TEACHING FOR UNDERSTANDING FOR TEACHING

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Although there has been much interest of late in teaching for understanding [1], mathematics educators started exploring this area as early as the 19th century (Price, 1975). Before long, understanding became a main interest, either as a prerequisite to learning and to teaching or as the desired end result of the teaching-learning process. What exactly constitutes 'understanding' in mathematics? More specifically, what is the stance that a teacher could hold and what are the experiences that a teacher could provide to her or his students in order for these students to 'understand' mathematics?

Many students cannot provide any explanation for why they 'flip' the $^1/_2$ or for what the 6 stands for in $3 \div {}^1/_2$. Frequently, students resort to the mantra "but I know how …".

This article describes a somewhat vigorous constructivist approach that I have employed in order to encourage my student teachers to unpack their fragile mathematical ideas and to re-construct a deeper and more flexible understanding of the different basic mathematical concepts (which for the most part they already 'know'). The main goal of this approach is to focus the attention of prospective elementary school teachers on understanding their mathematics-doings in the field of arithmetic, not to teach them the basic arithmetic concepts. This approach does not focus on the teaching of fractions but on the teaching of arithmetic as a whole. However, some of its applications, that I found to be more powerful and which I consider here, deal with fractions.

I begin with a conceptual framework, which draws from different theories to establish what constitutes understanding in mathematics and then turns to works that aim toward "teaching for understanding". After an overview of my approach, I describe it *vis-a-vis* a class engagement. A summary of its main attributes will conclude this section. After giving an example of one of its applications, in the discussion I highlight some of the key aspects of the proposed approach, and refer briefly to the possibility of employing it in elementary school.

Conceptual framework

When looking at the large scope of works that address the question "What constitutes understanding?" in mathematics, there seemed to be two main perspectives. One could be viewed as a *latitudinal* perspective and the other as a *depth* perspective. In the depth perspective, researchers identify different *levels* of understanding, as opposed to different *types* of understanding in the latitudinal perspective. In the latitudinal perspective, there are both *contextualised* and *de-contextualised* approaches. For example, among those who introduce de-contextualised types of understanding we find Skemp (symbolic, relational and instrumental, 1976, 1982), Greeno (structural and semantic, 1984), Lehman (operational and relational, 1977), Pirie (instrumental, relational and intuitive, 1988), Outhred and Mitchelmore (intuitive, 2000) and English (structural, 1999). In contrast, in the nineties, there were more studies that focused on mathematically contextualised types of understandings such as *geometrical understanding* (Pegg and Davey, 1991), *statistical understanding* (Roberts, 1999) and *stochastic understanding* (Truran, 2001).

There are also combined perspectives (see Buxton, 1978) where there is not only the traditional classification of different types of understanding but these categories are also put in some hierarchical order (see, also, Herscovics and Bergeron, 1983).

Attributing levels to understanding introduces understanding as a process (Sierpinska, 1990, p. 29; Pirie and Kieren, 1989) as opposed to understanding as an act. For Davis (2001), mathematical understanding "must be a matter of degree" (p. 137) and there is no such thing as "full" or "complete" understanding. The "dynamic of folding back to move out" that Pirie and Kieren (1989) attribute to their transcendentally recursive theory delineates understanding as an organizing process to entail action, perception of action, conception of percepts and conception of concepts.

The different theories that refer to understanding as a process differ in the number of levels that they attribute to the process of understanding and in the description of these levels. For example, Sierpinska (1990) introduces: *identification*, *discrimination*, *generalization* and *synthesis* (p. 29) and Pirie and Kieren (1989) present eight levels, beginning with *primitive knowing* and *image making* and concluding with *inventizing*. However, Thwaites (1979) brings up the dangers of making too precise a particular concept in educational psychology. Thus, the levels for him are of complexity – the number of contexts in which the knowledge can be applied and the number of relations made between them. It is about a relational approach with more/less understanding instead of naming stages.

