A COVARIATIONAL UNDERSTANDING OF FUNCTION: PUTTING A HORSE BEFORE THE CART

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> A **function** f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E. (Stewart, 2016, p. 10)

> What is a good definition? For the philosopher or the scientist, it is a definition which applies to all the objects to be defined, and applies only to them; it is that which satisfies the rules of logic. But in education it is not that; it is one that can be understood by the pupils. (Poincaré, 1908, quoted in Tall, 1988, p. 37)

Poincaré's criteria for a "good definition" are especially relevant to modern set-theoretic definitions of function often found in curricula. Definitions, such as Stewart's, emphasize two key properties, univalence explicitly and arbitrariness implicitly (Even, 1990). Univalence is the property that each element in the domain is associated with a unique element in the range. Arbitrariness refers to an association between sets that need not be defined by a known correspondence rule or curve representing a generalized regularity between sets. These properties allow modern function definitions to apply "to all the objects to be defined, and applies only to them" across several mathematical domains including real and complex numbers. However, researchers examining students' function understandings indicate modern function definitions are generally not "understood by the pupils". Based on this research, we infer the modern function definition as currently used for educational purposes is not a "good definition" per Poincaré's criteria.

In an effort to re-conceptualize the notion of function in school mathematics, Thompson and Carlson (2017) presented a covariational understanding of function, which resembles earlier mathematicians' understandings of function as representing constrained variation. In this paper, we draw on relevant literature on students' function understandings and Thompson and Carlson's description of a covariational meaning of function to illustrate how students' reasoning about covarying quantities can provide a foundation for more formal aspects of function. We include examples of a student's activity to highlight nuances in Thompson and Carlson's description of a covariational meaning of function and to illustrate how such a meaning can be productive for a student. We contend that students who develop meanings compatible with the covariational meaning of function presented by Thompson and Carlson likely have the horse (*i.e.,* foundational understandings) needed to pull a cart (*i.e.,* a formal definition of function).

Students' and teachers' function definition: words without meaning or motivation

Speaking on concepts associated with a formal function definition, almost a century ago Thorndike *et al.* (1926) contended:

> It is one of those fundamentally powerful conceptions whose elaboration has been one of the half dozen significant achievements of the [human] race, but to the high school student it is vague and tantalizing and stimulating rather than clarifying. (p. 82)

Thorndike and colleagues' assertion that students find the function concept "vague and tantalizing" rather than clarifying has been loudly echoed by research on students' function understandings (see Oehrtman, Carlson & Thompson, 2008).

Specific to a function definition, several researchers have distinguished between students' *concept definitions* and *concept images* (Tall, 1988; Vinner, 1983). Whereas a student's concept definition consists of the words used to describe a concept, a student's concept image is, "The total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (Tall & Vinner, 1981, p. 152). These researchers have indicated that often the imagery a student uses in one function situation is inconsistent or incongruent (from the researchers' perspective) with imagery in another situation. Vinner (1983) and Tall (1988) argued that although students can verbalize a definition of function (*i.e.,* have the cart), students almost exclusively rely on their concept images when addressing problems not explicitly asking for a definition. Hence, the extent to which introducing a formal function definition to students at an early age is beneficial to their mathematical development is an open question.

Several researchers have shown that students and teachers do not perceive a need for the univalence and arbitrariness properties of function. For example, Even (1990) indicated that pre-service secondary teachers' function definitions did not support their considering arbitrarily defined functions (*i.e.,* the Dirichlet function) as functions and noted, "some serious questions are raised by the fact that, without prompting, none of the subjects could come up with a reasonable explanation for the need for the property of univalence" (p. 531). Reflecting on this and other studies, Even and Bruckheimer (1998) questioned current approaches to teaching function that emphasize univalence, ultimately suggesting that researchers and educators be open to considering the historical development of function including its initial roots in relationships between variables.

Addressing function tasks: actions without understanding

Whereas the aforementioned studies highlight shortcomings in students' and teachers' understandings of a function definition, other studies underscore that students' and teachers' meanings for function are often constrained to engaging in specific actions. These actions often implicitly or explicitly require conventions commonly used in school mathematics be maintained. For example, researchers internationally have noted that students' meanings for function in graphical contexts can foreground the application of the *vertical line test* [1]. Montiel, Vidakovic, and Kabael (2008) described students applying the vertical line test to conclude the relationship defined by $r = 2$ in the polar coordinate system (*i.e.,* a circle) did not represent a function. As second example suggesting students' relying on the vertical line test, Breidenbach and colleagues (1992) demonstrated that only 11 of 59 students classified the graph in Figure 1a as representing a function (*i.e.,* the quantity's values represented along the horizontal axis as a function of the quantity's values represented along the vertical axis).

