Communications

In Praise of the Minus Sign

AMIR ASGHARI

I have told this story orally several times since the day it happened in a middle school mathematics class about twenty years ago when I taught negative numbers to a group of thirteen years old students. Until recently, I always thought of it just as a good example of interactive concept creation in a mathematics class. However, following a recent research on the history of negative numbers (Asghari, 2019), the taken for granted side of the story, the negative numbers themselves, came into a new light. In particular, I learned some of the advantages of the common representation of negative numbers with a minus sign in front.

It might seem strange to talk about the advantages of such a familiar representation, in particular, when the Glossary of the Common Core State Standards for Mathematics (2010) defines an integer as follows:

> **Integer.** A number expressible in the form a or $-a$ for some whole number a. [1]

So it seems it is defined as it is defined, and for that reason, we have to represent, say ‘negative three’ or ‘minus three’, by –3. But, we could use $^{-}3$ to distinguish between ‘minus three’ as in ‘five minus three’ and ‘negative three’ as a stand-alone number. Or we could even use different colours to distinguish positives from negatives, as the Chinese did more than two millennia ago, black for negatives, red for positives (Joseph, 2010). So why do we teach $-a$ instead of any other notation to represent negative numbers? After all, we know that it might be confusing for students to use the same symbol both for the operation of subtraction and a kind of number. Here I tell the story of some students who experienced negative numbers represented in another way and I then compare their invented notation with the usual one using the minus sign.

The students in my story were familiar with the notion of power (exponent) but not with negative numbers. The conversations are not verbatim, but in line with the real events.

The question

“Look at the table,” instructed the teacher, the younger me.

					3^1	3^2	3^3			
					3	9	27			

“From left to right, the cells are multiplied by three; from right to left the cells are divided by three. Thus, we can fill the cells of the second row in both directions.”

					3^1	3^2	3^3	3^4	3^5	
		$^1/_9$	$^1/_3$	1	3	9	27	81	243	

“We know that the cell above 81 is 3^4 and the cell above 243 is 3^5. But, what is the cell above 1, or above $^1/_3$? In other words, *what power of 3 goes with 1?*”

The first pattern

“Look,” replied some of the students (perhaps half the class), “when you move from 3^2 to 3^1, the power becomes half. So, the next cell, should be $3^{1/2}$.”

				$3^{1/2}$	3^1	3^2	3^3	3^4	3^5	
		$^1/_9$	$^1/_3$	1	3	9	27	81	243	

“It is true,” they admitted, “that this does not hold when we move from 3^3 to 3^2 or from 3^4 to 3^3 (see, 3 is not half of 4). But, it does not matter. The cells that matter to us now are the cells to the left of 3^1. So here is the table.

		$3^{1/8}$	$3^{1/4}$	$3^{1/2}$	3^1	3^2	3^3	3^4	3^5	
		$^1/_9$	$^1/_3$	1	3	9	27	81	243	

“The cells on the right of 3^2 follow one pattern (the powers become one more each time). The cells on the left of 3^2 follow another pattern (the powers become half each time).”

“If the point is only filling the cells,” I commented, “we have already succeeded. But then, we miss a very useful feature we had on the right of the table: to multiply two numbers on the top row (say, 3^{27} and 3^{10}) we can write the base and add the powers, so $3^{27} \cdot 3^{10} = 3^{37}$. This does not work on the left of the table. We might come up with a clever formula or method to find something like $3^{1/32} \cdot 3^{1/128}$ only based on knowing $^1/_{32}$ and $^1/_{128}$. But, whatever it might be, it is harder than just adding two numbers.” (I was also aware of another, more serious cost: later we will want to use $3^{1/2}$ with another meaning, so that $3^{1/2} \cdot 3^{1/2} = 3$.)

“But I see other students want to say something. Perhaps they have seen another pattern that is not so broken.”

The second pattern

“Look,” replied the others, “when you move from 3^2 to 3^1, the power becomes one less. So, the next cell, should be 3^0.”