On the other hand, there are those who view understanding as an act, such as the *ah-hah* instances. The metaphor of the jig-saw puzzle (Davis, 1992, p. 192) is useful in describing the "act" of understanding – the moment in which one piece "brings out" the entire picture – the picture that, a moment before, was nothing more than a collection of assembled colored pieces. However, this metaphor diminishes the importance of this part of the process – the effort that was put into assembling all the previous pieces together,

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that enabled this *ah-hah* instance. Thus, Sierpinska's (1990, p. 26) view of understanding as an act involved in a process of interpretation seems more appropriate.

The approach described here emphasizes the process rather than the act; it emphasizes the assembling of the many disconnected pieces of knowledge by finding the relations and the connections between them until the instant that the full "picture" (concept) "comes out". Time is yet another issue to be considered:

> In putting theory into practice, we gradually recognized perhaps the greatest challenge in teaching for understanding: it takes time to engage students in performances of understanding. (Unger, 1994, p. 9; see, also, Sierpinska, 1990, p. 29)

In incorporating constructivist approaches to my college level teaching and class practice, I have had to struggle to reconcile these approaches with the institutional demands, still influenced by the pre-reformed era of direct lecturing.

Lastly, the emergent constructivist movement places conceptual understanding as dichotomous to basic skills (Preston, 1975). Wu (1999) rejects this and calls it "a bogus dichotomy" (p. 14 in title). He describes mathematics, skills and understanding as being completely intertwined and argues that no-one can acquire conceptual understanding, problem-solving skills or basic skills individually; rather, they go hand in hand, and deep understanding of mathematics ultimately lies within the skills.

My approach assumes an integrated view, continuing Wu's perspective. I refer to it as "constructivist drill", which emphasizes conceptual understanding, individual constructs, multiple representations, verbalization, reasoning and making connections, but at the same time placing a significant emphasis on repetitive, gradually progressing assignments (*i.e.*, drill). This is done in the context of a basic mathematics course for pre-service elementary school teachers.

The teaching-learning-space approach

I perceive the core of the *Teaching Learning Space* (TLS) in which the teaching-learning experience is taking place as consisting of three sub-spaces: the *contextual*, the *abstract* and the *physical-abstract*, which differ in the kinds of reasoning that are employed in each of them, in the language that is used in each of them and in the kinds of activities that are performed in each of them. The transition among the three sub-spaces is effected by goal-driven means of re-phrasing in the language, which is used in the new sub-space. This TLS model provides a frame for relating various concepts across different representations and across different situations.

A *comprehensive support-system* surrounds the core of the TLS to address motivational issues (see Figure 1). This includes *verbal reinforcements* in order to boost the student's self-image and their confidence as well as to reduce their anxiety level. Also, a relaxed and informal atmosphere aims to encourage students to discuss, as openly and as frankly as they can, their mathematical ideas, their feelings and their attitudes towards the learning experience. Furthermore, the students have as many one-on-one instructional sessions as they need and many e-mail consultations.

In working with the space, in addition, I employed a supportive grading system in which the learning processes

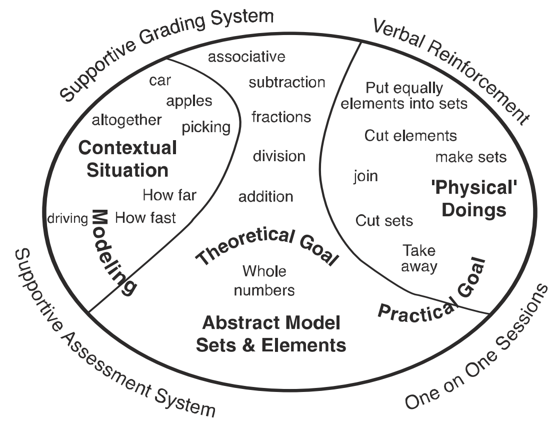

Figure 1: The Teaching-Learning Space (TLS).

