HEAPS, COMPLEXES AND CONCEPTS (PART 1)

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In this pair of articles, my focus is on how an individual appropriates notions from the socially-sanctioned body of knowledge which we call mathematics. Specifically, I am concerned with how students, to a greater or lesser extent, internalise mathematical ideas that exist in the social world (on the chalkboard, in textbooks, in the activities of their lecturers and fellow students) and make them their own. My motivation for focusing on the individual is that, at university level, mathematical learning largely takes place in a context which consists of an individual reading definitions and theorems and engaging in exercises or tasks suggested by a textbook or lecturer.

Over my years of teaching calculus and linear algebra to undergraduate university students, I have frequently had the sense that much of what I was observing was not easily or usefully explained in process-object terms:

> How do I explain, or even describe, what is happening when a student is introduced to a new mathematical object via a definition? In this situation there are no physical or graphical objects for the student to work with, and usually there are no explicit processes. There are only the signifiers (words and symbols) referring to an abstract object, which at that stage is unknown to the learner.
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> What is happening when students seem to muddle their way through new definitions and/or theorems making apparently arbitrary associations between different signifiers and between signifiers and hazy ideas? What is determining the student's activities and how do these activities enable or constrain the student? How do some of these 'initially-muddled' students get to use these new mathematical ideas meaningfully (which it appears that some of them do)?

Although many mathematics educators have considered this epistemological problem (*e.g.*, Tall, 1991, 1995, 1999; Dubinsky, 1991, 1997; Sfard, 1991, 1994, 2000), I have found that their theories do not describe or explain what I observe in mathematical activity at university level.

Tall, Dubinsky and early Sfard (1991, 1994) posit that in order for a mathematical concept to be known by the student, the student needs to transform a mathematical process into a mathematical object [1]. That is, they frame the problem of how a student constructs a new mathematical concept in terms of a distinction between 'process' and 'object' (Confrey and Costa, 1996). Although I regard this distinction as extremely useful, I contend that it is but an aspect of the mathematical learning process. That is, I regard a student who is able to use a particular mathematical process in an apparently meaningful way as already quite far along the path of mathematical object appropriation.

In this article I develop a "language of description" (Brown and Dowling, 1998, p. 2) for examining how an undergraduate university mathematics student appropriates a mathematical object (that is, constructs a mathematical concept) which is new to that student, but which is already part of the official mathematical discourse. I then describe and give exemplars of each of the stages – heaps, complexes and concepts – in the appropriation of a mathematical object, illustrating heaps in this article and complexes and concepts in part 2 [2].

A theory of concept formation

The language of description that I develop is based on Vygotsky's (1986, 1994) theory of concept formation. I suggest that the various activities of an undergraduate mathematics student engaging with a new mathematical object can be usefully interpreted as that student's (non-linear) movement through various pre-conceptual stages. These stages derive from the stages that Vygotsky elaborated in his theory of concept formation but are augmented by certain pre-conceptual stages specific to the mathematical domain. Furthermore, since the student is largely using processes other than logic and deduction in these pre-conceptual stages, the learner's mathematical behaviour may appear bizarre and idiosyncratic. My contention is that an interpretation of the learner's behaviour in terms of these pre-conceptual stages may help the educator to understand the student's 'strange' or muddled mathematical activities.

Sierpinska (1993), in her explication of Vygotsky's theory of development of concepts, similarly argues that:

> the theory could be used to explain some of the curious ways in which students understand mathematical notions, and why, at certain stages of their construction of these notions, they simply cannot understand in a different or more elaborate or more abstract way. (p. 87)

Before explicating and elaborating these stages to the mathematical domain, I will describe the mechanism, "functional use of the mathematical sign" (Berger, 2004), which propels a student through the various stages. The term "functional use" refers to Vygotsky's (1986) thesis that children use words for communication purposes and for organising their own activities before they have a full understanding of what that word means. Through this use of words in socially regulated discourse, the children come to develop mature understandings of these words [3]. In the mathematical domain, I similarly claim that learners start off using a mathematical sign [4] in communication and in

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activities before they have a mature or 'proper' understanding of the mathematical object referred to by the sign. (The role of a mathematical sign is analogous to the role of a word). Through this use of mathematical signs in socially regulated discourse and activities, the student comes to develop a personally meaningful concept that is also meaningful to the wider mathematical community.

That is, usage of the mathematical sign (no matter whether the usage is initially congruent with its usage by the mathematical community) gives the mechanism whereby the student appropriates a new mathematical object and this appropriation takes place in stages that resemble those stages described by Vygotsky in his theory of concept formation (1986).

