebraic sentence: none, in fact, even attempted to address the problem in this way.

Example: reading algebraic sentences

Unhappily the inability to produce an algebraic sentence has a converse. Students are often unable to read an algebraic sentence in the sense of extracting meaning from it More startling is that sometimes they cannot read it even in the sense of *seeing* all of it.

The next example comes from the numerical integration project done by each second semester calculus student at UNH. The techniques connected with estimating and controling the accuracy of numerical approximations to integrals are ones few students have seen before. The needed upper bounds for the derivatives of the integrand are called M in the explanatory material; many students complain about not understanding "the M stuff."

In some problems the relevant derivative's upper bound M must be calculated before it is used. But in each version of the problem set there is one problem in which the relevant function's fourth derivative, and an upper bound for it, are given as a labeled hint. A typical hint reads:

$$|f^4(x)|$$
$$= \left| \frac{6x \sin x + 32x \cos x - 72x \sin x - 120x \cos x + 105 \sin x}{16x^{9/2}} \right|$$
$$\leq 488, \text{ for } 0.5 \leq x \leq 1$$

Each term, eight or ten students will report that they *could not see* the numerical upper bound offered in the hint. This is not to say that they didn't see its significance, but that it was not part of their visual input.

Students are surprised when this happens to them. One such student, work in hand, asked suspiciously whether the number 488 might not be the needed upper bound. Expressing astonishment at not having seen the number initially, she said she saw it only when she "forced [herself] to look to the end of the line "

The Center's director and teaching assistant have also been greatly puzzled by this frequent visual failure on the part of the calculus students A possible explanation is suggested in the work of psychologists interested in the organization of memory in the human brain The idea is that the limitation on human memory

> strongly suggests that at the same time we listen to an utterance we are recoding its string of words into some other structured representation, and that this recoding occurs at a very rapid rate The original form of the utterance is typically lost in the process. Foss and Hakes [4, page 111]

Though these authors are describing spoken or written input of natural language sentences, it may well be that something similar happens with the input of a line of symbolic language. While reading a line of algebraic symbols from left to right, the student is executing a recoding of them in some way for storage in memory. When the complexity of the string of symbols overwhelms the recoding process, the student's visual input is effectively halted

Most students, of course, do see the number at the end of the line They are still apt not to want to use it They will often proceed with an algorithm for finding an expression's upper bound anyway, simply considering that the result of their calculation should be less than the number we specify. Their unwillingness to use the proffered number might mean one of two things. One possibility is that they don't realize that we've told them anything about a number they might want, and another possibility is that they aren't actually looking for a number.

Neither of these possibilities is as far-fetched as it might appear. The first is that the student has taken the meaning of the hint to be a procedure that would verify it, rather than the fact we meant to convey. Many semanticists would take a declarative sentence's meaning to be its *inten-*

sion, a function from possible worlds into the set [T, F] of truth values. Our calculus students would thus seem to have taken the hint's meaning as its intension — the means by which they can verify it in their own world. Odd as their behavior seems, they appear to be doing just what the philosophers would predict.

The other possibility is that they are not really looking for a number, but simply for an object known as M. For these students, the referent of M does not seem to be a number at all. Asking them what they are looking for almost never elicits an answer containing the phrase "a number." They will mention an intermediate goal ("I'm trying to find a bound") or the use they hope to make of it ("I'm trying to calculate my error"). But fluent conversation ceases if the student is asked what M itself is ("It's just . . . M").

For these students it often helps to rehearse the very argument that leads to the error expression without giving the derivative's upper bound a literal name ("Suppose the values of this derivative are never more than 7"). This awkward reference to "a number which might be 7, but probably isn't" does not seem to confuse students. It is the use of the literal for the number that causes the trouble.

Evidently if we want to say to students, "This integrand's fourth derivative has a long and messy expression, but its relevant values do not exceed 488," we are going to have to say so in English. Too many students think algebraic sentences do not convey meaning, but are only used to request computations.

Example: meaningless expressions

Just as our students fail to take our meaning when we write M but mean a particular fixed number, they sometimes have difficulty assigning meaning to algebraic variable expressions that are not part of a sentence. The simpler the expression, the more this problem stands out in relief. This apparently trivial limits problem is an example:

$$\lim_{x \to 0^+} \frac{1}{\sqrt{x}}$$

Many students have literal variables in elementary algebra chiefly in the context of equations which were to be solved. But to find this limit, the student cannot rely on any algorithm such as solving for x, thereby waiting for it to display itself as a found number. Instead, the student must imagine the expression $1/\sqrt{x}$ as a variable expression, think of x as a positive number close to 0, and picture the reciprocal of its square root.