(rather than the results or products) were assessed and graded and in which most of the points were given on "proven hard work", *e.g.*, the final examination was only 25% of the total grade; 10% was assigned to the weekly homework assignments (which were returned fully checked and with relevant comments); 12.5% was assigned to two papers of extensive reasoning tasks, more comprehensive than the usual one page homework tasks, where the student teachers collected their students' work and analyzed them and approximately 50% was assigned to quizzes and examinations that students could re-do.

There are many paths along which it is possible to walk through the different sub-spaces of the TLS and many ways to use the model to promote an understanding of mathematical concepts in the students.

The main path provides tools for studying one specific event: mathematical situation – contextual situation – abstract set model – physical abstract situation – contextual situation – mathematical situation (see Figure 2). Other paths are hierarchical: *whole number – fractions* or *additive – multiplicative* continuums. The latter two paths are interrelated and embedded in each other and in each of the applications of the main path.

Applying the model

Next, I describe the different sub-spaces *vis-a-vis* a description of a class engagement that aims to add meaning to the mathematical algorithm: $3\,{}^{1}/_{4} \div 2\,{}^{1}/_{5} = {}^{13}/_{4} \div {}^{11}/_{5} = {}^{13}/_{4} \times {}^{5}/_{11} = {}^{65}/_{44} = 1\,{}^{21}/_{44}$.

The contextual sub-space

By this I refer to the level of the story problem. Thus, the language here is *real* and is related to the 'story', and so are the activities, as well as the reasoning. The transition from the contextual to the abstract space is motivated by the problem, which is first stated in the contextual language.

In class: We began by constructing a *real problem* for the *mathematical problem*. Since we were dealing with division, we thought about whole number division, for example $6 \div 2$. We tried to come up with a *real* situation of sharing 6 between 2, or measuring how many 2s are in 6. We looked at real problems that also made sense in case of fractions, so "we have x pizzas to share among y people" is out of the question. Eventually we agreed on something like:

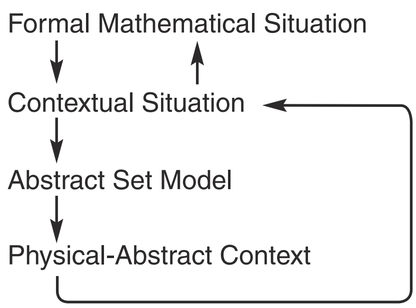

Figure 2: The pedagogical path.

> a gallon of ice cream is made with $2^{1}/_{5}$ cups of sugar. Liz has $3^{1}/_{4}$ cups of sugar. How much ice cream, in gallons, can she make?

This was followed by a *(verbal) discussion*: Will she make at least 1 gallon of ice cream? 2 gallons? More? Why? The *reasoning* here was based on

> she has more than 3 but less than 4 cups of sugar – so she'll make more than 1 gallon but less than 2 gallons of ice cream.

This discussion took place without any actual writing, drawing or calculating. We concluded the discussion with a *real strategy*, *whole number language*, to find out 'how many gallons' of ice cream she can make; for example,

> put the amount of sugar that is required for each gallon of ice cream in a separate heap, until there is no more sugar left, then count the 'number of heaps'.

The abstract sub-space and the abstract-set-model

By modelling the contextual situation we move to work in the *abstract sub-space* and to use *abstract sets-language*. Complex situations involve staging or breaking down procedures and using the model iteratively; but the basic model refers only to the basic binary operations and it has three components and a theoretical goal that is associated with it. The *model components* are: the *number of disjoint sets* that are involved in the situation (first); the *number of elements in each such whole set* (second); the *number of elements in the union set* (third). The *theoretical goal* stems from the desire to *complete* the model (*i.e.*, to find the missing component), or to *describe* the relations between two of the model's components. In the latter case, I would refer to it as a *relational-theoretical goal*.