Extending these studies, Moore, Silverman, Paoletti, Liss & Musgrave (2018) demonstrated both pre-service and in-service teachers' meanings for function are often restricted either to the vertical line test or to the convention [2] of representing a function's input on the horizontal axis in the Cartesian coordinate system. For example, the researchers prompted teachers with the graph in Figure 1b and the work of a hypothetical student who claimed the graph represented "x is a function of y." Only seven of the 25 pre-service and 11 of the 31 in-service teachers understood the student's statement as mathematically correct; a majority of pre-service and in-service teachers concluded the statement was invalid either because the graph failed the vertical line test or because the teacher understood that a function's input was necessarily represented on the horizontal axis. In this, and the aforementioned examples (Breidenbach *et al.,* 1992; Montiel *et al.,* 2008), the researchers presented graphical representations they intended to be representative of functions, yet the students' and teachers' meanings for functions and their graphs did not support them in concluding that the graphs were representative of functions.

In analytic representations (*i.e.,* equations), Sajka (2003) detailed how to a student function notation was more about what "we usually write" (2003, p. 247) than about how to represent her ideas and reasoning. Sajka indicated the student produced inconsistencies in her use of function notation (some only from the researcher's perspective and some she was aware of) because she conflated what "we usually write" and essential aspects of a mathematical idea. Further, and consistent with studies examining students' function understandings in graphical contexts, the student assimilated examples in ways that were consistent with her image but inconsistent with or inattentive to the researcher's intentions.

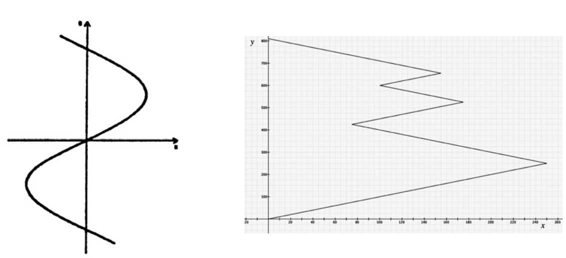

Figure 1. Graphs from (a) Breidenbach et al. (1992, p. 281) and (b) Moore et al. (2018).

The aforementioned research indicates that although teachers and students can recite a function definition, they do not apply this definition and its mathematical properties (*e.g.,* univalence) in flexible ways when addressing prompts related to function. Instead, students and teachers rely on repeating actions related to how graphs or analytic rules 'appear' when addressing questions regarding function in various representations. We take these collective findings to indicate that we give students and teachers a function definition (*i.e.,* the cart) without sophisticated mathematical experiences to support their flexibly understanding and applying the mathematical properties of a function definition (*i.e.,* the horse needed to pull the cart).

A covariational understanding of function

Modern set-theoretic function definitions are a culmination of centuries of mathematical development. Initial roots of the function definition include investigating deterministic relationships between values that vary in tandem and writing equations to relate these varying values (Boyer, 1946; Thompson & Carlson, 2017). These early developments evolved into determining a method to express *any* function, with function meaning a relationship between variable quantities that can be represented by a drawn curve or analytic expression. In response to Fourier's efforts in this regard, Dirichlet introduced the function named in his honor and a more formal function definition emphasizing a precise law of correspondence (Boyer, 1946). Dirichlet's definition opened the door for functions to be defined on objects other than (varying) real values. Subsequent mathematical developments ultimately emphasized a set and ordered pair focus commonly found in today's curricula while deemphasizing the variation and covariation foundations of function (Thompson & Carlson, 2017).

Drawing on the historical development of the function concept, their body of work, and the growing body of literature highlighting the importance of students reasoning about quantities that change in tandem (*e.g.,* Castillo-Garsow, Johnson & Moore, 2013; Confrey & Smith, 1995; Ellis, 2011), Thompson and Carlson proposed a definition of function that returns to its covariational roots. They described:

> A function, covariationally, is a conception of two quantities varying simultaneously such that there is an invariant relationship between their values that has the property that, in the person's conception, every value of one quantity determines exactly one value of the other. (2017, p. 444)

Rather than emphasizing univalence, we interpret Thompson and Carlson (2017) to foreground an individual first *constructing* an invariant relationship; once an individual has conceived of such a relationship (and potentially a network of quantities), she can begin to investigate properties of that relationship, including univalence. Hence univalence becomes a particular property of an invariant relationship. Similarly, the focus on constructing an invariant relationship lessens the importance of a known correspondence rule between quantities' values (*i.e.,* arbitrariness); constructing an invariant relationship involves understanding quantities' values as existing in tandem regardless if a rule exists.