				3^0	3^1	3^2	3^3	3^4	3^5	
		$^1/_9$	$^1/_3$	1	3	9	27	81	243	

“Nice!” I commented, “This holds also when we move from 3^3 to 3^2 or from 3^4 to 3^3 (see, 3 is one less than 4). Moreover, now we can add the powers in $3^0 \cdot 3^1$ to get to 3^1, that is what we want ($1 \cdot 3 = 3$; $3^0 \cdot 3^1 = 3^{0+1} = 3^1$). But then, what would be the power of the cell above $^1/_3$?”

“If we want to stick to the pattern,” they answered, “the power of 3 in this case should be one less than zero. But we cannot have less than zero, can we?”

(And suddenly came a suggestion, that I am still pleasantly surprised about after 20 years.)

"Look!" yelled one of the students, "3^1 is just 3, 3^2 is two 3's multiplied together. We can represent these with $3^{\times 1}$ and $3^{\times 2}$. Then, as $1/3$ is a division, we can represent it with $3^{\div 1}$, and with $3^{\div 2}$ and so on!"

Since we were already using 3^2 for $3^{\times 2}$, we agreed to continue using the familiar notation on the right of the table.

$3^{\div 4}$	$3^{\div 3}$	$3^{\div 2}$	$3^{\div 1}$	3^0	3^1	3^2	3^3	3^4	3^5	
$1/3^4$	$1/3^3$	$1/3^2$	$1/3$	1	3	9	27	81	243	

What we did with the invented notation

This notation (that I accepted for the time being) had certain similarities with the standard notation that I had in mind, the most crucial one being that the rule for multiplying by adding the powers works the same on both sides of the table. For example, to find $3^{\div 27} \cdot 3^{10}$, we can write the base and *add* the powers: $3^{\div 17}$, that is $1/3^{17}$. This remains true regardless of the base. So, we started exploring the arithmetic of these objects with one general principle in mind: we wanted to keep the rules we knew for the powers intact. Multiplication of powers gave us addition (of our new objects). Division of powers gave us subtraction. The power of a power gave us multiplication. And division came as the inverse of multiplication.

The evaluation of the power of a power is worth explaining as it provides a mathematical reason for one of the most famous rules of negative numbers.

We want when finding the power of a power, to keep the base and multiply the powers. Let us apply this rule to $(a^{\div 4})^{\div 5}$. $(a^{\div 4})^{\div 5}$ *should be* $a^{(\div 4)\cdot(\div 5)}$. But, what is $(\div 4) \cdot (\div 5)$? Let's see.

$$(a^{\div 4})^{\div 5} = \frac{1}{(a^{\div 4})^5} = \frac{1}{a^{\div 20}} = a^{20}$$

So we *have to have* $(\div 4) \cdot (\div 5) = 20$.

We did all of these calculations in the class. At this stage, the main question for me (as the teacher) was why and how I should introduce the standard symbol, the minus sign.

Why –

Socially, shared signs are the key to successful communication. If each of us had idiosyncratic notations, soon no one would understand anyone else, and mathematics would become a decryption game. So, with every concept comes a set of agreed upon notations, one of them being -1 for denoting the solution of $s + 1 = 0$, and not, say, $\div 1$. But, is this a wise choice, considering that the sign denoting negative numbers will be the same as the sign of subtraction? Surprisingly, this ambiguous use of the sign – turns out to be the main reason for 'choosing' it to denote negative numbers.

Negative numbers are not usually encountered for the first time when exploring exponents. And historically, the *rules of signs* were first practiced in the algebraisation of arithmetic, before a general admission of negative numbers (Asghari, 2019). For example, once operations on natural numbers are understood, it becomes possible to combine them, and, multiply binomials.

$$(a + b)(c + d) = ac + ad + bc + bd$$
$$(a + b)(c - d) = ac - ad + bc - bd$$
$$(a - b)(c - d) = ac - ad - bc + bd$$

Each equality can be justified geometrically. When multiplying $(a - b)(c - d)$, the term $+ bd$ comes from the rule that 'minus times minus is plus' (*not* 'negative times negative is positive'). No knowledge of integers is required. It is all signs.