Hence there are two basic types of the *abstract set model* (Figure 3): the *additive model* and the *multiplicative model*. While a *constant number* (2) of disjoint sets are involved in the situation characterizing the *additive model*, the *equal size* of the disjoint sets characterizes the *multiplicative model*. In both models, the third component is the number of elements of the union set of all the disjoint sets. The contextual space determines the theoretical goal for the abstract space, which is therefore expressed in set-language. If the theoretical goal is to expose the third component (*i.e.*, number of elements in the union set), then the model represents an addition (additive model) or multiplication (multiplicative model) situation, whereas if the theoretical goal is to expose

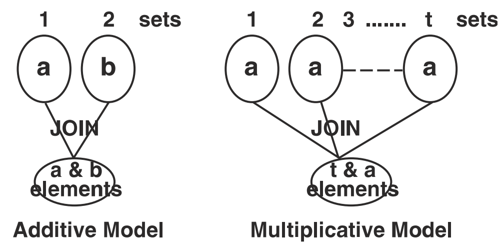

Figure 3: The abstract-set model.

the second (*i.e.*, number of elements in each whole set), then the model represents a subtraction (additive model) or a division (multiplicative model) situation. The latter is traditionally referred to as the *partitative approach* to division. If the theoretical goal is to expose the first component (number of sets), which is relevant only in a multiplicative model, then the model describes what is usually referred to as the *measurement division approach*. Relational-theoretical goals in the additive model could describe either additive relations (bigger-smaller) that are basically subtraction situations, or multiplicative relations, ratio (or proportion) situations.

In class: Since there are not 'exactly' two sets (bowls) and since all the sets (bowls) must have the same number of elements (same number of cups of sugar), the modelling of the real strategy leads to a *multiplicative model*; where the number of sets is unknown, there are $2^{1}/_{5}$ elements (rather than 2 and $^{1}/_{5}$ of an element for *whole number* language) in each *whole* set and $3^{1}/_{4}$ elements in the union set.
Thus the theoretical goal is to expose *the number of sets*. The reasoning here revolves around the appropriateness of the other models, too. For example, could it be that $2^{1}/_{5}$ is the number of sets and $3^{1}/_{4}$ is the number of elements in each whole set? If we have more than 2 sets each with more than 3 elements, in total we need to have more than 6, which we don't. What would this imply in the real situation?

The theoretical goal leads to a practical goal: I look for an activity to do 'physically' in order to achieve the theoretical goal, *i.e.*, to join sets, to take-away elements or to put-equally into sets.

In class: Set a practical goal, for instance, to use all our resources ($3^{1}/_{4}$ elements) to make as many sets as possible, each with $2^{1}/_{5}$ elements (whole number language), which makes the transition to where we actually *do* the mathematics.

The physical-abstract sub-space

The most significant feature of this space is the (mental) 'physical-doings' and language that are employed on the abstract objects (sets and elements). These physical-doings also serve to reason 'physically' about the situation. All the 'doings' shown here are accompanied by drawings to show how and words to explain why.

In class: The 'doings' of a student are described in Figure 4. The student starts making sets of the required size. He first cuts up one element into five equal pieces and then he puts 2 whole elements and one fifth of an element together to make up one set. Since there are not enough elements to make one more whole set the student puts all the leftovers together to make as "many" (whole number language) sets as possible. He needs to count the number of sets

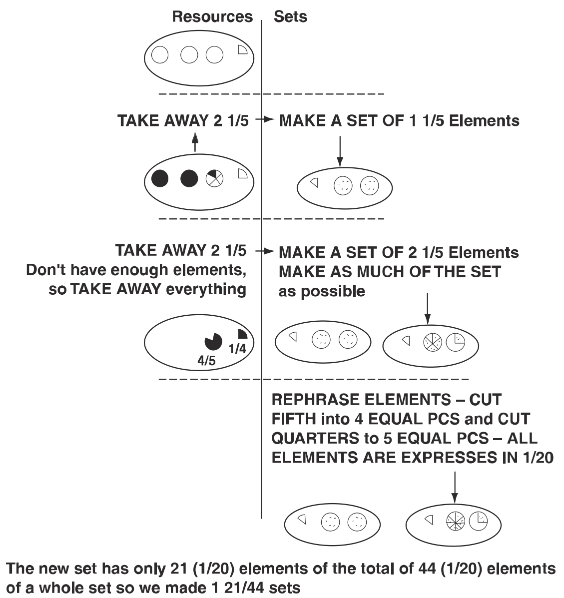

Figure 4: The physical doing of $3^1/_4 \div 2^1/_5$.

he made up. For this, he rephrases the "names" of the elements (uses different units, given that the original unit of 1 element is not practical anymore), by cutting up the fourth into five equal pieces and each fifth into four equal pieces to get many "new" elements, such as $^1/_{20}$.