Relatedly, Thompson and Carlson (2017) avoid referencing notions of dependence and independence in their definition. They explained, "What is independent and what is dependent will depend entirely on the person's conception of the situation and which way they envision dependence, if they envision dependence at all" (p. 444). For instance, in conceiving of a relationship between one's height and shoe size, it is not necessary to think of one quantity as dependent upon the other; both quantities exist and vary simultaneously. However, Thompson and Carlson also indicated actively conceiving a relationship entails some cognitive sense of dependency of one quantity to another, as an individual must think of one quantity before the other. Thompson and Carlson (2017) added, "it is through covariation that the dependency becomes crystalized in her thinking as being invariant across quantities' values" (p. 444). However, the extent to which students absolutely maintain a dependency of one quantity in relation to another when conceiving of an invariant relationship between two quantities remains an open question.

Constructing a covariational understanding of function: the case of Arya

In order to provide an example of a student who maintained meanings consistent with those described by Thompson and Carlson, we draw from data collected during a semester-long teaching experiment with an undergraduate student, Arya. Arya was beginning her first pair of courses (content and pedagogy) in a secondary mathematics teacher preparation program in her third year of university studies. Arya had completed a calculus sequence and at least two additional mathematics courses (*e.g.,* linear algebra or differential equations) with a minimum grade of a C in each course. At the outset of the teaching experiment, Arya exhibited function understandings consistent with those reported elsewhere. For instance, when presented with the hypothetical student's statement that the graph in Figure 1b represented a function because "x is a function of y", Arya argued the student was incorrect as she presumed the input (y) must be represented on the horizontal axis for the student's statement to be viable.

Throughout the teaching experiment, Arya repeatedly conceived of and constructed relationships between covarying quantities in dynamic situations and represented these relationships graphically. For instance, Arya modeled phenomena including a rider's displacement on an amusement park ride (Moore, Silverman, Paoletti & LaForest, 2014), the height and volume of liquid in a bottle as liquid enters and leaves the bottle (Paoletti & Moore, 2017), and various quantities in circular motion contexts (Moore, 2014). During these activities we intentionally prompted little to no focus on univalence and arbitrariness. In what follows, we focus on Arya's activity towards the end of the teaching experiment that is particular to Thompson and Carlson's description of a covariational meaning of function and refer the reader to Paoletti (2015) for more detail on the study and her specific activities in each setting.

Conceiving of an invariant relationship in the Car Problem

Arya addressed an adaptation of the *Car Problem* designed by Saldanha and Thompson (1998). Consistent with Saldanha and Thompson's use of the task, we asked Arya to graphically represent the relationship between an individual's (Homer's) distances from two cities (Shelbyville and Springfield) as he travels back-and-forth on a road (Figure 2). We adapted the task by asking about 'function' *after* Arya constructed her graph; we address her response to this prompt in the next section.

Arya's activity throughout the task suggests she conceived of "two quantities varying simultaneously such that there is an invariant relationship between their values" (Thompson & Carlson, 2017, p. 444). Arya consistently focused on the relationship between Homer's distances from the two cities through constructing the relationship in the situation and then representing her conceived relationship graphically. To illustrate, she first described the directional covariation of Homer's distance from each city (*e.g.,* as Homer moves from the beginning of his trip, the distance from each city decreases), and then drew a segment from right to left corresponding to decreasing ordinate and