And we can apply the same rules to something like $(2 - 5)(3 - 7)$ that does not make sense in the realm of natural numbers, and which the usual geometric models fail to represent. We can blindly apply the rules of signs to get a 'result':

$$(2 - 5)(3 - 7) = 2 \cdot 3 - 2 \cdot 7 - 5 \cdot 3 + 5 \cdot 7 = 6 - 14 - 15 + 35 = 12$$

Apart from the original multiplication, everything else in this chain has meaning in the realm of natural numbers. So, if $(2 - 5)(3 - 7)$ is going to have any meaning, any result, it should be equal to 12.

We could have performed the multiplication using my students' invented notation. $(2 - 5)(3 - 7) = (2 + \div 5)(3 + \div 7)$, because $(2 - 5)$ is the exponent of $a^2 \div a^5$, and so is $(2 + \div 5)$. The process of multiplication goes as follows:

$$(2 - 5)(3 - 7) =$$
$$(2 + \div 5)(3 + \div 7) =$$
$$2 \cdot 3 + 2 \cdot \div 7 + \div 5 \cdot 3 + \div 5 \cdot \div 7 =$$
$$6 + \div 14 + \div 15 + 35 =$$
$$6 - 14 - 15 + 35 =$$
$$12$$

With the sign –, we do not even need to write $(2 - 5)(3 - 7)$ as $(2 + -5)(3 + -7)$. We only need to know when we multiply the minus signs, we do not multiply them as subtraction signs; we multiply them as the signs preceding 5 and 7. Simply, the pre-integer knowledge of the signs suffices for all the calculations and there is no need to learn new rules.

Concluding the story

Thinking of negative numbers as minus numbers brings a procedural flexibility. However, pedagogically we should be cautious about thinking of negative numbers as minus numbers, as it might create conceptual obstacles when our students move to the algebra of variables. Knowing why the sign – works and why it is procedurally better than the rival signs, may help students to see a negative number as a number in itself, not simply a whole number preceded by a minus sign. I later told my students to use $-a$ simply so they could communicate with the others mathematically. So, then I just said, "From now on, we use –, rather than $\div 1$." Now, with the benefit of hindsight, I could do a better job moving from our invented symbol to the standard one.

Note

[1] Online at http://www.corestandards.org/Math/Content/mathematics-glossary/glossary/

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A post publishing discussion in mathematics education research is needed

MARIO SÁNCHEZ AGUILAR

I read with great interest the article in **39**(2) by Mogens Niss (2019) in which he identifies an archetype of research article that tends to dominate current publications in mathematics education research journals. It prompted this comment on one of the opportunities that we as a community are missing in relation to academic publishing in mathematics education: the lack of an organized post publishing discussion. While some post publication discussions take place in our community—in informal talks among colleagues, in seminars with students, and in formal reactions to papers published in journals (*e.g.,* Bakker, 2019)—I think such discussions still do not achieve the status and scope that they deserve.

Opportunities for post publication discussions in mathematics education

We are living in a technological age where collective evaluation—through 'likes' and comments—is an inherent part of the dynamics of social networks; however, the scholars in the field of mathematics education research have not fully exploited this potential for collective assessment. Social networking sites such as *ResearchGate* are starting to fill this space in mathematics education research. Through public recommendations, reads, and comments, sites like ResearchGate offer a space for post publication peer evaluation and discussion. However, we as a community could take the lead by generating post publication discussion spaces to promote the development of our field.

Research journals and their editors could play a fundamental role in the establishment and administration of such post publication discussion spaces. For instance, articles published on the websites of research journals could have a public space within the journal's site for readers to assess and comment on the significance of the published work. In turn, journal editors could act as 'curators' of the public comments, by highlighting the most relevant and significant ones. Journal editors could even issue formal invitations to scholars to participate in the comments section of specific articles.