I also suggest and discuss alternative ways, for example, changing the unit of reference in the beginning:

> in cutting each $^1/_4$ of an element into 5 equal pieces and each $^1/_5$ into 4 equal pieces, work with $^1/_{20}$ of an element instead of one whole element. Thus, we *rephrase* the problem to the whole number problem of: 65 ÷ 44.

Contextual realm revisited

For closure, we take our 'products' back to the contextual realm. To *insert/relate our products/results* into the real situation; $1^{21}/_{44}$ what? Does it make sense? Did we get a negative number for the number of gallons? More than 1 gallon and less than 2 gallons, as we estimated in the beginning? We also discuss our results in terms of the *measurement* approach. For example, we *measure* how much sugar Liz has by the "number of gallons of ice cream it can make" (not by cups) or by a "special measuring cup" of a size of $2^1/_5$ cups;

The mathematical realm

Then back to the mathematical realm to discuss the mathematical principles that underlie our doings, such as, issues of commutative, associative, distributive, multiplicative and additive inverses; What are the 'physical' sources or the 'physical' indications for them in our physical-doings?; Where did the 44 come from?; What does the 'flipping' mean?

The main principles of the TLS approach

Thinking through the previous example it is possible to focus on the main principles of the TLS approach:

1. constructing a natural-intuitive understanding of the mathematical concepts by using 'natural' operations, which require no formal learning of a procedure such as 'to join', and by using visualization tools, such as drawings

2. building on a deep understanding of simple whole number situations as a basis for all further learning: a) using a whole number language 'we have *half* groups', rather than 'we have half a group'; b) using whole ('natural') number models to deal with 'new' non-natural kinds of numbers, for example, a division like this $6 \div {}^1/_2$ is much the same as a division like this, $10 \div 2$, so *do* the same!; sequencing a continuum from whole number situations to fractions [2]

3. using an abstract model, which conceptualizes all basic mathematical situations as the same, to channel and to mold the reasoning.

4. promoting a reasoning attitude: a) any statement is accompanied by reasons for, such as, "Why is it true/false?" and "Why is it important/not important?"; b) making as many statements as possible about any given situation, such as, declarative, descriptive and relational statements [3]; c) exploring each situation thoroughly, rather than employing a 'task-oriented' exploration [4]; d) massive reasoning tasks (using this model as well as others).

More on the TLS approach

There are many more applications of the TLS that make mathematics more meaningful and more accessible to the students. For example, the *physical-abstract* realm together with the *abstract set* model can be used to accentuate the differences between additive and multiplicative structures and to channel the student's reasoning in distinguishing between these structures – an area that is long known to be one of great difficulty for students. Also, making the different characteristics of the additive and the multiplicative models explicit, in the many different situations that we explore in class, helps the students to see the similarities and the differences between additive and multiplicative situations; this, in turn, helps them to make the distinction when necessary.

One of the most significant applications of the TLS approach is in the understanding of the understanding of others, for example, I describe how students make sense of an unusual partial solution for a long division problem (Figure 5). The students were asked to justify, to reason and to 'finish' Anonymous's work.

30 100's is 3000, 6 100's is 600,
130+130+130=390, 6 13's is 2 39's
is 2 40's less 2 is 78, 390 and 78 is 400
and 68, 468 and 74 is 500 and 42, 542...

Figure 5: Anonymous's doings 3674 ÷ 87.

Only those who wrote down both options for the multiplicative model (see Figure 6), and searched for 'clues' in Anonymous's work to try and understand which model support their doings, succeeded.