Its very simplicity can make this problem difficult, because there is no way to embed it in an algorithmic context. It occurs at the beginning of the course when students have no alternative other than the correct one of imagining values of the variable expression itself. As the course proceeds, more algorithms become available: substitute an appropriate constant value into the expression; divide by the highest power; use l'Hôpital's Rule. An algorithmic method, whether or not appropriate, is more familiar to the beginner than one that relies on consideration of the expression itself.

Algebra as a semantic-free language

The students in these examples seem unable to encode meaning from natural language, where interesting problems arise, into algebraic symbolic language. And they seem not to be able to recognize meaning in an algebraic sentence either. Their remaining option is to use algebraic language simply to carry out formal manipulations on patterns of symbols. Their algebraic language is empty, having only syntax.

These students, filling a page with symbols, do not think of themselves as communicating in a language. Their very writing styles make this plain. For example, one student's evaluation of an integral by means of an algebraic substitution begins more or less at the top of the page and ends more or less at the bottom, but in each free spot on the page auxiliary thoughts blend into the central statements. There are three conclusions: one is simply circled, one is in a box, and one (alas, not the correct one) is in a circle with three arrows pointing to it. It is beyond imagination that the student's letters home mirror this style.

Many students, like this one, do not see that algebra functions as a language. Yet how could we call theirs an unreasonable view? No one would call a system a language, if it had no semantic component.

In another view, almost as impoverished, the system of algebraic symbols is regarded as a language, but only as

> a "language" of formulas, and the job of the mathematician as explaining which combinations of symbols represent true propositions . . . [such as] "$(x + 1)^2 = x^2 + 2x + 1$" . . . Winograd [11, page 12]

Though we might imagine a language this self-contained, it could surely support few interesting sentences. The topics we could discuss using algebraic expressions in this way would simply be other algebraic expressions.

And no real language is used simply to manipulate its own words. The power of language is not in the words themselves, but in the use that we make of them to communicate with each other. The words of our language support communication because they are symbols, pointing beyond themselves to things we experience in our world. To be real, language has to be *about* something.

Most people do assume that the system of algebraic symbols is a language of some kind. It can be seen written in books and papers, for instance, so that it must support some sort of written communication between at least some people. Students sometimes toy with the idea that the symbol system functions as a *spoken* language as well, perhaps being the lingua franca of mathematicians at supper. And in frivolously exercising the idea they even ask the right question: "What do you talk about?"

Algebra as a variant of English

It is reasonable to view the system of algebraic symbols not as a separate language at all, but rather as a telegraphically written subset of natural language. Sentences written in the algebraic notation are to be understood within the natural language, in which the variables are to be understood as quantity *names,* even when denoted by letters. The sentence written "$i = prt$" is to be understood, within English, as "Interest is principal times interest-rate times time." Yet the student whose algebraic language is divorced from natural language may well apprehend only the string of five symbols "$i = prt$."

In trying to account for the beginner's muteness in producing algebraic sentences, an interesting place to begin is to consider which English sentences have algebraic translations. One linguist writes, in introducing an example of how young children acquire language,

> For each English active sentence with a certain sort of transitive verb (including *eat, bite, catch, ...* etc but excluding *cost, weigh, ...* etc.) there exists a corresponding passive sentence with the same verb. Fodor [3]

This author was not describing algebraic language, yet it is noteworthy that the verbs excluded in the example at hand are just ones that could occur in elementary algebra problems. We should take the hint; only certain verbs will occur in an English sentence if it has an algebraic translation.

The English sentences that can become algebraic statements are those that have to do with quantities. Their nouns are quantifiable entities. And notice that the example verbs *cost* and *weigh* correspond to *measure functions.* Though they appear as verbs in an English sentence, in its algebraic version they would be rendered as part of a nominal, a kind of algebraic noun, rather than as the main verb. Corresponding to each of these English verbs is a measure function *cost-of* and *weight-of,* so that sentences "The coat costs $225" and "The dog weighs 74 pounds" will be rewritten "cost-of (coat) is 225" and "weight-of (dog) is 74."

By the time encoding is complete, the English sentence will have had all its nominals treated in this way. A measure function, whose value is a real number, will have been applied to each. And the English sentence's verb may function as a measuring modifier in the algebraic translation.