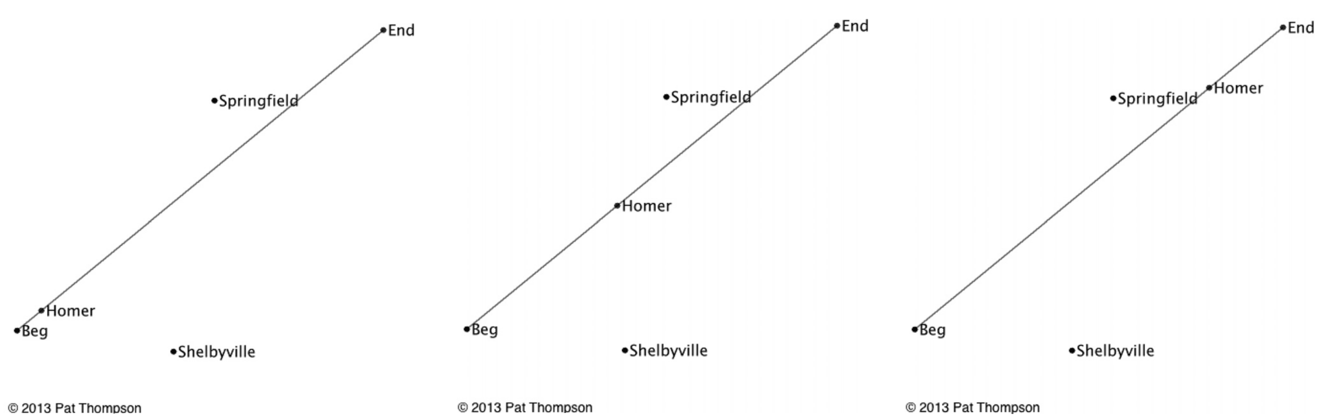

Figure 2. Several screenshots from the Car Problem applet. [3]

abscissa magnitudes (indicated by ① in Figure 3a). Arya pointed to the applet and described:

> We start off *[...]* far from Springfield and pretty close to Shelbyville *[points to 'Beg' (see Figure 2) then traces along road].* Then *[...]* you're getting closer to Shelbyville for a little ways and closer to Springfield as we're moving along the road.

With respect to her graph, Arya marked horizontal and vertical dashed lines from each graphed point to the vertical and horizontal axes, respectively, to verify that she represented distances from Shelbyville and Springfield each decreasing (indicated by ② and ③ in Figure 3a). Arya continued such actions to construct and justify the other two segments in her graph (Figure 3b).

Consistent with Thompson and Carlson's (2017) description, Arya did conceive of one quantity first when constructing a covariational relationship in the situation and graph; however, Arya alternated which quantity she considered first as she progressed through the task. As described above, before drawing her first segment (seen in Figure 3a) Arya first described how Homer's *distance from Springfield* varied. Before drawing her second segment (seen in Figure 3b), Arya first described how Homer's *distance from Shelbyville* varied (*i.e.,* "We're moving away from Shelbyville after that and closer to Springfield"). Finally, before drawing the third segment, Arya again first described how Homer's *distance from Springfield* varied (*i.e.,* "And then, we move away from Springfield again, and away from Shelbyville"). Arya's actions were a contraindication that she conceived an explicit dependent-independent relationship of either distance to the other. Arya instead maintained a focus on two quantities simultaneously covarying whilst alternating which quantity she considered first in order to accurately construct and represent the relationship she conceived.

Addressing questions about 'function' in the Car Problem

After conceiving of and representing a relationship between covarying quantities, we asked Arya if she could "talk about anything in this situation in terms of things being functions?" Because the relationship is such that distance from Shelbyville is not a function of distance from Springfield or vice versa, we conjectured Arya may spontaneously consider other quantities in the situation that were not directly represented in the graph.

In contrast to the students and teachers reported on by others, rather than being restricted to applying the vertical line test or presuming the horizontal axis must represent the input of a graphed function in the Cartesian coordinate system, Arya first considered whether either graphed quantity was a function of the other graphed quantity. She stated, "If you take either like the distance from Springfield or the distance from Shelby[ville] as your input you're going to have more than one output in some places *[...]* So neither are really functions." Intending to investigate if Arya conceived of any functional relationships in the situation, the first author asked, "Is there anything else we could have asked you about that may or may not have represented a function?" In response, Arya considered either distance from a city "and how it travels over total distance." Furthermore, for both distance from Springfield and distance from Shelbyville, Arya considered using either distance from a city *or* total distance as the input quantity to make conclusions regarding the 'function-ness' of each possible input-output pair. For example, she concluded that Homer's distance from Springfield is a function of his total distance traveled, but his total distance traveled would not be a function of his distance from Springfield. Notably, Arya referred only to the dynamic image of the situation (Figure 2) as she considered if each distance from a city corresponded to exactly one total distance traveled and vice versa; Arya did not need a graph, equation, or table to make conclusions regarding the 'function-ness' of the relationships she conceived in the situation.