Number of Sets in Situation	87	Number of Sets in Situation	?
Number of elements in Each Whole Set	?	Number of elements in Each Whole Set	87
Total Number of Elements in the Situation	3674	Total Number of Elements in the Situation	3674
Theoretical Goal: Number of Elements in Each Whole Set	A	Theoretical Goal: Number of Sets in Situation	B

Figure 6: Two models – model A, partitative and model B, measuring.

Each model leads to a different practical goal and setting (see Figure 7): Model A leads to "put equally all your 3674 elements into 87 empty sets" and Model B leads to "make as many sets as possible, 87 elements each, with all your 3674 elements". If Anonymous was thinking about Model A, we need 'to see' 87 sets (or something close) filling up. Since nowhere in Anonymous's work do we see 87 'empty sets' or something close, nor do we see anything that might be understood as 'putting equally', so we can not really convince ourselves that he or she is thinking about Model A.

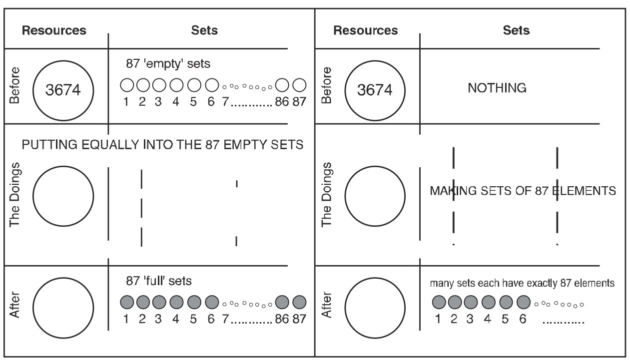

Figure 7: Physical settings of model A and model B.

That leaves us with Model B. Here, Anonymous is supposed to start only with that resource set, and we need to see some sort of 'making of sets', each with 87 elements. Using these glasses 'to look' at Anonymous's work we 'see' that there are "30 100's", which are 30 sets of a 'fixed' number of elements. We can now relate this to the practical goal in Model B, "make sets of size *x*". With this in mind, we can follow the work when Anonymous makes sets of hundreds and then 'corrects' them into the desired size. This also 'explains' the 13s that we see in the work and which many of the students had no clue where they came from at the beginning. Now we can 'finish' the work in the same way (see Figure 8).

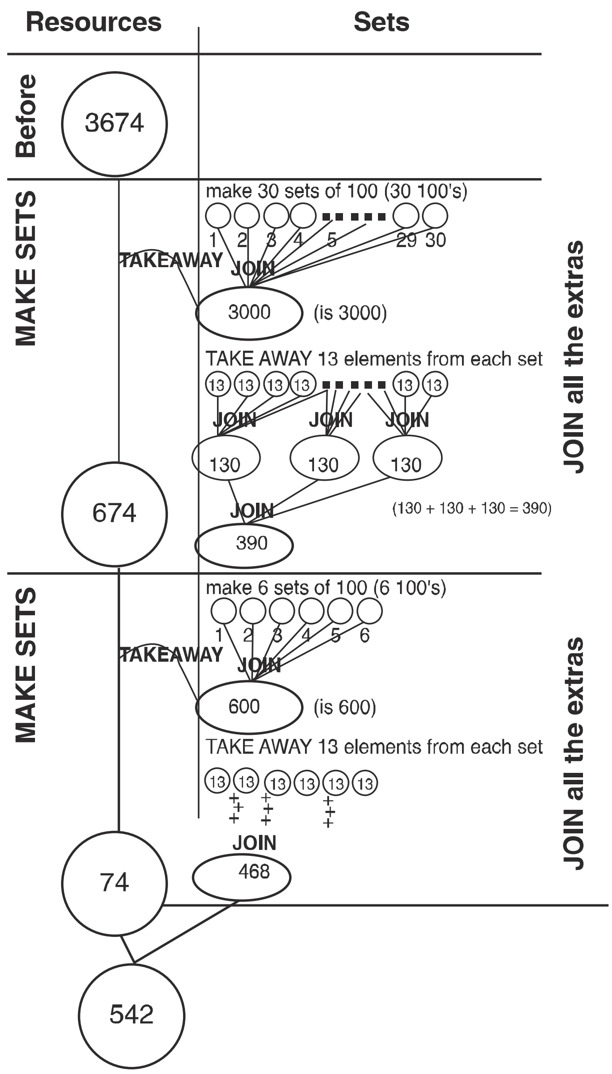

Figure 8: Physical reasoning of model B.