Mathematics and the individual

It is important to note that this focus on the individual (be it with a textbook or in consultation with a lecturer) does not contradict the fundamental Vygotskian notion that "social relations or relations among people genetically underlie all higher functions and their relationships" (1981, p. 163). After all, a situation consisting of a learner with a text is necessarily social; the textbook or exercises have been written by an expert (and can be regarded as a reification of the expert's ideas) and the text has been prescribed by the lecturer with pedagogic intent [5]. Thus, a focus on the individual does not diminish the significance of the social. As Sfard (2003) puts it:

> Learning is social regardless of the way it occurs. Indeed, learning does not have to be interactive to be social.

The interpreters and translators of Vygotsky, van der Veer and Valsiner (1994), claim that the use of Vygotsky in the West has been highly selective. In particular they lament that:

> *the focus on the individual developing person* which Vygotsky clearly had [...] has been persistently overlooked (p. 6; italics in original).

Hopefully this article will contribute in some small measure to remedying this situation.

My primary concern is with a student:

> reconstructing "old" mathematical means and meanings (van Oers, 2000, p. 137),

particularly from definitions and theorems (possibly guided by specific mathematical activities). Such an approach to mathematical learning at undergraduate level is prevalent in my institution, and to some extent at many institutions of higher learning. Certainly the theory that I develop here would need to be extended and elaborated on if it were to take account of students developing 'new' mathematics, *i.e.*, mathematical ideas which are not yet part of the codified body of mathematical knowledge, or if it were to take account of a more experimental approach to mathematical learning as afforded by, say, the extensive use of a Computer Algebra System or Dynamic Geometry software. However, my aim is to explicate mathematical learning in the here and now of a traditional university undergraduate environment.

Vygotskian stages

According to Sierpinska (1993), Vygotsky's theory around the genesis of concepts is a theory around the genesis of intellectual operations such as generalization of objects and situations, identification of features of objects, their comparison and discrimination (that is, their abstraction), and the synthesis of thoughts. Furthermore, the development of these processes of generalization and abstraction occurs in stages. Vygotsky spoke of three types of pre-conceptual thinking (heaps, complexes and potential concepts) each of which roughly corresponds to a different stage of the development of generalization and abstraction in the individual.

According to Kozulin (1990), Vygotsky's position was that these:

> preconceptual types of representation are retained by older children and adults, who quite often revert to these more "primitive" forms depending on their interpretation of a given task and on their chosen strategy for solution. (p. 159)

It is in this latter sense that I maintain that university mathematics students use heap and complex thinking when dealing with new ideas (I explain in part 2 that I exclude potential concepts from my considerations). In a similar vein, Sierpinska argues that:

> the general pattern of development of conceptual thinking from early childhood to adolescence seems to be recapitulated each time a student embarks on the project of understanding something new or constructing a new concept. (1993, p. 88)

In a very general sense, each of these stages can be described as follows:

> *With heap thinking*, the person links ideas or objects together as a result of an idiosyncratic association. This form of thinking (also called thinking in 'syncretic images') is an initial stage in the child's development of the process of generalisation. Very young children often use heap thinking when first encountering a new object or concept and regard everything vaguely connected with that new object as being of the same category. For example, little children will classify various objects that they associate with water as fish. I have observed how an eighteen-month-old child, after being told that the 'things' she saw swimming in the river were 'fish', pointed at a bird flying above the river and said "fish". A few minutes later she touched a reed and said "fish". Such a child was not yet able to abstract objectively significant attributes of objects; she seemed to group objects together as a result of their proximity or for some other such circumstantial reason.
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> *With complex thinking*, ideas are based on experience and associations rather than on logic or a system but the learner is able to abstract actual attributes of the idea (for instance, 'lives in water'). In Vygotsky's theory, complex thinking involves the development of generalisations upon which further and more refined generalisations and abstractions may be based. For

example, a complex-thinking child (or adult) may describe anything that lives in the sea or river as a fish. According to such a person, an oyster, a crocodile and a whale are all fish. Of course, this is not correct in terms of the scientific definition of a fish.

With conceptual thinking, the bonds between the parts of an idea and between different ideas are logical and the ideas form part of a socially-accepted system of hierarchical knowledge. For example, a person who is able to classify an animal as a fish according to specific and systematic attributes contained in its scientific definition is using conceptual thinking. (According to the 1964 third edition of the *Shorter Oxford English Dictionary*, the scientific definition of a fish is that an animal is a fish only if it lives in water, has a backbone and gills, is cold-blooded and has fins rather than limbs.)