What would we gain?

In addition to expanding and nurturing the academic discussion related to specific research articles, the spaces for post publication discussion would have other potential benefits for the field. Some of these potential benefits are: (1) to minimize the possible biases inherent in the peer review system, (2) to assist in the identification of works that are considered relevant and meaningful to the community, and (3) to function as a mechanism for the self-correction of our field as a scientific discipline.

Several authors have pointed out the limitations of the peer review system, which is at the heart of scientific publication (*e.g.,* Haffar, Bazerbachi & Murad, 2019). Inconsistencies that may emerge during a peer review process have been identified (for example, when different evaluators have opposite opinions about the quality of a manuscript), and biases that affect certain groups of authors have been highlighted (*e.g.,* Fox & Paine, 2019). I think that a post publishing discussion in which other academics participate—academics different from the reviewers and editors responsible for the editorial decision—would undoubtedly enrich the valuation of the scientific work at stake, and would help to counteract the possible biases and inconsistencies that could emerge during the peer review process. The responsibility for assessing the significance of an academic work would not only fall on a limited number of specialists, but would extend to a broader group of scholars who would have a voice and opinion about the importance of the work, even after it was published.

Giving voice to a larger number of mathematics educators about the quality and importance of a published article could reveal in a clearer way how the mathematics education research community feels about the published work. A post publication discussion would accelerate the identification of works that are considered important for the community. It would not be necessary to wait for a published work to accumulate a large number of citations to be able to assess its significance and influence.

Finally, post publication discussion spaces could become complementary peer review systems that help to maintain the integrity of the research published. An open and collective post publication discussion could help to identify works published in our field that could be candidates to be retracted. Thus, the post publication discussion could become a new cornerstone in the self-correcting mechanism of science (Peterson, 2018).

I am aware that several conditions must be in place in order to institutionalize the post publishing discussion as an element in our scientific publishing practices; however, I think the benefits it would bring to our community are significant. A post publishing discussion in mathematics education has the potential to become a collective learning site, with the capacity to transform the way we interact with published research and its authors.

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Editor's Note

For the Learning of Mathematics was founded as a venue to explore ideas and foster discussion. I am always pleased to receive communications, like the one above, that take part in the conversation by responding to a previously published article. Such communications are FLM's main medium for post-publication discussions. As David Wheeler wrote in an editorial in **2**(1), they are "a start, at least, on laying down a feedback loop that will, I hope, be increasingly used." I would be interested to receive suggestions of other ways we could fulfil our aims: "to generate productive discussion; [...] to promote criticism and evaluation of ideas and procedures current in the field."

The construction of regular polygons between geometry and algebra: a didactic-historical example

EMILIA FLORIO, LUIGI MAIERU', GIUSEPPINA FENAROLI

In this short communication we consider the first proposition of the chapter on the regular pentagon and decagon in Abū Kāmil's treatise on algebra, *Kitāb fī al-jabr wa al-muqābala,* as a way to show students an example of the connections between the 'knowledge' of geometry and that of algebra, considering knowledge as potential that emerges from human activity in the process of becoming 'materialized' into knowing (D'Amore & Radford, 2017). We propose a didactic presentation of this proposition, with some semiotic observations (Duval, 2018), to help students observe how Arabic algebra, as a numerical problem solving technique, uses the properties of the figure considered, expressed by Euclid in the geometric language of the Elements, in order to write the equation that solves the proposed problem.

The first of twenty problems in geometry that Abū Kāmil solves in the chapter on the pentagon and decagon is to determine the measure of a chord of a fifth of a circle from its diameter [1]. While the question formulation itself contains no specific numbers, Abū Kāmil's solution begins by setting the diameter to 10, making this a question in arithmetic that contains no algebraic terms.

To solve this problem, Abū Kāmil considers a circle of diameter $EH = 10$ and a regular pentagon *ABDEC* inscribed within it (see Figure 1).

