Theory of Mathematics Education (TME): an Introduction*

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1. Complexity, interrelations, a systemic view

Mathematics education is a field whose domains of reference and action are characterized by an *extreme complexity*: the complex phenomenon "mathematics" in its historical and actual development and its interrelation with other sciences, areas of practice, technology and culture; the complex structure of teaching and schooling within our society; the highly differentiated conditions and factors in the learner's individual cognitive and social development, etc. In this connection the great variety of different groups of people involved in the total process plays an important role and represents another specific aspect to the given complexity

Within the whole system several *sub-systems* have evolved. They do not always operate sufficiently well: especially, they often lack mutual interconnection and cooperation. With respect to certain aspects and tasks mathematics education itself as a discipline and a professional field is one of these sub-systems On the other hand, it is also the only scientific field to be concerned with the total system

A *systems approach* with its self-referent tasks can be understood as an organizing *meta-paradigm* for mathematics education. It seems to be a necessity in order to cope with complexity at large, but also because the *systemic character* shows up in each particular problem in the field. J KILPATRICK has described the situation in a similar way:

> We must take full account in our research of the multiple contexts in which both learning and thinking occur. Each is embedded in interacting systems of the pupil's cognitions, the subject matter, and the social setting. We have tended to concentrate on at most one of these systems, and we have neglected interactions within the system, not to mention interactions between the systems.... To design and conduct studies that can handle the complexities of these multiple contexts is perhaps the biggest challenge we face. ([9], p. 24)

With particular emphasis on the various social groups involved, their actions and interactions, M. OTTE has suggested the following characterization of *the didactics of mathematics* as a scientific field:

> The central problem of the didactics of mathematics, i.e. its scientific object, is the content related and accordingly organized system of relations between all the partners who take part in the realization of mathematics education and its integration into the comprehensive educational and societal contexts additionally determined by the aspect and task of optimization. ([12], p. 7. Transl. by the author)

2. Different views of mathematics education as a science

As a reaction to the extreme complexity of the problems in mathematics education one often finds the opinion that it is impossible to attack these problems in a scientific way and that therefore mathematics education can *never become a science* or a field with scientific foundations. The field is then left open for highly subjective views and beliefs, for short range pragmatism and an interpretation of mathematics teaching as primarily an art

Another reaction is the *systematic reduction of complexity* by selecting and favoring a *special aspect* such as content analysis, curriculum construction, teaching methods, the development of abilities in children, classroom-interactions, etc., and making this specificity the determining center of the whole field. Often this goes together with assigning one of the various referential disciplines, such as mathematics, epistemology, pedagogy, psychology, sociology, or selected theories and methods in one of the disciplines, a preferred and dominating role in establishing the basic orientations and research methods of mathematics education. Accordingly, among those who think that mathematics education as a science is possible and does exist, one finds a variety of *different definitions*, e.g. the study of relations between mathematics, the individual and society; the reconstruction of present time mathematics at an elementary level; the development and evaluation of teachable mathematics courses; the study of mathematical knowledge, its types, representation and growth; the study of children's mathematical learning behavior; the study and development of teachers' competences; the study of communication and interactions in the classroom, etc. In connection with these and other interpretations mathematical education is *differentially classified*: as a special field of mathematics, as a special branch of epistemology, as an engineering science, as a subdomain of pedagogy or general didactics, as a social science, as a borderline science, as an applied science, as a foundational science, etc.

Apparently there is a need for a *theoretical basis* which allows us to better understand and identify the various positions, aspects and intentions which underly the different definitions of mathematics education in use, to analyse the relations between these positions, and to put them together in *a dialectic understanding* of the total field. In addition to a systemic view a *complementarist philosophy* together with *activity theory* seem to provide adequate conceptual tools for coping with this problem

3. Mathematics education on its way to "normal science"?

While the foregoing remarks concern rather global views, the actual debate on the status of mathematics education as a science is focussed more on questions like the degree to which mathematics education has come close to *"normal science"* in KUHN's sense, the *relation between theory and practice* and the problem of *interdisciplinarity*. In the Federal Republic of Germany, the institutional establishment of the didactics of mathematics took place basically between 1960 and 1975 through a considerable increase in the number of chairs at Teachers Colleges, totalling more than 100. Recently most of these colleges have been integrated into universities and most of the chairs have been assigned to mathematical departments. As a consequence it has been articulated by colleagues that, as far as research is concerned, the didactics of mathematics is now in a situation where it has to *keep up with the traditional sciences*. In terms of *Kuhn's concept of theory development* [10] this means that mathematics education as a field of research is to be measured by means of a scale that extends from pre-paradigmatic activities through multi-paradigmatic to mono-paradigmatic sciences. First attempts to analyze by these means the situation in West Germany came to the conclusion that the didactics of mathematics has reached the state of a *multi-paradigmatic* science (see [3]).