Appropriation theory

As previously indicated, I have extrapolated Vygotsky's theory of concept formation to the mathematical domain. To do this, I have had to make various modifications to his stages of concept formation (see Figure 1 for a representation of my elaborated theory, which I have named 'Appropriation Theory'. Those stages, described by Vygotsky (but which I have extrapolated to the mathematical domain), are indicated by thick lines; those stages that I have developed specifically for the mathematical domain are indicated by dotted lines. I have shown which stages are signifier-orientated by using italics and underlining and I have shown which of the stages are signified-orientated by using a regular non-underlined font).

What are the nature and motivations for these modifications?

Vygotsky's classification of the different stages in thinking derives from an experiment in which people of all ages had to group together concrete objects (blocks of different shapes and colours) for which two sets of stimuli were given – the block itself and the signs (words) attached to each block. Thus Vygotsky's classification does not address certain activities that are prevalent when abstract objects (mathematical objects) with concrete representations (the

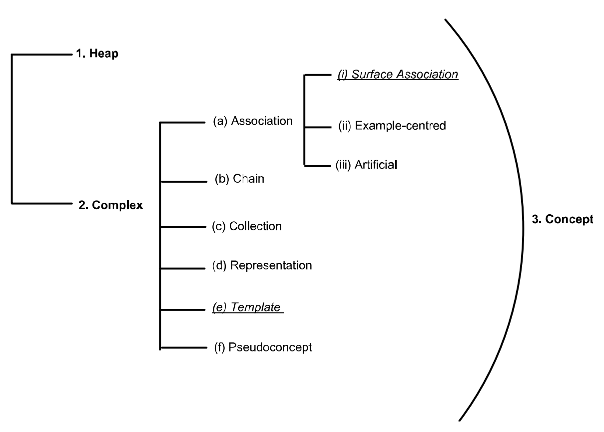

Figure 1: Stages in the appropriation of a mathematical object.

signifiers, *i.e.*, symbols or words) are presented to learners. I have, therefore, introduced several new categories of sign usage that are specific to the construction of a mathematical concept.

Also, unlike the construction of concepts based on the abstraction of attributes of the concrete blocks and their organisation via signs, the construction of a mathematical concept is based on the abstraction of attributes of the sign (signifier or signified depending on the stage of learning) and their organisation via those same signs. Thus I have had to particularise each of Vygotsky's stages in terms relevant to this dual role of the sign in mathematics.

Given that mathematical concepts (like any abstract concept) are represented in terms of signifiers or words rather than concrete objects or experiential sensations, I have distinguished between the signifier-orientated aspects of object appropriation (where the student's primary focus is on the symbol) and signified-orientated aspects of object appropriation (where the student's primary focus is on an idea conjured up by the symbol).

Finally, it is important to note that progression through the various stages is neither linear nor hierarchical; a student may zigzag through the various stages, omitting some altogether (depending on the student and the learning task) and returning to previously visited stages (for shorter or longer periods) at other points. Indeed, a professional mathematician may also revert to complex thinking when dealing with a new or unfamiliar mathematical idea.

Description and exemplification of the stages

Before I elaborate and exemplify the stages in Appropriation Theory, I am going to make explicit my epistemological stance. In line with many philosophers, I assume that observation is always theory-laden:

> Due largely to the work of N. R. Hanson [...] [r]esearchers are aware that when they make observations they cannot argue that these are objective in the sense of being "pure," free from the influence of background theories or hypotheses or personal hopes and desires. (Phillips, 1990, p. 25)

I assume that order in the world does not exist independently of the human mind; rather we impose order on the universe through our various theoretical constructions. Any account that I (or any other researcher) give is an interpretation of what is happening out there; an attempt to structure and organise an understanding of 'reality'. Therefore, any interpretation of the world is necessarily filtered by a particular theoretical perspective, in my case a Vygotskian paradigm.

I do not claim to know what anyone is thinking or why. All I can do is observe usage of signs and classify these usages according to empirical indicators that derive from an explicit scheme (in my case Appropriation Theory). So, although in this article I classify usages of signs as indicative of heap thinking or complex thinking or conceptual thinking, these are not intended as statements of fact. Rather, these are my interpretations of the learner's activities with the mathematical signs, inferred through my (hopefully consistent and rigorous) use of empirical indicators.