Returning to the graph, and because Arya had added an arrow to it (Figure 3b), we conjectured she may reason about the trace of her graph as being defined parametrically and thus representative of a function. The second author asked, "What if your input was total distance traveled and your output was *[...]* a pair of values. Where that pair is your distance from Springfield and your distance from Shelbyville *[...]* What do you think about that case?" Arya identified this relationship as having a "two-dimensional

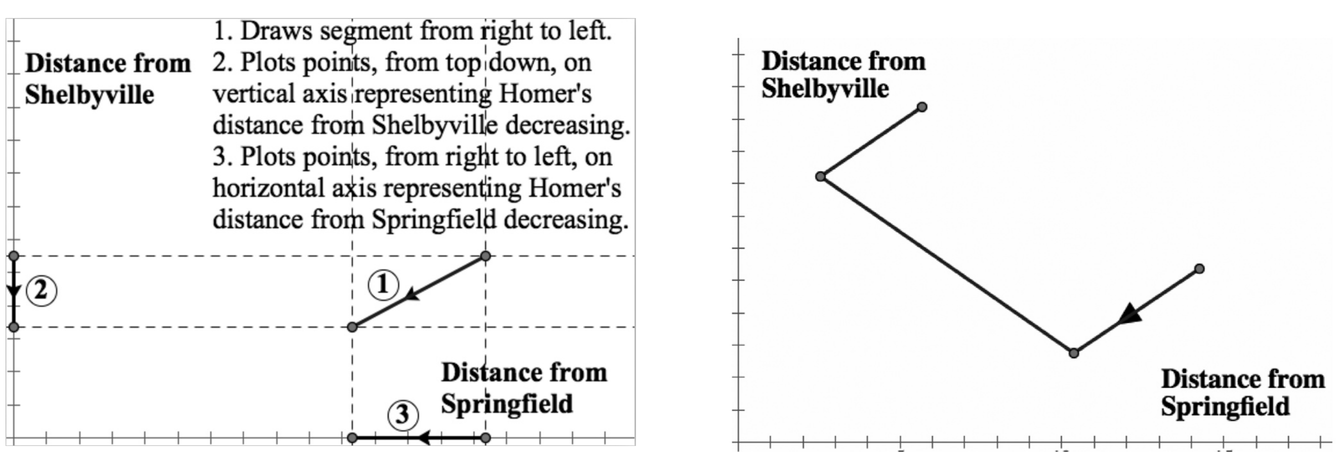

Figure 3. (a) A re-creation of Arya's initial work, and (b) a re-creation of Arya's final graph.

output" with an input "not on the graph." With respect to the 'input', she discussed two quantitative interpretations of 'total distance': (1) Homer making one trip from Beg to End (see Figure 2) and (2) Homer traveling back and forth along the road accumulating total distance throughout. Arya stated that in either case the relationship represented a function. With respect to the latter case, she explained, "They *[referring to two different total distances at the same location on the road]* can hit the same output it's just *[…]* A single input can't have multiple outputs."

The first author next asked Arya to consider the case where she starts with a two-dimensional input and an output of accumulated total distance as Homer travels back and forth.

Arya — Okay hold on. *[13 second pause]* No that wouldn't work, right? Because we're always doing, this would be our input again *[motioning over the curve on the computer screen]*, like our input pairs, and then we're going to have more than one total distance because we're just moving back and forth *[motioning over the curve on the computer screen]* across those input pairs so we're going to hit those more than once for total distance would just be moving back and forth. So you'd have, for each input *[pointing and holding her finger on a specific location on the curve on the computer screen]* that could correspond to a number of distances *[…]*

TP — Okay and so what if we restricted?

Arya — To once?

TP — To once like you were doing *[referring to Homer taking one trip from Beg to End]*.

Arya — *[5 second pause]* Then I think we're okay. Hold on. Each input just has one *[11 second pause, looking at screen and making hand motions as if tracing curve]*. No right. *[looking at screen intently and motioning towards screen, 13 second pause]* Yeah I think we're okay. 'cause each input value still in one *[trip from beginning to end]* each input value *[tracing over the graph on the computer screen]*, we're using the input values, no we're not. Okay each ordered pair, cause it's an ordered pair, that's going to mess me up. Each input value that corresponds to one total distance as you move along *[tracing over curve on computer screen]*. That's good. Right? *[8 second pause]* Yeah.