With a more international view ROMBERG reports about trends towards "a research consensus in some problem areas in the learning and teaching of mathematics" and toward " 'normal science' in some mathematics education research" ([16], [17]). In his 6 stage "route to normal science", he describes mathematics education as having partly reached stage 3, which he characterizes by *"model building."* It is followed by 3 more stages not yet achieved by mathematics education: 4: paradigm selection, 5: normal science, 6: scientific revolution. For him

> the key step in the evolutionary sequence from myth to tradition to theory is model building. The purpose of building a model is to unpack the broad assertions about a problem in order to elucidate the key variables and the relationships between them. Going from macro-assertions to *micro analysis* is the critical step. ([16], p. 34)

According to Romberg "the best example of research that is nearing 'normal science' is the current work on addition and subtraction" in the study of children's mathematical learning. [17] As another example primarily located in the US research scene, he emphasizes "time-on-task" models such as have been used in the Beginning Teacher Evaluation Study (BTES) and other projects ([16]).

Apparently there is a *need for more and broader investigations* of the status of theory development, model building, use of paradigms, etc., in mathematics education, also taking into account different developments in different countries and performing related *comparative studies*. It would be interesting, e.g., to compare the situation say between the US and France in this respect (see [24]). Also it seems necessary to discuss the *epistemological instruments* by means of which such studies have been and should be pursued. An important improvement of Kuhn's theory concept has been developed by J.D. SNEED ([120]) who makes clear that the *statement view* of theories has to be abandoned in favor of a combination of a *theoretical kernel* and a related set of *intended applications*. Not only can this insight be transferred with important methodological implications from theoretical physics to mathematics, evidencing in a new and profound way the *quasi-empirical nature of mathematics* (see [7]), it also seems to prove useful in better conceptualizing the role of theories and their *applications in mathematics education* (see [1]).

4. The integrative tasks of mathematics education: its role as a science at a university

The discussions in West Germany on the comparability of mathematics education with established sciences and on its expected development towards "normal science" has led to the question whether such goals would be at all desirable for mathematics education. On the one hand, the respective arguments were related to the *role mathematics education should have within a university* (see [21]). Here the answer depends on the goals a university should aim at. Traditionally the *purpose of a university* comprises more than just providing a setting for various kinds of specialized research. One of its important tasks has always been *education* and *teaching*. However, while those tasks could still be pursued under highly specialized and relatively isolated aspects, as in many ways — even in teacher education — is actually the case, a more comprehensive goal for a university in our time would be to *integrate knowledge* from different disciplines and to contribute to a *comprehensive understanding of reality*. Indeed, the university seems to be the only place in society where the *multiple function of the sciences as culture, formation and reflection* can be realized. Therefore, the role of a science in society and within a university is not adequately described only by its purposes as a field of research.

At least with respect to the situation in West Germany, the sciences, especially mathematics, as they are established and functioning in the universities, do not always fulfil the expectations just explained. On the other hand one can say that the didactics of mathematics has developed a considerable sensitivity with respect to the underlined societal goals. Therefore it would be wrong if the didactics of mathematics tried to keep up with the established sciences only from the perspective of specialized research instead of proving its ability by being an example and adopting a *linkage function* between mathematics and society. This is possible and necessary especially by means of its contribution to the *elaboration and actualization of the many neglected dimensions of mathematics: the philosophical, the historical, the human, the social* and — comprising all this — the *didactical dimension*. This implies that mathematics education should not exclusively be determined by its role in teacher education but have a broader didactical task as an orientation both for its research and its teaching.