It follows that there are basically only two verbs that can occur in an algebraic sentence: they are *is* and *exceeds.* Combinations and rearrangements of these verbs, such as "less than or equal to," can be left to the algebraic grammar's transformational component.

Once an English sentence has been encoded algebraically, it has become an algebraic language statement about real numbers. The student has achieved the equation that summarizes the desired word problem. The equation can now be treated with the syntactic transformations appropriate to the algebraic language, which will produce a numeric solution. For the transformations appropriate to the algebraic language's grammar are simply calls on the axiom system for the real numbers.

Writing a sentence in algebraic language

It is time to return to the original problem. Are there language-based ways to help students distill an algebraic language summary sentence from a natural language problem description? If, as suggested, algebraic language is a thinly disguised subset of the student's natural language, the first requirement for the algebraic solution of word a problem is to write a summary sentence. And the sentence should be in English.

This is not the way students usually proceed. They are apt instead to begin a word problem by making a small glossary of variable assignments: "let $x = \ldots$, let $y = \ldots$" Often enough this formal beginning gets them nowhere. Yet the students are working in good faith, proceeding according to commonly-issued instructions that the way to initiate a problem's solution is to read the problem carefully, draw a sketch if possible, assign variables (often letters toward the end of the alphabet are specifically advocated) ... and write an equation.

It is not to be wondered at if the student's progress ends with the acquisition of this temporary algebraic lexicon. It offers no insight into the problem, yet the student must now keep track of it. And descriptions of quantities in terms of letter variables lose the semantic force the quantity names enjoyed in natural language.

Probably more decisive is that the student, trying to encode the problem description into algebraic terms, now has a list of correspondences between only the nominals in English and the nominals in algebraic symbols. With only nominals in hand, the student cannot yet know what to say about them. In fact, the verb is missing.

> It is predication that turns nominal designations into sentences — that enables people to put language to use. Miller [7, page 85]

Until the student has a verb for the problem statement, possession of algebraic words for all the appropriate nominals cannot lead to an equation.

There are essentially two ways to obtain a verb for the new sentence: the first is to insert the verb into the sentence in its algebraic context. The student must construct appropriate expressions from the literal variables in the glossary, and relate them with the appropriate form of the verb *is* or *exceeds.* Doubtless this is what authors have in mind when they instruct students to assign variables and write an equation.

A better way to find the verb for an algebraic sentence is to assemble the whole sentence in English first. That way the student has the advantage of using his or her natural language for thinking about the problem requirements. Further, the sentence comes already equipped with its verb.

In the second method, the translation of English sentence to algebraic equation occurs piecemeal as the solution proceeds, and reference to variables is delayed.

Example: two beginnings to a problem

Let us use a simple example from an algebra text to illustrate the different ways of beginning a solution:

A wallet contains $460 in $5, $10, and $20 bills. The number of $5 bills exceeds twice the number of $10 bills by 4, while the number of $20 bills is 6 fewer than the number of $10 bills How many bills of each type are there?

To start the problem's solution in its natural langauge context, we begin with a sentence in (at least telegraphic) English:

TOTAL MONEY in wallet is $460

The subsequent procedure has a distinctly tree-like nature. The beginning sentence is the root, and note that it already contains the verb. Expanding nodes in the tree, we begin by replacing the reference to TOTAL MONEY by more specific descriptions:

VALUE OF FIVES + VALUE OF TENS + VALUE OF TWENTIES = $460
$5 (FIVES) + $10 (TENS) + $20 (TWENTIES) = $460

At the next level we expand nodes by replacing the references to FIVES and TWENTIES by the equivalents given in the problem: FIVES becomes 2 (TENS) + 4, while TWENTIES becomes TENS - 6

5[2 (TENS) + 4] + 10[TENS] + 20[TENS - 6] = 460

Having arrived at a level in the tree that shows only one variable, we pause. We have an equation in one unknown, TENS, which students will be able to solve. By this time, of course, the student is writing:

$$5(2t + 4) + 10t + 20(t - 6) = 460$$

It is certainly not suggested that students must think of this process as expanding nodes in a tree They would be horrified But it should be noted that the procedure has a top-down nature; that because it begins with a meaningful English sentence the evolving equation, at each stage, carries the semantic weight we want to preserve; and that each node expansion calls for a strategy of filling lexical gaps that speakers of the language, even small children, are already known to use. [2]

By contrast, the traditional strategy of beginning by constructing a temporary lexicon . . .

let x = number of $5 bills,
y = number of $10 bills,
z = number of $20 bills

is a bottom-up approach. It proceeds by assembling descriptions of the nominals in the problem, elaborated as the problem's side conditions are introduced.