With respect to considering the distance pairs as the input and the two interpretations of total distance as the output, Arya identified there was a unique total distance for each coordinate pair in case (1) but not for each coordinate pair in case (2).

We interpret Arya's activity to highlight how a student conceiving of an invariant relationship between covarying quantities consistent with Thompson and Carlson's description can use this understanding to determine if there was a "relationship between their values that has the property that *[…]* every value of one quantity determines exactly one value of the other" (2017 p. 444). Additionally, Arya did not require correspondence rules or a descriptive mapping in order to consider the 'function-ness' of each relationship. We highlight how Arya's rich quantitative conception of the situation enabled her to consider the notion of function more broadly than between a single pair of covarying quantities; in total, Arya's image of the situation enabled her to consider 10 different possible functional relationships (see Table 1), regardless if the relationship was explicitly represented by a graph, equation, or table.

Table 1: The relationships considered by Arya as possibly representing functions.

Input	**Output**	**Function?**
Distance from Springfield	Distance from Shelbyville	No
Distance from Shelbyville	Distance from Springfield	No
Total distance traveled	Distance from Springfield	Yes
Distance from Springfield	Total distance traveled	No
Total distance traveled	Distance from Shelbyville	Yes
Distance from Shelbyville	Total distance traveled	No
Total distance traveled (one trip)	(Distance from Shelbyville, Distance from Springfield)	Yes
Total distance traveled (accumulated)	(Distance from Shelbyville, Distance from Springfield)	Yes
(Distance from Shelbyville, Distance from Springfield)	Total distance traveled (one trip)	Yes
(Distance from Shelbyville, Distance from Springfield)	Total distance traveled (accumulated)	No

A covariational understanding supporting unconventional analytical representations

We present Arya's activity in the *Cylinder Problem* to highlight how maintaining a covariational meaning of function also provided Arya with flexibility in interpreting analytic rules as representing an invariant relationship involving quantities' covariation. In the *Cylinder Problem*, we asked Arya to graph the relationship between surface area and height of a dynamic cylinder with a constant radius and varying height (see Figure 4a). Arya described imagining surface area in terms of "unwrapping" the cylinder and defined surface area as, "Circumference times the height plus *[...]* two times the area of the base." Arya wrote the equation $SA = 2\pi rh + 2\pi r^2$ (Figure 4b), noted that r was constant and h varied, and concluded the relationship between surface area and height was linear.

Shortly after Arya wrote her original rule, the first author asked Arya "to write the inverse" of her rule, by which he meant she should take the height as the input of the relation. He wished to examine how Arya may use a rule written in a different form that implied surface area as the input. Arya determined the rule $h = (SA - 2\pi r^2)/(2\pi r)$ (Figure 4b). The following conversation ensued:

TP Okay so in this one *[pointing to* $h = (SA - 2\pi r^2)/(2\pi r)$] you're doing what?

Arya Inputting surface area outputting height *[pointing to SA and h, respectively, in* $h = (SA - 2\pi r^2)/(2\pi r)$*]*.

TP Okay and in this one *[pointing to* $SA = 2\pi rh + 2\pi r^2$]?

Arya Inputting height and outputting surface area *[pointing to h and SA, respectively, in* $SA = 2\pi rh + 2\pi r^2$*]*. Ah it doesn't really matter either way but yes.

TP So what do you mean by it doesn't really matter either way?

Arya I mean it doesn't, it doesn't matter *[pointing to* $SA = 2\pi rh + 2\pi r^2$*]*, you change it but you'd have to solve for this *[pointing to* $h = (SA - 2\pi r^2)/(2\pi r)$*]* to do the math so. I mean it doesn't matter which one your inputting *[pointing back and forth between h and SA in* $SA = 2\pi rh + 2\pi r^2$*]* if you, if you're changing surface area *[pointing to SA in* $SA = 2\pi rh + 2\pi r^2$*]*, you can still solve for h *[pointing to h in* $SA = 2\pi rh + 2\pi r^2$] using this formula but it ends up being this *[pointing to* $h = (SA - 2\pi r^2)/(2\pi r)$*]* so it doesn't matter I don't think.