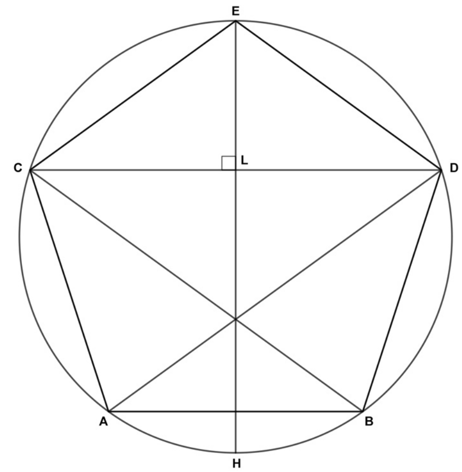

Figure 1. The regular pentagon inscribed within a circle.

Abū Kāmil starts from Euclid's construction of a regular pentagon in a circle (*Elements* IV, 11). This allows him to think of the given figure as 'built', which gives meaning to the search for the length of its side. To discover the length, Abū Kāmil draws the line *CLD*, (where *L* is the point of intersection between the chord *CD* and the diameter *EH*). Both the drawn figure and the beginning of Abū Kāmil's text are in the Euclidean register, as can be seen from the terms introduced, the use of capital letters to indicate points and segments, and the construction of the pentagon and the line *CLD*.

There is then a shift to the algebraic register. Abū Kāmil names the unknown straight line *ED* 'a thing' (in modern symbols 'x'). It is this fact that an unknown quantity is named and that later an equation is set up and solved that makes this a solution by algebra ("*al-jabr wa'l-muqābala* or sometimes just *al-jabr*", Oaks 2014, p. 28). Drawing the line *CLD* implies a movement made in Euclid's 'space' in which the construction takes place, while naming the straight line *ED* as 'a thing' introduces the reader to the algebraic 'narrative'.

Abū Kāmil continues in the algebraic register by stating that *EL* is one-tenth of the 'māl' [2] (in modern notation $\frac{1}{10}x^2$), but he justifies this in the Euclidean register as the product of *ED* by itself is equal to the product of *HE* by *EL* (*Elements* VI, 8).

The transformation of objects from the Euclidean register to the algebraic register continues, justified step by step by propositions from the *Elements*. *DL* becomes *the root of a māl minus one tenth of one tenth of a māl-māl* (Elements I, 47). In modern notation:

$$\left(\sqrt{x^2-\frac{1}{10}\frac{1}{10}x^2x^2}\right)$$

This is equal to *DL.* The same happens with *CL*; because its value is double that of *DL, CD* becomes *the root of four māls minus two fifths of one tenth of a māl-māl.*

There is then a passage entirely in the Euclidean register: since the product of *AB* and *CD* plus the product of *AC* and *BD* is equal to the product of *AD* and *BC*, it is possible to write that the sum of the product of *AB* and *CD* and the product of *AB* by itself is equal to the product of *CD* by itself.

Abū Kāmil returns to the algebraic register, evaluating the product of *CD* by itself as *four māls minus two fifths of one tenth of a māl-māl.* By eliminating (subtracting) the product of *AB* by itself, which is a māl, Abū Kāmil obtains for product of *CD* by *AB* a second expression in the algebraic register, *three māls minus two fifths of one tenth of a māl-māl.* Dividing by the straight line *AB*, which is *a thing,* results in a second expression for *CD*: *three things minus two fifths of a tenth of a cube.* In modern notation:

$$\left(3x-\frac{2}{5}\frac{1}{10}x^3\right)$$

At this point, Abū Kāmil completes the operation that, several centuries later, will be taken up by Descartes in his *Géométrie* (1637). That is, he writes the equation that solves the problem by equating the two different 'narratives' that 'tell' the story of the same object, *CD* multiplied by itself. In modern notation:

$$\left(4x^2 - \frac{x^4}{25} = 9x^2 + \frac{x^6}{625} - \frac{6}{25}x^4\right)$$