Side conditions: $x = 2y + 4$, $x = y - 6$.

Value of money in $5 bills = $5x = 5(2y + 4)$

Value of money in $10 bills = $10y$

Value of money in $20 bills = $20z = 20(y - 6)$

Notice that the expressions representing side conditions look like equations Students can be sidetracked into "solving" them Yet none of these mini-equations is the desired central equation needed for solving the problem.

Some authors seek to avoid confusion between equations representing assignment of variables, those representing side conditions from the problem, and the central equation by having the student enter representations of quantities in a table:

Type of bill	Number of bills	*	Value per bill	=	Dollar Value
$5	$2y + 4$		5		$5(2y + 4)$
$10	y		10		$10y$
$20	$y - 6$		20		$20(y - 6)$

This strategy is meant to clarify the problem for the student, and sometimes it does, though students are sometimes then diverted by the intermediate task of filling in the table. A more serious difficulty is that choosing an appropriate form for the table requires some prior understanding of the form of the desired central equation for the problem The lack of this understanding was probably the student's difficulty in the first place

Whether the student uses a table or not in the bottom-up approach of naming all the quantities first, it can produce only a more or less carefully elaborated description of the problem's nominals At the end of the preliminary procedure the student still has no central equation, because the verb for the sentence is still missing

Origins of the student's algebraic language

It may seem natural to assume that remedial needs of beginning calculus students stem from material imperfectly learned during the high school years But troubles residing in the inability to use variables meaningfully have earlier origins, probably in the important transition from middle school general mathematics to the first course in elementary algebra.

The transition from general mathematics to elementary algebra is marked by the student's acquisition of an algebraic symbolic language. It is important at this stage that both the semantic and syntactic components of the new language become available to the student. If the young student receives only a quick and abstract initial encounter with variables followed by practice in the essentially syntactic skills of manipulating algebraic expressions, the semantic component of the new language may never be realized.

The acquisition of algebraic language on the basis of a known natural language and many known instances of arithmetic is temptingly reminiscent of first-language acquisition, in which language competence arises in the very young child out of the many instances of speech that form the language corpus the young child hears. Though prudence dictates that only the most tentative analogies with first-language acquisition be made, the importance of an existing language corpus intrudes itself on the mind.

First-language acquisition in the very young child is not really a matter of teaching. We should rather say that the young child *acquires* language on the basis of the language corpus it hears. Certainly it is not possible to teach grammar to a young child, though there are entertaining quotes in the literature of people trying to do so. The young child constructs its own grammar.

The child of an age to learn algebra in school is no longer acquiring language in this same way. The algebraic language is properly taught and learned, as well as acquired And we specifically teach an algebraic grammar to students in their first course in elementary algebra, which is to say we offer students a formal, axiomatic summary of the properties of arithmetic operations.

It is unhappily common for students to learn the axiomatic, algorithmic system in a formal spirit only, with a poorly semanticized language resulting. Yet the semantic starvation occurs in the midst of plenty, for the student starting to learn elementary algebra already possesses a language corpus relevant to the symbolic language of algebra. It includes everything the student knows about numbers, as well as those portions of the student's natural language having to do with quantitative description.

A trivial but important thing to notice abut the young child's acquired natural language is that it includes the original language corpus that induced it. We might similarly expect that the student's algebraic language should include its own proper corpus Clearly this is often not the case In fact, to say that a particular student cannot access meaning in the algebraic language is another way of saying that that student's algebraic language does not contain its proper corpus.

The middle school student's development of an algebraic symbolic language that fulfills early experience with numbers and quantities is a major step towards mathematical maturity It is important that at each stage of learning it, the algebraic language be seen by the student to convey meaning. To allow young students to learn their algebra as a semantically impoverished formal language is to prejudice their subsequent mathematical learning

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The Right to Make Mistakes

The preceding issue of the journal, Volume 7, Number 3, contained three very fine articles on errors and mistakes It also contained, alas, a larger number than usual of setting, editing, and proofing errors. The following is one of the more egregious

In Guershon HAREL'S article, page 30, second column, fourth paragraph: on the fifth line of the paragraph, "the ovem" should read "theorem". All the words from "Some textbooks . . " on the eigth line to " . approach" on the sixteenth line should be deleted.

There were a number of errors in Gert SCHUBRING'S article: corrections will be printed in our next issue.

Apologies to our authors and to our readers for these mistakes and for all those others as yet unnoticed.