Heap

According to Vygotsky (1986) the heap stage is characterised by the grouping together of disparate, inherently unrelated objects that are linked by chance in the child's perception. For example, in a classification task requiring a child to group together all 'similar' objects (in terms of shape, size or colour), a heap-thinking child may group together all objects that are physically close to each other. The learner does not isolate particular attributes of the objects; rather they link objects according to circumstantial or chance criteria, which may not be relevant to the task at hand.

In the context of mathematics, I interpret the heap stage as that stage in which the learner associates one sign with another because of physical context or circumstance rather than through any inherent or mathematical property of the signs. An indicator of this stage is the learner's primary use of a physical context (such as layout on the page) or other non-mathematical context, in their justification or activities with a mathematical statement or question.

An example of heap thinking derives from a set of interviews I conducted with several first-year university students in the year 2000 (Berger, 2002). In this interview, the student, Tom, was asked to interpret the following definition of improper integrals with infinite integration limits. He had not seen this definition before. It is presented here (see Figure 2) with a similar layout to that in the textbook (Larson, 1994, p. 536).

Definition of improper integrals with Infinite Integration Limits

1. If f is continuous on the interval [a, ∞) then

$$\int_a^{\infty} f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx .$$

2. If f is continuous on the interval [−∞, b) then

$$\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx .$$

3. If f is continuous on the interval (−∞, ∞) then

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx$$

where c is any real number.

In the first two cases, the improper integral **converges** if the limit exists– otherwise, the improper integral **diverges**. In the third case, the improper integral on the left diverges if either of the improper integrals on the right diverges.

Figure 2: Definition of improper integrals with infinite integration limits.

Tom read this definition to himself and, at my prompting, explained what he had gathered from the definition.

Pointing to Cases 1 and 2 in the definition and glancing down to the last paragraph, he told me that if one of the limit values is infinite then the integral converges. Pointing to Case 3, he explained that if both limit values are infinite then the improper integral diverges.

Tom's explanation was clearly not correct. He seemed to have created an idiosyncratic link between a single infinite limit and convergence and two infinite limits and divergence. I suggest that this link corresponds to the *font* and the *order* of the presentation of information about convergence and divergence in the last paragraph and the order in which the three cases (one with upper limit infinite, one with lower limit infinite and one with two infinite limits) are presented.

It is as if Tom read the definition as: "In the first two cases, the improper integral **converges** [...] – otherwise the improper integral **diverges**", moving from one bold-type word to another and conveniently omitting the phrase "if the limit exists" after the word converges. He also ignored the last sentence focusing on the third case.

I classify this example as 'thinking in heaps' since there is no actual (concrete or abstract) association between the order of information given in the last paragraph and the convergence or not of the three cases of improper integrals, Type 1. In fact the link is entirely dependent on the font and layout of the information.

In part 2 of this article, I continue with this elaboration of Vygotsky's theory by describing and exemplifying complexes and concepts in the mathematical domain.

Notes

[1] Dubinsky (1991) calls this transformation, "encapsulation"; Sfard (1994) calls it "reification"; Gray and Tall (1994) speak of an amalgam of a process, a mathematical object and a symbol as a "procept".

[2] Part 2 of this article will appear in FLM **24**(3), November, 2004.

[3] In Vygotskian terms a word embodies a concept. Thus the meaning of a word is a concept.

[4] Sign = signifier (*e.g.*, mathematical symbol, alphanumeric symbols) + signified (object 'referred' to by signifier).

[5] Elsewhere (Berger, 2004) I argue for a conception of the zone of proximal development (ZPD) that includes those artefacts (such as texts and computer software) which have been written by people with particular pedagogic intent. After all, such artefacts are social in their origins (they are the ideas which exist in the social arena) and they are designed to guide the student in their appropriation of socially-sanctioned knowledge.

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Let us go back to the original statement for which Korzybski [1941] is most famous – the statement that *the map is not the territory*. This statement came out of a very wide range of philosophic thinking, going back to Greece, and wriggling through the history of European thought over the last 2000 years. In this history, there has been a sort of rough dichotomy and often deep controversy. [...] It all starts, I suppose, with the Pythagoreans versus their predecessors, and the argument that took the shape of "Do you ask what it's made of – earth, fire, water, etc?" Or do you ask, "What is its *pattern*?" Pythagoras stood for inquiry into pattern rather than inquiry into *substance*. [Footnote 1] R.G. Collingwood has given a clear account of the Pythagorean position in *The Idea of Nature,* Oxford, 1945.

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