It now remains to deal with this equation. Abū Kāmil manipulates and solves it in the algebraic register by the methods he learned from al-Khāwrizmī's *al-Kitāb al-mukhtaar fī hisāb al-jabr wa al-muqābala.*

In the exposition of this first proposition Abū Kāmil seeks the measurement of an object (the side of the pentagon) from another object related to it (the diameter of the circumscribed circle). Students can see how Abū Kāmil obtains his result by first working in the Euclidean register; he uses theorems from the *Elements* that express geometric properties of the figure under consideration. From these, he obtains statements that he converts to the algebraic register using terms for powers: *thing, māl, cube.* Here, it is possible to identify the three stages defined by Oaks (2014, p. 28). The converted statements are combined to set up an equation (Oaks' Stage 1). Abū Kāmil simplifies the equation using methods laid out by al-Khwārizmī (Oaks' Stage 2) to obtain an equation that can be solved by the rules of al-Khwārizmī, thus obtaining the root that satisfies the problem statement (Oaks' Stage 3).

The fact of having constructed a figure geometrically and the properties of the figure that are determined in the Euclidean register play a key role. These properties are the basis for the 'objectification' (Radford, 2014) of the problem in a new way. This is fundamental for learning, through the way objects are 'seen' and then 'told' through a new sign, with epistemological consequences that are sometimes surprising and unpredictable.

Notes

[1] We have not analyzed Abū Kāmil's proposition in the original Arabic, however, we have referred to translations into a number of languages (Sacerdote, 1896; Levey, 1966; Lorch, 1993; Chalhoub, 2004; Rashed, 2012), which we believe gives us insight into the style of the original.

[2] Here we use the transliteration of the original Arabic word. Literally, this word means a quantity of money (among other things). In modern mathematical translations it is usually translated as 'square' but this suggests an identification with the geometrical object, intended by Euclid but not, we believe, by Abū Kāmil', who is referring to 'the second power of an unknown thing'.

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From the Archives

The following is an edited excerpt from Rosamund Sutherland's article *Some Unanswered Research Questions on the Teaching and Learning of Algebra,* which appeared in issue **11**(3).

As I have begun discussing in this paper, the predominant view of the role of algebraic symbolism seems to be that pupils first have to 'understand' a mathematical problem and then have to 'translate' this understanding into symbolic language. "The essential feature of algebraic representation and symbolic manipulation, then, is that it should proceed from an understanding of the semantics of referential meaning that underlie it." (Booth & Johnson, 1984, p. 58). Kieran suggests that the difficulties pupils have with translating are still likely to persist even within computer-based environments: "I believe that the result of much of the research discussed in this chapter would be applicable to computer intensive algebra learning situations for the following reason: In this hypothetical algebra programme, there would probably still be the need to represent formal mathematical methods, that is, to formalise procedures and to symbolise them." (1989, p. 53). The implication is that the formalisation process is in some way an add-on and final stage of the algebraic process, a position which I have already suggested is influenced by the Piagetian view that language is grafted on to understanding. In contrast, Vygotsky views language as a crucial mediator of inter-psychological functioning and an essential agent in intra-psychological functioning. Vygotsky was predominantly concerned with the role of natural language, but he also suggested that, "The new higher concepts in turn transform the meaning of the lower. The adolescent who has mastered algebraic concepts has gained a vantage point from which he sees arithmetic concepts in a broader perspective." [(1934/1986, p. 202)]. If algebra is a language which can structure thinking then we might predict that methods which present the algebraic language as a final translation of an already understood process will restrict pupils in their development.

When we consider certain interactive computer-based languages (*e.g.,* Logo and spreadsheet packages), then pupils' learning of these languages seems to be much more related to how they learn natural language than to the way they are traditionally expected to learn algebra. For many pupils the computer-based symbolism is an essential tool in their negotiation of a generalisation, and the way they come to progressively formalise informal methods. The process is dialectical. Writing a general Logo procedure involves both identifying the mathematical relationships within a problem and making these relationships explicit with a formal language. There is evidence that naming and declaring the variables in the title line of a Logo procedure helps pupils come to terms with the algebraic relationships within the problem. The language is used as a means of expressing and exploring mathematical ideas, and the learning of syntax and semantics are inextricably. [...]