Although Arya initially rewrote the equation in '$h =$' form to represent the relationship when she is considering first "changing surface area", she immediately argued she can consider either quantity (height or surface area) as changing first in either equation she had written. In contrast with the student reported by Sajka (2003), Arya understood that regardless of which quantity she considered as the input, an invariant relationship existed, and that conception applied to

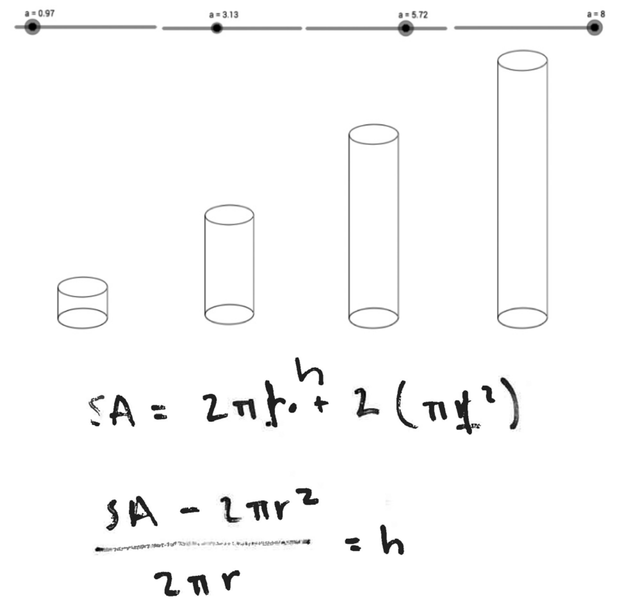

Figure 4. *(a) Screenshots from the Cylinder Problem and (b) Arya's work determining the inverse of* $SA = 2\pi rh + 2\pi r^2$.

the analytic rule regardless of how the equation is written with respect to an implied dependency.

Concluding remarks

Echoing Poincaré's criteria of a "good definition", Jayakody and Zazkis (2015) synthesized mathematics education literature to argue there is an abundance of evidence of the importance of developmentally and mathematically appropriate definitions. By providing an example of a student maintaining a covariational definition of function as described by Thompson and Carlson, we highlight how this definition can be both mathematically and developmentally powerful relative to formal properties typically associated with a modern definition of function. Arya's activity conveys how a student who understands "two quantities varying simultaneously such that there is an invariant relationship between their values" can leverage this understanding to determine if "every value of one quantity determines exactly one value of the other" (2017, p. 444). Further, Arya did not need a known correspondence rule or curve to draw conclusions about the 'function-ness' of each relationship. Hence, Thompson and Carlson's definition meets the criteria set forth by Poincaré, Jayakody, and Zazkis.

In addition to highlighting why maintaining a covariational understanding of function is powerful for students, Arya's activity addressing the *Car Problem* allowed us to clarify Thompson and Carlson's notion that an individual must conceive of one quantity varying first, but this does not imply the student's reasoning necessarily entails a direction of dependency between two quantities. The quantity Arya considered first switched throughout her activity, which illustrates that a student can move flexibly between considering either of two quantities first as she conceives of and represents a relationship between quantities.

Further, Arya's activity in both the *Car Problem* and *Cylinder Problem* underscore how a conceived relationship between covarying quantities can provide students with something more to reason about than applying the vertical line test or focusing on "what we usually write" (Sajka, 2003). In essence, a structure of covarying quantities and, more broadly, an image of a situation entailing a robust quantitative structure can serve as the *something* (Thompson, 1994) that provides students flexibility in reasoning about 'function' and associated mathematical properties (*i.e.,* univalence and arbitrariness) in multiple representations. Arya leveraged reasoning about relationships between quantities to determine whether numerous relationships, most of which were represented neither by a known analytic rule or a graph (*i.e.,* arbitrariness), had the property that for each value of one quantity there was exactly one value of the second quantity (*i.e.,* univalence). Returning to the opening analogy, we contend that supporting students in constructing invariant relationships between covarying quantities (and, more broadly, a quantitatively sophisticated image of a dynamic situation) has the potential to provide them with the horse needed to pull the cart that contains mathematical properties important for the formal definition of function in school mathematics.

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Notes

[1] The vertical line test is a commonly taught technique in several countries, and is pervasive in US instruction and curricula, to determine if a curve in a Cartesian coordinate system represents a function. The technique involves imagining a vertical line sweeping horizontally across a curve in a Cartesian coordinate. If the vertical line intersects the curve at more than one point, the curve does not represent a function.

[2] We note that our use of convention here is from our perspective, as the teachers' actions suggest that such a practice was *not* a convention from their perspective, but instead critical to the mathematics.

[3] Reproduced with the kind permission of Pat Thompson.

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