5. Micro- and macro-models

Another criticism of a predominant orientation of mathematics education towards "normal science" concerns — as was indicated in the beginning — the need of a *comprehensive understanding* of the interrelation between the various aspects and contexts and thus a need of a *theoretical frame* or *meta-paradigm* which *combines selectivity and unity* ([23]). This has also been discussed with respect to the *importance of macro-models as compared to micro-models.* Here we have Richard SKEMP's reaction to ROMBERG's judgments:

> A year or two ago, the view was expressed by a prominent mathematics educator that what we need is not a global model but a collection of relatively specific models for specific situations. I do not share this view. I think that we need micro- *and* macro-models ... We certainly need investigations of this kind, leading to micro-models giving much detail in small areas of children's mathematical learning.... But we do also need a macro-model within which all these micro-models will eventually fit together, and for which at any particular point, one of these micro-models provides interiority. I see two complementary ways in which this may come about. The first is the provision of a macro-framework, relating meaningfully the many aspects of mathematical education: cognitive, emotional, interpersonal, and also the goals of learners, the goals of teachers, parents, taxpayers, employers.... The second is by mutual assimilation of these micro-models..... As a contribution to helping this come about, I would like to make explicit what I see as three requirements, necessary but not sufficient. These are: models are of the same type; researchers have similar goals; researchers meet and discuss ([19], p. 113)

One will have to investigate what examples of macro-models we have and what purposes they fulfil, whether sameness of types and similarity of goals are really necessary conditions as opposed to *dialectic* and *complementarist ways* of handling seemingly contradictory models and controversial positions, whether meeting and discussion of researchers hints at the direction that was indicated by the systems view outlined in the beginning of this paper.

6. Home-grown theories vs. interdisciplinarity

In his talk on "Research on Mathematical Learning and Thinking in the United States" given at the 1981 PME conference in Grenoble, KILPATRICK not only complained about an observable lack of attention to theory in much US research in the field of his report, he also criticized the strong *borrowing of theories from other disciplines* and the lack of what he calls *"home-grown"* theories. In quoting a paper by D. SANDERS (published in *Educational Researcher* 10 [1981] 3: 8-13), he states:

> Further, when theoretical constructs and contexts *are* used they are not "home-grown" — they are today, as they have been for years, borrowed from outside mathematics education. As Donald Sanders (1981) has noted: "[We have a] tendency to approach educating through constructs rooted in psychology or the social sciences rather than through theories or constructs fitting the phenomena as they appear in educational settings. ... Educational research rooted in the theories and paradigms of related disciplines may advance those disciplines, but it does not necessarily advance scientific knowledge of the process of education." ([9], p. 23)

I think there would be a *danger of inadequate restrictions* if one insisted in mathematics education on the use of home-grown theories. The nature of the subject and its problems ask for *interdisciplinary approaches* and it would be wrong not to make meaningful use of the knowledge that other disciplines have already produced about specific aspects of those problems or would be able to contribute in an interdisciplinary cooperation. Actually, interdisciplinarity does not primarily mean borrowing ready-made theories from the outside and adapting them to the condition of the mathematical school subject. There exist much deeper interrelations between disciplines.

In an OECD-CERI-Seminar about interdisciplinarity held in Nice in 1970 (see [11]), Jean PIAGET in his talk on "The Epistemology of Interdisciplinary Relationships" which was based on his structuralist philosophy, gave the following short analysis:

> ...a structure (of a science) extends beyond the boundary of phenomena. Only its manifestations are observable; as a system it is grasped only by deduction, therefore by connections not observable as such. ....insofar as a structure extends beyond the observable, it leads to a profound change in our concept of reality. ..... We no longer have to divide reality into watertight compartments or mere superimposed stages corresponding to the apparent boundaries of our scientific disciplines. On the contrary, we are compelled to look for interactions and common mechanisms. Interdisciplinarity becomes a prerequisite of progress in research, instead of being a luxury or bargain article.
>
> An obvious consequence of the evolution ....is not that science develops at one level only; each comprises various levels of conceptualisation or structuralisation. Hence, each discipline sooner or later has to work out its own epistemology. But if the search for "structures" in the sense of underlying transformation systems is already a basic factor of interdisciplinarity, it is clear that any internal epistemology, aiming in particular at characterising existing relationships between observables and the models used in a science, will very soon be an integral part of the epistemology of the neighbouring sciences, not only because the epistemological problems are found everywhere but because the relationship between subject and object can only be discovered by comparative means. ([15], p. 128/129)

Coming back to mathematics education, we have to observe that for its particular purposes there not only exists a level at which the various referential disciplines are or should be cooperating and interrelated. For the *problem-identification* and the *coordination* of interdisciplinary work mathematics education has an essential *regulating and organizing function* which is located at another level and seems to be indicated by what Piaget called *transdisciplinarity*:

> Finally we may hope to see a higher stage succeeding the stage of interdisciplinary relationships. This would be "transdisciplinarity", which would not only cover interactions or reciprocities between specialized research projects, but would place these relationships within a total system without any firm boundaries between disciplines ... As for defining what such a concept should cover, it would obviously be a general theory of systems or structures including operative structures and probabilist systems, and linking these various possibilities by means of regulated and definite transformations. ([15], p. 138/139)

Apparently mathematics education has not sufficiently reflected and practiced these indicated relations between disciplines. Rather than restricting its search for theoretical foundations to *home-grown theories* it should develop more professionality in formulating *home-grown demands* to the cooperating disciplines. This means that in addition to its subject-specific competencies, mathematics education has to bring in a *transdisciplinary regulative function* as a kind of indispensable *meta-competency*. We have — I think — a few examples where interdisciplinarity and transdisciplinarity around mathematics education have been exercised in a successful way and we definitely need more exchange of experiences and discussions on these matters.

7. Systems approach, values, and the management philosophy of science

Piaget's ideas on transdisciplinarity have been taken up and further developed in a specific direction by Erich JANTSCH in his paper "Towards Interdisciplinarity and Transdisciplinarity in Education and Innovation" presented at the same OECD-CERI-Seminar (see [11]). Jantsch tries to design an *"integrated systems view of science, education and innovation"* which uses as a *value base* and a *purpose* for this dynamic system the notion of *"creating an anthromorphic world"*. Bringing in these values and goals is only possible if science and its external and internal relations are understood not independently of societal and *social processes of decision making and transmission.* Here, Jantsch refers to CHURCHMAN's *management philosophy as the true foundation of science*:

> A systems approach — as it is proposed in this paper — would consider science, education and innovation, above all, as general instances of purposeful human activity, whose dynamic interactions have come to exert a dominant influence on the development of society and its environment. *Knowledge* would be viewed here as *a way of doing, "a certain way of management of affairs,"* (Churchman) ([18], p. 99)

I have quoted Jantsch here for two reasons. First, I think that *value and goal orientation* should also be important factors of every *systems view of mathematics education* commencing with the fact that mathematics itself — like other sciences — is not a value-free human activity (see [22]). I have the impression that the discussion on values and goals, especially the *ethical, social and political aspects* of mathematics education, have been unduly neglected and separated from other research problems, as if rational arguments on these matters were not possible. It is probably one of the *essential weakness* of mathematics education that it has not sufficiently developed the related type of rationality with respect to its own field and domain of responsibility, thus bringing about all the *negative consequences for its self-concept* and *social status*.

Secondly, I believe that CHURCHMAN's *management philosophy of science* especially applies to the role the *didactics of mathematics* as a *self-reflective discipline* has to play in taking regulating functions both in *interdisciplinary research* affairs as well as in the *theory-practice interplay*. Churchman has described his role as a systems scientist when trying to study fundamental problems of sectors of our society and I repeat here a probably well known quotation:

> I can well remember my attempts to advise railroad managers. Those were men who had to come up through the ranks and clearly knew far more than I would ever hope to about the intricacies of railroad operations. They were correct in saying that what I could contribute would be naive. They were wrong, however, in saying that a naive system-science approach to railrods was useless. The approach provided another way of looking at a railroad — as a system, not as a physical instance of transportation. Since then, I have had the same experience with managers in health, law, education, defense and production. They all wanted to know how an operations researcher in six months, or a year, or ten years, could ever hope to 'solve' their problems. Of course, he can't, for he is no more of an expert than they are about the really fundamental problems. But he can provide a link of the *maximum loop,* a way to reflect that no profession by itself can ever hope to provide ([4], p. 115)

The reference made here to the *maximum loop principle* as opposed to the minimum loop idea which erroneously suggests that a thing can just be measured by itself, indicates a profound insight into the *importance of the consideration of large systems,* the *nature of self-reflectedness* and how these two are *interrelated*:

> The principle is fantastic. It says that self-reflection is possible only if one returns to the self after the longest possible journey. ([4], p. 113)