Kaput (1989) believes that the current mathematics curriculum is misdirected, having an undue focus on syntax as opposed to semantics. He believes, along with others, that if we increase the referential teaching of algebra most of the difficulties pupils have with learning algebra will disappear. [...] Kaput has been involved with the development of a number of computer programs which reflect these ideas. [...] Kirshner strongly disagrees with Kaput's perspective. "I believe that the human mind is uniquely fashioned to learn syntax as syntax and that current syntactic instruction fails not because it is syntactic, but because research has not begun to fathom the depth of complexity and intricacy required of syntactic performance in algebra. I believe that the natural predisposition of the mind is to approach new, structured domains syntactically. Regretfully, I will not be surprised to find that the computer environments designed to promote meaningful conceptual development result first and foremost in learning about button pushing on computer keyboards." (Kirshner, 1989, p. 197). Our experience with Logo and spreadsheet packages is that if syntax is introduced within an accessible, motivating, and interactive problem solving situation then syntax is learned with surprising ease. If on the other hand pupils are allowed to explore the syntax with no clear goal (either devised by themselves or provided by the teacher) then a button-pushing phenomenon does develop. [...] So it seems to me that it is the welding together of syntax and semantics within a motivating problem situation which is critical to the learning of the computer programming language Logo.

Conclusions

At a time when most mathematics educators are still clinging strongly to a Piagetian perspective, psychologists are focusing on the role of the social context as a critical and fundamental factor in cognitive development (see, for example, Light, 1986). Theories which stress the social construction of meaning seem particularly relevant for an understanding of the classroom situation. For some [...] knowledge is developed within practices and from this point of view into algebra is not initiation into decontextualised knowledge but initiation into another social practice. This position predicts that pupils will not easily make links between one practice and another, which fits well with the findings that pupils make very few connections between their arithmetical and their algebraic practices (Lee & Wheeler, 1989).

As Light has pointed out, "The focus of attention is thus shifted away from the abstract 'epistemic subject' of Piaget's structuralist approach, towards the real child's experience in specific social contexts. Paradoxically the achievement of abstract thought is seen as context dependent and context driven." (1988, p. 235). Light goes on to suggest that a pragmatic approach [...] to understanding might offer as much as the universalistic cognitive development approach: "The child is apprenticed to a language and culture which are grounded in practical human purposes." (p. 236). He also points out that the mathematical tools developed by a particular culture are very related to the social practices of that culture. [...]

We know that pupils can and do solve mathematical problems without using algebraic symbolism, as did Diophantus. Can we develop a school algebra culture in which pupils find a need for algebraic symbolism to express and explore their mathematical ideas? The technology is now available to design many new mathematical environments. I suspect that these new environments will not replace the need for school algebra, but quite what the nature of this algebra should be is still an open question. My main concern is that a research community which firmly believes in the absolute nature of pupils' algebraic errors and their inevitable relationship to cognitive development is likely to reject the real potential of these computer environments and will not be able to constructively inform the creation of such environments. I do not wish to suggest that cognitive development is not a factor influencing pupils' developing understanding but merely that we need to know more about the influence of both context and language on pupils' development.

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Remembering Ros

LAURINDA BROWN

The extract from Rosamund Sutherland's writing above is from an article that I used as the first reading for a Master's course at the University of Bristol. The important focus for me in Ros's writing in **11**(3) is the idea of 'need', in this case, 'need for algebra'.