Discussion

Addressing the challenge of mathematics education in the elementary schools must begin with understanding of mathematics by the teachers themselves (see also, Brandt, 1994, p. 4). The title of this article suggests that the goal of the proposed teaching approach is a special kind of understanding – *understanding for teaching*. This, in turn, implies that the desired understanding that I require from my student teachers might be different from the understanding that we require from any other student (see also Cohen *et al.*, 1993, pp. 2-3; Sowder *et al.*, 1991).

Indeed, by the time they finish high school, prospective elementary school teachers 'know' most, if not all, of the mathematics they are required to teach their future students. However, their degree of understanding (Davis, 2001, p. 137) is not enough for 'teaching for understanding' for their future students. They do not know the 'whys', so how would they explain it to their students?

The TLS model presents the prospective teachers with the opportunity to increase the degree of complexity of their understanding of the basic arithmetic concepts (Thwaites, 1979). It offers them more contexts in which to apply their knowledge, as well as guiding them in making explicit the relations between the different contexts, the different languages, the different situations, the different concepts and the different modes of reasoning.

The underlying premise of the TLS approach is of understanding as a process (Sierpinska, 1990, p. 29; Pirie and Kieren, 1989). It steers the dynamic of folding back to move out from one subspace to another (Pirie and Kieren, 1989). The main path through the TLS is seen as leading the student teachers through Kieren's (1990) levels of understanding: beginning with primitive knowing (the contextual subspace), image making (the abstract-set model) and concluding with inventizing (the physical-abstract subspace).

The use of the abstract-set model to model the contextual situation involves identification, discrimination and generalization processes while the 'going back' from the physical-abstract realm to the mathematical realm, through the contextual realm, propels a synthesis process (Sierpinska 1990, p. 29).

The synthesis process leads to the *Ah-hah* instance – the moment in which one piece brings out the entire picture in the metaphor of the jig-saw puzzle (Davis, 1992, p. 192).

The TLS approach incorporates many elements that were recommended by mathematics educators to facilitate learning for understanding; "activities that ask them to generalize, find new examples, carry out applications" (Perkins and Blythe 1994, p. 6); production of knowledge through discourse, or performance, disciplined inquiry and personal value for the student (Newsman and Archbald, 1992); showing relationships between different modes of representation, such as, pictures, spoken symbols, manipulative aids, real-world situations, written symbols (Cramer and Bezuk, 1991, based on Lesh's model); modelling (Simon and Blume, 1994); oral and written communication (Senn, 1995); and concrete materials (Thompson and Lambdin, 1994).

How am I to know when the students "understand"?

> I consider an individual to have understood when he or she can take knowledge, concepts, skills and facts and apply them in new situations where they are appropriate. (Gardner, in Steinberger, 1994, p. 26-27)

Using 'conceptual drills' (see the section on conceptual framework) offers the students repetitive experiences of gradually increasing complexity to apply their knowledge and to test their understanding.

In addition, the use of multiple representations helps to lessen the authoritative nature of education, which inhibits the development of critical thinking on the part of the learner. Using multiple representations for a concept, showing that any situation could be resolved in many ways and that many seemingly different views are actually compatible, not only enriches students' understanding, but also helps them to develop 'a mind of their own', encouraging them to contribute their own ideas to the discussion, even if they are erroneous.

Another important aspect of this proposed approach is the informal 'proofs' or justifications that it offers for the knowledge that the students already possess, but are unable to explain or to justify. The abstract-set model, the physical-abstract doings' and the physical-language serve instead of the formal theorems and logic, which are used by mathematicians to prove/understand mathematical theorems. By 'physically' tracing each step of the 'statement' (*e.g.*, solution, algorithm, commutative rule) the student proves it is true or false. The physical-abstract-doing serves as what mathematicians refer to as insightful proof, a proof that offers a 'deep' understanding of the situation at hand.