8. Complementarity, human activity, meta-knowledge: the role of practice

All our attempts thus far to exhibit various aspects of a comprehensive approach to mathematics education could yield the impression that there is an underlying structural *coherence* and *homogenicity* as an essential base for the intended integration and synthesis. However, there are *fundamental epistemological phenomena* which show us that the *opposite is the case*. As in physics, relativity theory and structuralism were followed by *quantum theory* with its *relation of indeterminateness*, and in N. BOHR's famous *principle of complementarilty* it became clear in a much broader sense that

> every relevant piece of theoretical knowledge being part of some idea or model of the real world, will in some way or another have to take into account that the person having the knowledge is part of the sytem represented by the knowledge. All knowledge presupposes a subject, an object and relations between them (which are established by means of the subject's activity). Therefore, all knowledge has an incoherent structure with metaphorical and strictly operative connections. ([13], p. 45/46)

In more recent times, this phenomenon has also been identified and confirmed with respect to mathematics and mathematics education (see [6], [23]). For example, most of the so called *"false dichotomies"* Peter HILTON was dealing with in his plenary talk at the Karlsruhe congress ([5]), such as "skill vs. understanding", "structure building vs. problem solving", "axiomatics vs. constructivism", "pure vs. applied mathematics", represent *pairs of semingly opposing positions* which can be pursued through the history of mathematics and mathematics education. In educational practice and theory one has often tried to straighten out these paradoxes in a *reductionist way*: one has either given *absolute dominance* and principal importance to one of the two sides or one has taken a so called *multi-aspect position* which just says "do both" without really understanding and operationalizing the underlying antagonistic relationships which are actually connected in a fundamental way with the epistemological problem of the *relation between knowledge and activity* as the *kernel of all the complementarities*.

The concept of complementarity also turns out to be an adequate tool for better understanding the *relations between different types* and *levels of knowledge and activity* as the appear in contrapositions like "scientific theory vs. everyday knowledge", "meta-knowledge vs primary knowledge", "empirical vs. formal", "the personal vs. the social", "perception vs. cognition", etc., and also as they come up in the *regulation and control problem of systems-theory*. It plays an important role in the *foundations of cognitive psychology*. In his article titled "The Need for Complementarity in Models of Cognitive Behavior", PATTEE writes:

> The classical idea that we can explain control in cognitive systems without complementary modes of description verges on a self-contradiction, or at least a conceptual paradox. Complementarity may be viewed as a recognition of the paradox. It has its roots in the subject-object dualism and in the basic paradox of determinism and free will. ... An idea that I would promote is that psychologists make the difficult effort to assimilate the basic concept of complementarity as an epistemological principle. It is by no means a clear and distinct concept, but it is rich and suggestive. The complementarity principle does not promote resolutions of the central binary oppositions of psychology: mind and body, structure and process, subject and object, determinism and free will, laws and controls, etc. On the contrary, ..., the principle of complementarity requires simultaneous use of descriptive modes that are formally incompatible. Instead of trying to resolve apparent contradictions, the strategy is to accept them as an irreducible aspect of reality. ([14], p. 26/27)

This seems to be an adequate *description at a phenomenological level*. However, there are *deeper mechanisms* behind. Their analysis and reconstruction need conceptual tools as they are available from *activity theory*, which tries to build an understanding of cognition primarily on a concept of *"objective human activity"* rather than on "knowledge". Human object-related cooperation with its *practical wisdom, habitual features* and its *socio-historical reality* in BOURDIEU's sense (see [2]) plays a special fundamental role as a *"structuring structure"* in the regulation and control problem. M. OTTE has described the interrelation between complementarity and human activity in the following way:

> Only within activity theory can the epistemological need for complementarity be productively developed and applied. On the other hand, a complementarist viewpoint ... should prevent activity theory from detrimental reductionism and at the same time provide possibilities for the necessary and unavoidable relative reductionism. As far as the problem of cognition is concerned, we have to accept that we cannot know without knowing that we know. We cannot learn a particular theoretical concept without acquiring knowledge about theoretical concepts (their categorical characteristics, their function for cognition etc.). We cannot gain knowledge without acquiring meta-knowledge. But meta-knowledge is at one point the product of the evolution and at another its indispensable condition. Therefore, knowledge and meta-knowledge can neither be completely expressed nor represented as a closed and coherent system and in a uniform description. ([13], p. 53/54)

The *consequences for the development of* TME should be clear: It can only be done successfully if it proceeds at a very general level and at the same time through very concrete examples; if it performs the systems approach simultaneously for the large system and for particular problem domains understood as sub-systems; if it is aware of and contributes to the elaboration of the inherent complementarities and related types of activities; if it is heading towards a simultaneous development of "the practical sense" and of "meta-knowledge", observing their fundamental interrelation.