When Ros wrote this article she was at the Institute of Education, London, and in 1995 she joined me at the University of Bristol. In 2001 Ros was the chair of one of the working groups of the twelfth ICMI Study on *The Future of the Teaching and Learning of Algebra.* The *Approaches to Algebra* group, of which I was a member, allowed me to experience, in an intense few days of work, Ros's skills in energising people from different backgrounds and cultures with their own aims on joining the ICMI study. She wrote, in a late draft of the introduction to the chapter in the study volume, "[…] we worked as a community of inquiry with everyone openly and creatively contributing, asking questions and moving the ideas forward. As a result, it does not seem appropriate to only attribute this chapter to one or two authors. For this reason, we decided to call our ourselves the APPA group". Such an atmosphere would not have been created without Ros and remains, for me, the most creative and fun experience of being in an ICMI study working group.

Even though Ros had periods of being Head of School, with many other responsibilities, she retained her commitment to mathematics education and to taking part in the activities of the group. A recent memory is of her facilitating a meeting of a group who were working on a European research bid. We were invited to her house in Branscombe. It was summer, and she created a relaxed environment, with swimming in the sea, cooking meals or walking to the local pub, yet, we worked intensively and energetically, in a collaborative way, for many hours every day.

A typical Ros conversation would move between sociocultural theory to the latest photos of her grandchildren. She was always learning, and re-learning, including knitting when her first grandchild was born.

Enduring challenges

CELIA HOYLES

Revisiting this article gave me a chance to look again at some of the research taking place in the late 80's and early 90's. The mathematics education research scene had already illuminated some of the major 'obstacles' to learning across the mathematics domain. But also, there was work specifically in algebra, a domain regarded as the 'gateway' to more advanced mathematics, where seeing structure, capturing it in symbolic form and manipulating symbols were regarded as critical for success.

Simultaneously, around that time was the advent and spread of programming, among mathematicians and also more gradually in schools. A community was born inspired by Seymour Papert who looked to use the programming language Logo to catalyse a change in the relationship of children to learning, and a group in England worked to harness this change to promote mathematics learning. Ros was a leading member of this group, along with myself and Richard Noss (I mention two other early publications that complement this one written by Ros, namely, Noss, 1986, and Hoyles, 1987.) What Ros makes explicit in her paper is the need for a shift in thinking, away from seeing the formalisation process as 'an add-on and final stage of the algebraic process', to one where it is *part* of that process - so, students would experience the *power* of symbols and algebra would become a *meaningful* way to express of generality.

All of us in London, along with many others abroad, were convinced that an interactive programming environment, appropriately designed, supported and mobilised could serve as a vehicle for achieving this shift. There was an energy and optimism among the group at that time, as we grappled together with some of the critically important issues raised in Ros' paper; issues that are still centrally important for mathematics learning and, more generally the development of an appreciation of the major challenges facing us in 2020.

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Long collaborations and lasting impacts

TERESA ROJANO

Ros Sutherland's experience and deep knowledge of the role of the Logo environment in learning mathematics coalesced with the line of research on algebraic thinking that was cultivated in Cinvestav (The Center for Research and Advanced Studies) in Mexico. The joint work between the Institute of Education and Cinvestav evolved, including other technology environments, such as spreadsheets. Those research projects showed that students of different ages and school levels were able to carry out activities using spreadsheets, activities that involved powerful ideas in mathematics, such as *variation, generalization* and *infinite processes.* In a like manner, other collaborative studies provided evidence about algebra-resistant secondary school pupils (pupils that knew algebra but that did not use it to solve word problems) who were able to overcome that resistance by performing activities with spreadsheets.

That academic exchange continued into the beginning of the new millennium. In the year 2000, the Spencer Foundation of Chicago provided the funding for the Anglo-Mexican project entitled "The role of spreadsheets within school-based mathematical practices in sciences" developed in collaboration between the University of Bristol and Cinvestav. The results obtained in the multiple joint studies were able to influence both the field of research of mathematics education at the international level, as well as the design and implementation of government programs in Mexico and other Latin American countries, programs that were pioneers in incorporating the use of technology environments in educational systems, for the teaching and learning of mathematics and sciences.

Editor's note

Ros's death in 2019 was a great loss to the community that is *for the learning of mathematics.* We invite further reflections on her life and work for publication in the next issue.