For the most part, these prospective elementary school teachers do not possess the 'big picture'. Hence, the situating of their prior arithmetic knowledge in a meta-frame enables them to see part of it, and to better appreciate the kind of processes that make mathematics what it is. The proposed approach grants them a higher degree of flexibility in dealing with mathematical problems, even in complex situations.

Thus, they might design rich, insightful and enjoyable experiences for their future young students. They would be able to offer their young students concrete explanations, accompanied with relevant problem-stories and concrete ways of solving arithmetic problems by means of 'physical-abstract doings'. They might even use the TLS approach itself with their students. Though, the abstract levels might not be suitable for very young students (K-4, 5-10 years old) or for students with learning difficulties.

This, in turn, will help their students learn mathematics 'for understanding' rather than for knowing the algorithms. By employing the TLS approach we teach 'for understanding' the mathematics by our prospective elementary school teachers; we teach for understanding 'for teaching' (the deep understanding that is required from a teacher); and we teach for understanding for 'teaching for understanding' (the desired end results is that the future students of our students will 'understand' their mathematics).

The identification of learning difficulties of their future students is among the most important tasks of our future mathematics teachers. The proposed approach provides a tool that makes tracking down and pinpointing these difficulties easier, helping the teacher deal with the students' difficulties. For example, in one of the activities in class, students were asked to analyze an erroneous solution to a long division problem. Using the physical-abstract tools we noticed that the student who wrote the solution had a good understanding of the concept of division, and even of the long division algorithm, but he had difficulties subtracting numbers.

Notes

[1] A simple analysis of the titles in the ERIC database shows an increase of about 200% from the seventies to the nineties in the appearance of the word 'understanding' and an increase of only about 50% in the volume of articles that are concerned with learning or with teaching.

[2] Something like:

$4 \div 1$; $4 \div 2$; $4 \div 4$; $4 \div 8$; $1 \div 4$; $\frac{1}{2} \div 1$; $\frac{1}{2} \div 2$; $1 \div \frac{1}{2}$ ….

[3] For example, declarative statements: 'we have 3 sets', descriptive statements: 'there are 3, 2 and 3 elements in the sets', or relational statements: 'these 2 sets are of equal size' or 'this is larger than that'.

[4] All the 'Why?', and 'How' questions as well as 'What does this mean?', 'What if we change this?', 'Which units are used?' and so on.

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[The rest of the references can be found on page 38 (ed.)]

of ideas can further emphasize the role of reasoning to develop a sense of the correctness of answers. During our study, Dallas and Lloyd moved away from unproductive ways of dealing with errors, such as attributing errors to non-specific causes, toward identifying strategies and rules that led to their errors. As they discussed and were questioned about their errors, the social culture influenced their ways of operating with these errors.

We analyzed how two students recognized their errors, the factors they attributed their errors to, and the strategies they used to reconcile these errors using the framework we call the *reflective cycle of error analysis*. As such, this framework builds upon the instructional recommendations of Hiebert *et al*. (1997) and Borasi (1987) by providing insight into the ways students address the errors they recognize. Current instructional recommendations place increased emphasis on assisting students in becoming autonomous problem solvers who can diagnose and use their errors in productive ways. Our framework can provide guidance to teachers in using student errors to deepen understanding of mathematical ideas. The process of recognizing, attributing, and reconciling errors is a complex one that has often been underemphasized in the mathematics classroom. However, this process is critical in many careers where mathematics plays an important role, such as the domains of computer science and engineering. We must further understand how we can encourage students to make use of their errors productively rather than simply to ignore them.

Notes

[1] Quotation attributed to "Anon." found at: http://www.worldofquotes.com/topic/Error/1/index.html, accessed 25th August, 2006.

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[These references follow on from page 29 of the article "Teaching for understanding for teaching" that starts on page 24 (ed.)]

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