9. IME as a developmental program

I can see basically *three components of TME*, which are, of course, interrelated:

— *meta-research* and development of *meta-knowledge* with respect to mathematics education as a discipline

— development of a *comprehensive view* of mathematics education comprising research, development, and practice by means of a *systems approach*

— *development of the dynamic regulating role* of mathematics education as a discipline with respect to the *theory-practice interplay* and *interdisciplinary cooperation*

Thus far I have not explicitly emphasized *meta-research* as a component of TME. One of the reasons is that it is just what this paper is trying to do, or at least trying to outline how it could be done. In his PME talk at Grenoble already referred to, KIPATRICK viewed meta-research in the following way:

> A final need is to devote some attention to scholarly inquiry into and reflection on our own research activities. Scriven (1980) terms this "self-referent research", and at first it may seem just another gimmick to find something else to do research on But just as meta-cognition — cognition about one's own cognitions — is indispensable for intellectual growth, so some meta-research efforts are required for the growth of our field. ([9], p. 24)

This is much in accordance with what we have called the development of meta-knowledge. However if one takes Kilpatrick's reference to SCRIVEN's paper ([18]) seriously one would also have to include under "self-referent research":

> ...research dealing with *vehicles* of ..research, for example, the printed word, the machinery for its processing and production, and the journals which select and package it,... studies of research practice, of the proportion of women presenters at meetings of professional organizations (such as AERA) compared to the proportion of women members, ... *policy* research on, for example, the policies of research journals, ...*cost*-effectiveness evaluations... : of systematic efforts to replace review panels with linear regression equations; .. using teleconferencing rather than traveling for the peer panel discussions, .. developing indexes of cost-effectiveness in research, and so on. ([18], p. 7/8)

I hope that as a component of TME meta-research is put into a *broader context* which helps to identify goals, to set priorities, and to develop strategies for the work to be done. I think we could start with discussions on a variety of points which have been touched upon in this paper. From these discussions we may go over to designing some first plans for further work at a national and international level.

The points suggested for a first discussion are:

1. Different definitions of mathematics education as a discipline.
2. The use of models, paradigms, theories in mathematic education research. The state of the art. Tools for analysis.
3. Micro- vs. macro-models.
4. "Home-grown theories" vs interdisciplinarity, transdisciplinarity.
5. Relations between theory and practice.
6. The place and role of mathematics education in academic institutions, especially universities.
7. The ethical, societal and political aspects of mathematics education.
8. The need for comprehensive approaches. Self-referential and self-applicable theories. The role of a systems view.
9. Complementarity and activity theory.
10. Types of meta-research.

Let me underline again: The *primary goals* of TME are to give mathematics education a *higher degree of self-reflectedness* and *self-assertiveness*, to *promote another way of thinking* and of *looking* at the problems and their interrelations. It is an *open developmental program* with much emphasis on the *process* and their partly *indirect effects*.

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Why, then, did the Cambridge mathematicians such as Peacock, Babbage, the young Whewell, De Morgan and the other followers of the analytic movement treat symbols as "the all in all" of algebra? Why did it seem natural to treat symbols as self-sufficient, as if mathematics were essentially about marks on paper? ... the answer lies in the social meaning and use that was attached to the doctrine. The mathematicians imputed self-sufficiency to their symbols when they, their users, were asserting their own self-sufficiency and impressing that fact on others. Formalism was useful to the emerging "professional" mathematicians of Cambridge and London because it brought mathematics entirely within their grasp. It made it out to be an internal system of meanings in which no one else had a legitimate interest. It celebrated the self-sufficient character of mathematics, and hence the self-sufficient character of mathematicians.... Conversely, symbols were denied autonomy and were portrayed as standing in need of reference to something ideal when their users — like Hamilton — wanted to impress on others the need for an analogous dependence in the social realm.... These doctrines were, therefore, ways of rejecting or endorsing the established institutions of social control and spiritual guidance, and the established hierarchy of learned professions and intellectual callings. Attitudes to symobls were themselves symbolic, and the messages they carried were about the autonomy and dependence of the groups which adopted them.

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